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August 1975 No. 75-45

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A Diagram for Transforming Duopoly Output Levels into Duopoly Profit Levels

by Dermot Gately and Suk-Mo Koo*

In this short paper we explain a four-quadrant diagram used to depict the relationship between duopolists' output levels and their corresponding profit levels. The diagram provides graphic insight into several standard problems in duopoly: the incentives for each producer to expand or restrict output; the incentives for them to form market-share agreements and to abide by or cheat on such agreements, and the credibility and potency of certain threats of unilateral output expansion to enforce market-share discipline; and the profit-level analogues of the Cournot reaction functions and the dynamics of the movement to the Cournot equilibrium.

Using a linear demand curve and average cost functions that are either constant or quadratic (listed in Table 1), we depict the effects of various changes in the duopolists' output levels upon their respective profit levels (Figures 1 and 3); we also depict the implications for profit levels in a Cournot world (Figures 2 and 3). Though our illustrations are based on particular numerical functions, the method is of course independent of these examples.

Table 1

Numerical Functions Used[1]

	Case 1 (Figures 1 and 2)	Case 2 (Figure 3)
demand function	$P = 10-2Q$	same as for Case 1
	$= 10-2(Q_1+Q_2)$	
average cost functions	$AC_1 = 1$	$AC_1 = 4-Q_1+Q_1^2$
	$AC_2 = 1$	$AC_2 = 5-Q_2+Q_2^2$
profit functions	$\pi_1 = 9Q_1-2Q_1Q_2-2Q_1^2$	$\pi_1 = 6Q_1-2Q_1Q_2-Q_1^2-Q_1^3$
	$\pi_2 = 9Q_2-2Q_1Q_2-2Q_2^2$	$\pi_2 = 5Q_2-2Q_1Q_2-Q_2^2-Q_2^3$
Cournot reaction functions	firm I: $Q_1 = 9/4 - .5Q_2$	implicit form: firm I: $6-2Q_1-3Q_1^2-2Q_2 = 0$
	firm II: $Q_2 = 9/4 - .5Q_1$	firm II: $5-2Q_2-3Q_2^2-2Q_1 = 0$

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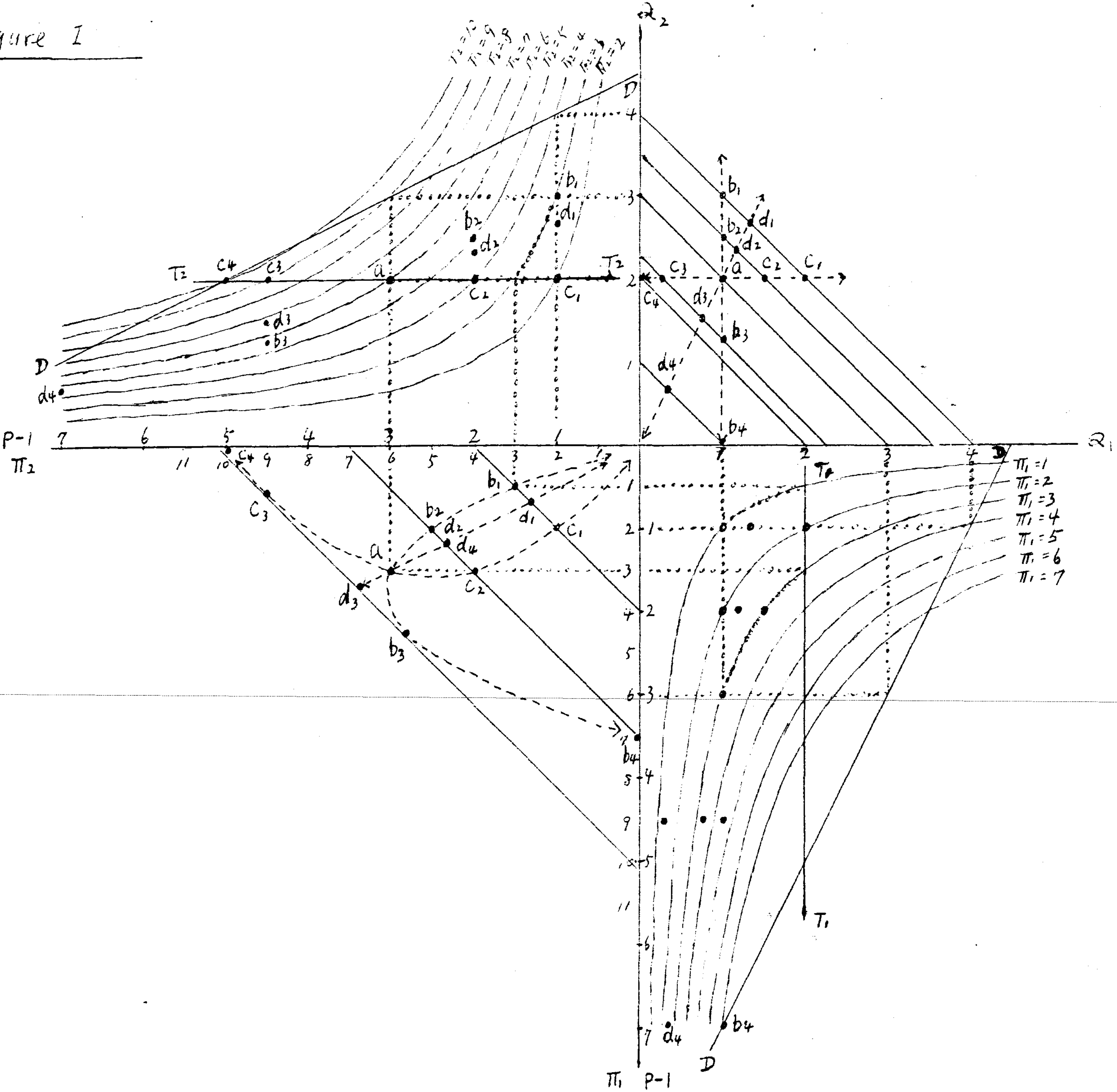
* New York University. The authors would like to thank William Baumol for helpful comments on an earlier draft.

Figure I



Consider the demand and cost functions of Case 1, illustrated in Figure 1. The NE quadrant, with axes Q_1 and Q_2, depicts various pairs of output levels for firm I and firm II; six total-output iso-Q diagonal lines are shown: Q=1, Q=2, Q=2.25, Q=3, Q=3.5, Q=4. The SW quadrant, with π_1 on the left side of the South axis and π_2 on the bottom side of the West axis, shows the profit levels for each firm; three total-profit iso-π diagonal lines are depicted: $\pi=4$, $\pi=7$, $\pi=10.125$. Starting with point a in the NE quadrant ($Q_1=1$, $Q_2=2$), three types of output change are shown: horizontal movement, changing only Q_1; vertical movement, changing only Q_2; and proportional change in Q_1 and Q_2 along a ray from the origin. The resulting changes in profit levels are traced in the SW quadrant; each pair of profit levels is denoted by the letter of the corresponding pair of output levels.

The procedure for determining π_1 as a function of Q_1 and Q_2 is described in the SE quadrant, which has Q_1 on the East axis and P-1 (where $1=AC_1$) on the right side of the South axis. An analogous procedure, left to the reader, is used to determine π_2 as a function of Q_1 and Q_2 via the NW quadrant, which has Q_2 on the North axis and P-1 (where $1=AC_2$) on the top side of the West axis. In the SE quadrant are depicted seven iso-π_1 curves; these are rectangular hyperbolas.[2] Superimposed on the SE quadrant is the demand curve DD; from it we can determine P as a function of[3] Q. Consider point a in the NE quadrant, at which $Q_1=1$, $Q_2=2$, Q=3. The price associated with Q=3 may be found by moving diagonally along the iso-Q line Q=3 until it intersects the Q_1 axis, and from that point move vertically down to the demand curve DD in the SE quandrant,

[1] The numerical examples were taken from M. Shubik, Strategy and Market Structure (New York: John Wiley and Sons, Inc., 1959), Chapter 4.

[2] Since $\pi_1=PQ_1-(AC_1)Q_1=(P-AC_1)Q_1$, the iso-$\pi_1$ curves will always be rectangular hyperbolas if the axes are Q_1 and $P-AC_1$. Similarly for the iso-π_2 curves.

[3] The demand curve P=10-2Q is superimposed on the SE quadrant, with axes P-1 and Q; note that the East axis, for purposes of the demand curve, should be read as Q_1. It is a normal demand curve except that the East axis has been shifted up by one unit, so that the right side of the South axis is P-1, not P. The demand curve intersects at p-1=9 (not shown) and at Q=4.5, rather than at P=10 and Q=5 in a normal diagram.

where that vertical distance below the horizontal axis measure P-1; in this case P-1=3, that is, P=4. The resulting profit level π_1 is simply the product of Q_1 and P-1; for point a we have $\pi_1=3$, on the iso-π_1 curve $\pi_1=3$ in the SE quadrant. By an analogous procedure we find that $\pi_2=6$ for point a. For a different point, b_1 ($Q_1=1$, $Q_2=3$, Q=4), we have P-1=1, $\pi_1=1$ and $\pi_2=3$.

Now for the final step in mapping a pair of output levels in the NE quadrant onto a pair of profit levels in the SW quadrant. We have, for each pair of output levels in the NE quadrant, a point on some iso-π_1 curve in the SE quadrant and a point on some iso-π_2 curve in the NW quadrant; from these two points, using the transformation lines T_1T_1 and T_2T_2 respectively, we can determine the π_1 and π_2 coordinates for the corresponding point in the SW quadrant. We shall describe the procedure for determining the π_1 coordinate; derivation of the π_2 coordinate procedes analogously. For any iso-π_1 curve the points are at various vertical distances below the horizontal axis, and they can all be transformed into the same vertical distance (π_1 coordinate in the SW quadrant), at the point at which that iso-π_1 curve intersects the transformation line[4] T_1T_1. For point a or any other point on the same iso-π_1 curve, $\pi_1=3$, the π_1 coordinate in the SW quadrant is given by the vertical distance below the horizontal axis at the intersection of the iso-π_1 curve $\pi_1=3$ and the transformation line T_1T_1.

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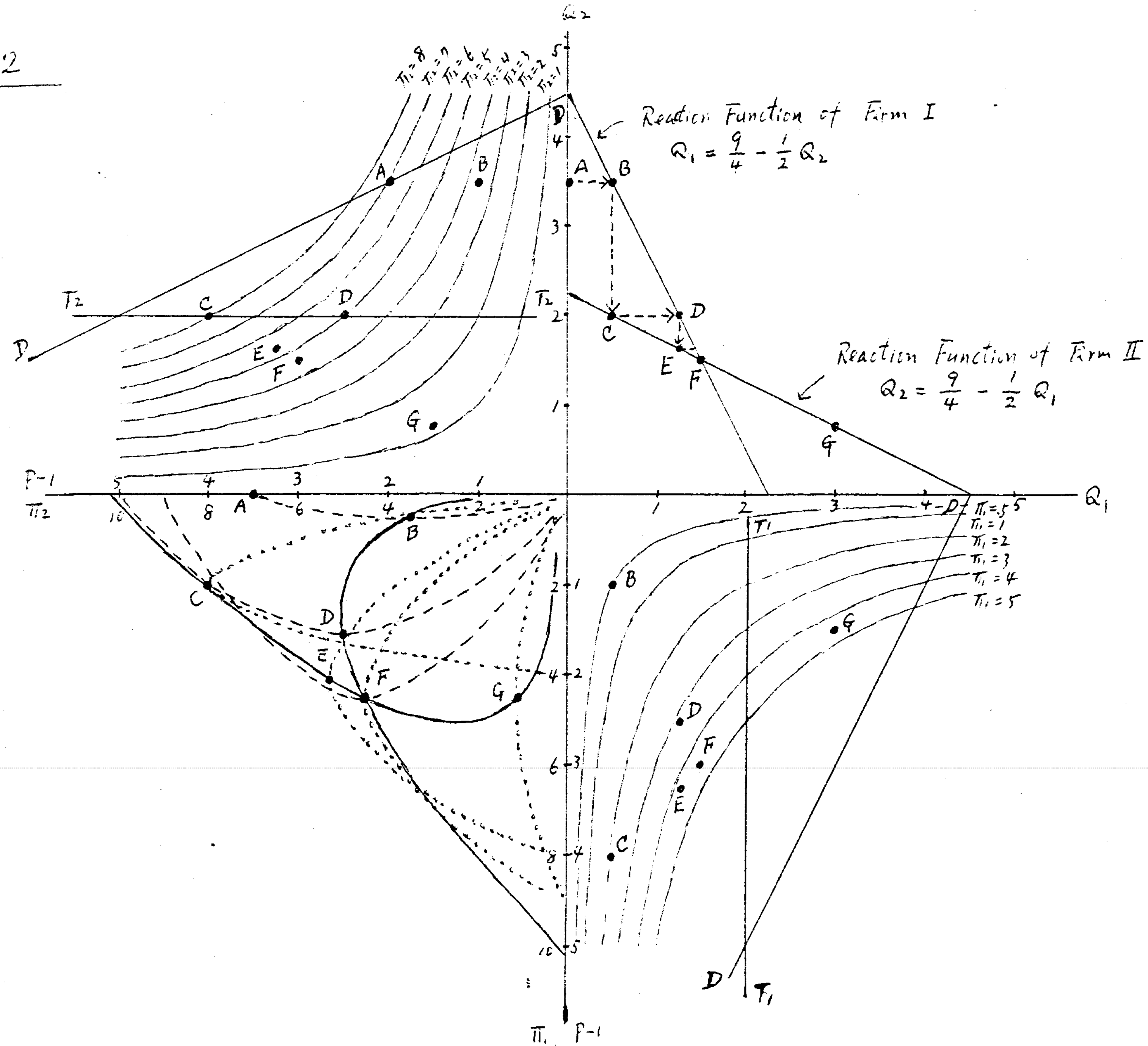
[4] Any straight line parallel to the P-AC axis can be used as the transformation line. In Figure 1 line T_1T_1 is straight and parallel to the P-1 axis at the point $Q_1=2$. (The position of the line determines the scale of π_1 in the SW quadrant; if the line shifts from the point $Q_1=2$ to $Q_1=1$, the scale will be doubled.) Suppose that π_{11}, π_{12}, and π_{13},...denote the vertical distances from the axis to the points where line T_1T_1 intersects the iso-$\pi_1$curves $\pi_1=1$ $\pi_1=2$,and $\pi_1=3$,---,respectively. Then the ratios $\pi_{11}:\pi_{12}:\pi_{13}$---=1:2:3---, since all the values of π_{11},π_{12},and π_{13} are determined by 1/2 π_1 (where $2=Q_1$); $\pi_{11}=1/2$, $\pi_{12}=2/2$, and $\pi_{13}=3/2$. These proportionate relations among π_{11},π_{12},π_{13}---hold for any value of Q_1,i.e.,regardless of the position of line T_1T_1.

The entire procedure is depicted for points a and b_1. Given a pair of output levels at a point in the NE quadrant, two points are found in the SE and NW quadrants, from which are found the respective coordinates of π_1 and π_2 for the resulting point in the SW quadrant.

Movements in the output quadrant from point a are traced out in the profit quadrant. It should be noted that the mapping is not one-to-one: points d_2 and d_4 in the output quadrant are mapped onto the same point in the profit quadrant. Output levels on the iso-Q line Q=2.25 in the output quadrant result in joint-profit maximization, on the iso-π line π=10.125 in the profit quadrant.

Some different aspects of the same duopoly problem (Case 1) are presented in Figure 2. The Cournot reaction functions are graphed in the output quadrant and the traces of the points on these reaction functions appear in the profit quadrant. The intersection of these functions (point F) represents the Cournot equilibrium point. It can be seen that firm I's reaction function gives the π_1-maximizing output Q_1 for any given output Q_2; three points on this reaction function are listed: B,D, and F. Given output Q_2=2, the choices of Q_1 open to firm I would be represented by a horizontal line at Q_2=2 in the NE quadrant, passing through points C and D. The trace of these choices of Q_1 is the dashed parabola passing through points C and D in the profit quadrant; since point D maximizes π_1 on this parabola, firm I should choose the associated output level Q_1=1.25 given that Q_2=2 (point D, on its reaction function in the output quadrant). Similarly for points B and F and for all other points on firm I's Cournot reaction function: the output level Q_1 is such that it maximizes π_1, with respect to Q_1, given the output level Q_2. The trace of firm I's Cournot reaction function in the profit (SW) quadrant is the locus of π_1-maximizing points on the dashed parabolas, each of which parabola represents the profit implications of the Q_1-choices for any given value of Q_2. Analogously for firm II: each dotted parabola in the profit (SW) quadrant represents the profit implications of various choices

Figure 2


Reaction Function of Firm I
$Q_1 = \frac{9}{4} - \frac{1}{2} Q_2$
Reaction Function of Firm II
$Q_2 = \frac{9}{4} - \frac{1}{2} Q_1$
Q_1
Q_2
π_1
π_2
P-1
T_1
T_2
D
$\pi_1 = 1$
$\pi_1 = 2$
$\pi_1 = 3$
$\pi_1 = 4$
$\pi_1 = 5$
$\pi_2 = 1$
$\pi_2 = 2$
$\pi_2 = 8$

for Q_2, given the output level Q_1; for all points on firm II's reaction function, the output level Q_2 maximizes π_2, with respect to Q_2, given the output level Q_1.

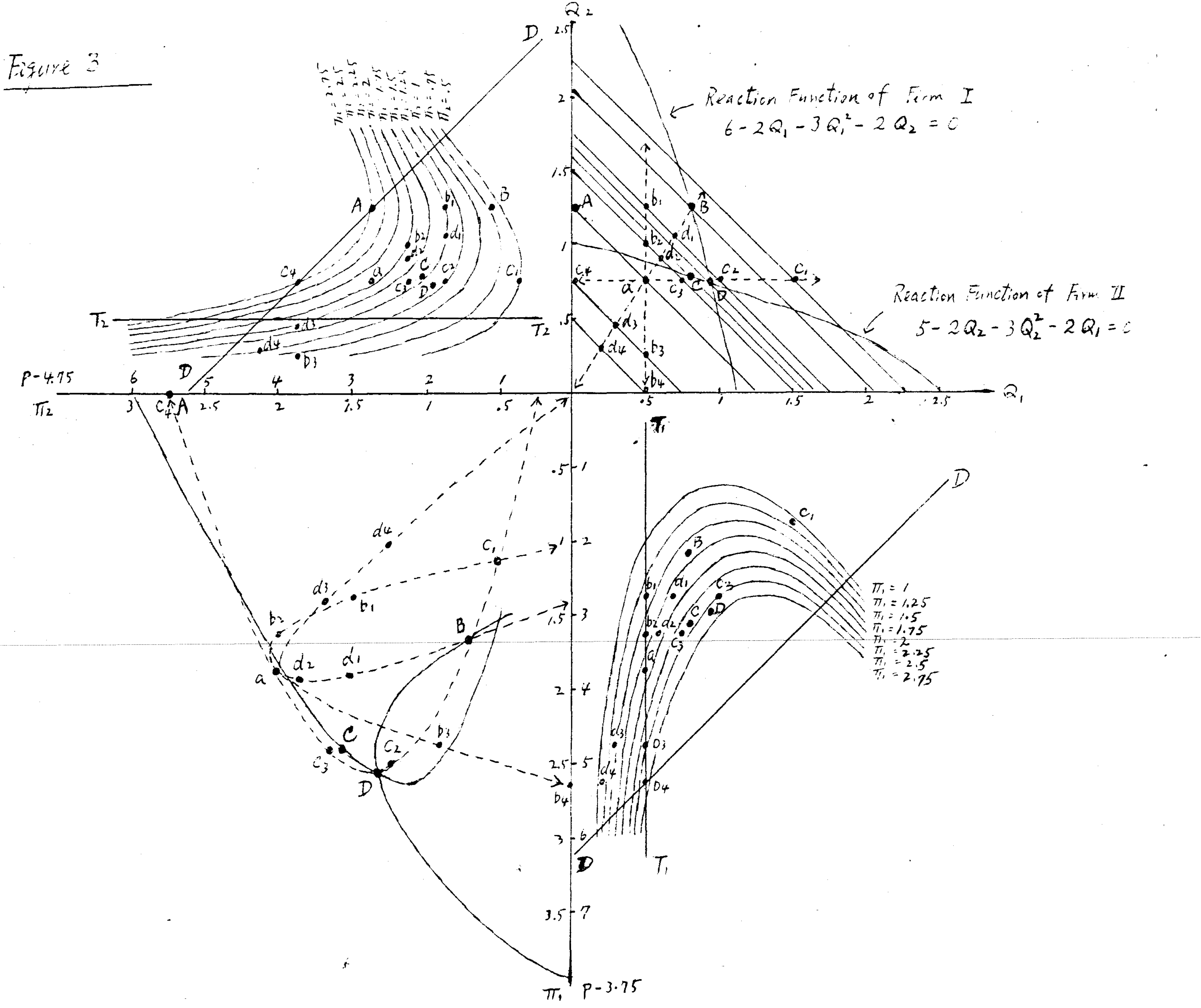
The dynamics of the movement to the Cournot equilibrium are also depicted in Figure 2, in both the output quadrant and the profit quadrant. Starting from point A (Q_1=0, Q_2=3.5), firm I moves to the point B (Q_1=.5, Q_2=3.5) on its reaction function, maximizing π_1 on the dashed profit parabola at point B. Given Q_1=.5, firm II moves to point C (Q_1=.5, Q_2=2) on <u>its</u> reaction function, maximizing π_2 on the corresponding dotted profit parabola at point C. And so forth, until the Cournot equilibrium point F (Q_1=1.5, Q_2=1.5; π_1=4.5, π_2=4.5) is reached. Point F in the output quadrant is the intersection of the Cournot reaction functions and point F in the profit quadrant is the intersection of the respective traces of the Cournot reaction functions. Point F in the profit quadrant is the only point at which both of the following occur: firm I is maximizing with respect to π_1 on its dashed profit parabola and firm II is maximizing with respect to π_2 on its dotted profit parabola.

For Case 2, in which two firms have different quadratic average cost functions (Table 1), the effect of changes in output levels upon profit levels and the dynamic process of the Cournot solution are depicted in Figure 3. This is, unfortunately, more complicated than that for Case 1.

There are two main differences: the iso-π_1 and iso-π_2 curves are no longer rectangular hyperbolas, and the values of AC_1 and AC_2 are no longer constant, which creates a problem in scaling the price axes P-AC_1 and P-AC_2 and in locating the transformation lines T_1T_1 and T_2T_2.

The iso-π_1 (and iso-π_2) curves are no longer rectangular hyperbolas when the axes are Q_1 and P-AC_1 because, in the equations $\pi_1 = (P-AC_1)\, Q_1$, the average cost AC_1 is itself a (quadratic) function of Q_1. Consider the iso-π_1 curves in Figure 3, drawn in the SE quadrant, with axes Q_1 and P-3.75; the choice of 3.75

Figure 3



Reaction Function of Firm I
6 - 2Q₁ - 3Q₁² - 2Q₂ = 0
Reaction Function of Firm II
5 - 2Q₂ - 3Q₂² - 2Q₁ = 0
Q₂
Q₁
π₂
π₁
P-4.75
P-3.75
T₂
T₁
D
π₁ = 1
π₁ = 1.25
π₁ = 1.5
π₁ = 1.75
π₁ = 2
π₁ = 2.25
π₁ = 2.5
π₁ = 2.75

will be explained below. When any iso-π_1 curve is drawn, for example $\pi_1=2$, it will lie vertically below the corresponding rectangular hyperbola, $2 = (P-3.75)Q_1$ (not drawn), at all points except at $Q_1=.5$, where $AC_1=3.75$ (the AC_1 minimizing point), and where the iso-π_1 curve and the rectangular hyperbola touch. As Q_1 gets further from $Q_1 = .5$ the AC_1 is increasing and the iso-π_1 bends further away from a constant-average-cost rectangular-hyperbola iso-π_1 curve. For the iso-π_2 curves the situation is similar except that the axes are Q_2 and P-4.75 (explained below).

The choice of the price axes as P-3.75 and P-4.75 is related to the location of the transformation lines, T_1T_1 at $Q_1 = .5$ and T_2T_2 at $Q_2 = .5$. Only when the scale on the P-AC_1 axis is consistent with the position of the transformation line can a straight line serve as a transformation line.[5] For, once the scale on the vertical axis is fixed as P-3.75, the ratio of the vertical distances to points at which any two iso-π_1 curves intersect the straight line T_1T_1 would be equal to the ratio of the corresponding values of iso-π_1 curves, only if the straight line is drawn at Q = .5. Similarly for the axis P-4.75 and the line T_2T_2.

While we have not done so, the diagram could also be used to analyze other problems in duopoly theory. In addition, some of the techniques used in transforming output levels into profit levels might also be adapted to other problems in economics.

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[5] We could choose various positions for line T_1T_1, for example at Q=0.25. In this case the scale of P-AC_1 axis should be fixed as P-3.81 (P-4.81 for firm II), where AC_1=3.81 at Q_1=.25 (AC_2=4.81 at Q_2=.25). Then straight lines at Q=.25 can function as the transformation lines T_1T_1 and T_2T_2. Of course, the scale of the profit axes in the SW quadrant becomes larger than in Figure 3.