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MEASURES OF TECHNICAL EFFICIENCY

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ABSTRACT

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Färe and Lovell (J. Econ. Theory, 1978) suggest four properties that a measure of technical efficiency should satisfy; the commonly employed Debreu/Farrell measure fails three of them. I provide necessary and sufficient conditions for a measure to satisfy the four conditions and analyze recently proposed measures in the light of this result. I also argue that the Debreu/Farrell measure has several desirable properties when one takes account of (1) (perhaps unknown) market prices and the relationship between technical and economic efficiency and (2) its characterization as a measure of economic efficiency using the shadow prices implicit in the production technology. J. Econ. Theory (English). New York University, New York.

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1. Introductory Remarks

Measures of technical efficiency were first proposed by Debreu [2] and Farrell [6]. These measures are useful where there is reason to distinguish between technical and economic efficiency, most notably when market prices for inputs are unavailable.

The Debreu/Farrell measure is simply the inverse of the Malmquist [10]/ Shephard [11] distance function (restricted to feasible input/output combinations). That is, it measures the maximum amount that an input vector can be shrunk along a ray while holding the output levels constant. More recently, Färe and Lovell [4] have noted some problems with the Debreu/Farrell measure if the production technology does not satisfy a strict monotonicity condition (strictly positive marginal products).

In particular, Färe and Lovell suggest four properties that a measure of technical efficiency should satisfy:

(1) indication of efficient input vectors (the measure is equal to one if and only if the input bundle is technically efficient in the sense of Koopmans [8]),

(2) homogeneity of degree minus one (e.g., doubling all input quantities cuts the measure in half),

(3) strict monotonicity (increasing one input quantity while holding all others constant lowers the measure), and

(4) comparison to efficient input vectors (the measure compares each feasible input vector to an efficient input vector).

The Debreu/Farrell measure satisfies the homogeneity property, but fails the other three criteria for production technologies that do not satisfy strict monotonicity.

Färe and Lovell proposed an alternative measure that, instead of shrinking the input vector along a ray, shrinks the input vector in coordinate directions (maximizing the sum of the proportionate reductions of input quantities) until an efficient point is reached. Färe, Lovell, and Zeischang [5] later noted, however, that the Färe/Lovell measure does not satisfy the homogeneity property, but instead satisfies the much weaker property of "sub-homogeneity of degree minus one." In this paper, I show that the Färe/Lovell measure also fails to satisfy the strict monotonicity condition.

More constructively, I provide necessary and sufficient conditions for a measure to satisfy the four conditions proposed by Färe and Lovell. In particular, I identify the class of functions that satisfy the first three of the Färe/Lovell conditions. I then argue that the fourth condition is ill defined ("compares to" is not a mathematical concept) and, when well defined (in terms of a "proportional distance metric"), is implied by the other three conditions.

The characterization of the class of functions satisfying the Färe/Lovell conditions is not a closed form. Indeed, whether there exists a measure satisfying these conditions for the broad class of technologies considered in this paper remains an open question. Zieschang [14] recently proposed an amalgam of the Debreu/Farrell and Färe/Lovell measures (using the former to scale down input vectors to the isoquant and the latter to shrink the vectors along an isoquant), which satisfies the homogeneity property but, as he notes, satisfies monotonicity only for restricted classes of technologies. In this paper, I prove that a broader class of measures (containing Zieschang's as a special case) does not contain a member satisfying the four conditions.

Finally, I conclude with a suggested resurrection of the Debreu/Farrell measure. This measure has several desirable properties when one takes account of (perhaps unknown) market prices and the relationship between technical and economic efficiency emphasized by Farrell [6]. Moreover, even if market prices do not exist, the Debreu/Farrell measure is a natural one because it has an evocative characterization as a measure of economic efficiency using the shadow prices implicit in the production technology.

2. Preliminaries

Let $\mathbf{R}^n_+$ be the nonnegative Euclidean n-orthant and adopt the following notations for $(x,\hat{x})\in\mathbf{R}^{2n}_+$: $x \geqq \hat{x}$ if $x_i \geqq \hat{x}_i\ \forall\ i$, $x \geq \hat{x}$ if $x \geqq \hat{x}$ and $x \neq \hat{x}$, and $x > \hat{x}$ if $x_i > \hat{x}_i\ \forall\ i$. Also, $0^{(n)}\in\mathbf{R}^n_+$ is the n-dimensional zero vector.

I characterize the production technology by the input correspondence, $L:\mathbf{R}^m_+ \rightarrow P(\mathbf{R}^n_+)$, where $P(\mathbf{R}^n_+)$ is the power set of $\mathbf{R}^n_+$. L(u) is the set of input vectors $x\in\mathbf{R}^n_+$ that can produce the output vector $u\in\mathbf{R}^m_+$. Assume that L satisfies the conditions (L):

$$L(0^{(m)}) = \mathbf{R}^n_+. \tag{L1}$$

$$\text{If } u \geq 0^{(m)},\ 0^{(n)} \notin L(u). \tag{L2}$$

$$\text{If } x\in L(u),\ \lambda x\in L(u)\ \forall\ \lambda\in[1,\infty),\ \forall\ u\in U. \tag{L3}$$

$$L(u) \text{ is closed } \forall\ u\in U. \tag{L4}$$

The range restrictions in (L1) and (L2), while quite reasonable and required for the results of this paper (as stated), could be dropped at the cost of more complicated restrictions and extensions of the domains and ranges of the mappings employed below. The third condition is a very weak monotonicity condition (called weak disposability of inputs by Shephard [12]). Taken together, the conditions are very weak. For example, regions of local or global satiation are not precluded, convexity is not required, and only limited continuity (continuity from above for fixed output vectors) is posited.

Define

$$U = \{u \in \mathbf{R}_+^m | u \geq 0^{(m)} \wedge L(u) \neq \emptyset\} ,$$

$$\Lambda = \{(u,x) | u \in U \wedge x \in L(u)\} ,$$

and

$$\Lambda' = \{(u,x,\hat{x}) | u \in U \wedge x \in L(u) \wedge \hat{x} \in L(u)\} .$$

(Note that, for all $u \in U$, $(u,0^{(n)}) \notin \Lambda$.) The (restricted) distance function, $D : \Lambda \to \mathbf{R}_+$, is defined by

$$D(u,x) = \max\{\lambda | x/\lambda \in L(u) \wedge \lambda \in \mathbf{R}_+\}.$$

(This restriction of the Malmquist/Shephard distance function to Λ, rather than $U \times \mathbf{R}_+^n$, avoids some messy complications and is of no consequence for what follows.) Given (L), D is well defined and D(u,x) is strictly positive and satisfies the homogeneity property,

$$D(u,\kappa x) = \kappa D(u,x) \;\forall\; \kappa \in [D(u,x)^{-1},\infty). \tag{H}$$

The input isoquants are defined by

$$\mathrm{Isoq}(u) = \{x \in \mathbf{R}^n_+ | x \in L(u) \wedge \lambda x \notin L(u) \ \forall \ \lambda \in [0,1)\}$$

and the efficient subset is defined by

$$\mathrm{Eff}(u) = \{x \in \mathbf{R}^n_+ | (x \in L(u) \wedge \hat{x} \leq x) \Rightarrow \hat{x} \notin L(u)\}.$$

Clearly, $\mathrm{Eff}(u) \subseteq \mathrm{Isoq}(u)$. Moreover, $x/D(u,x)$ is in $\mathrm{Isoq}(u)$ but not necessarily in $\mathrm{Eff}(u)$.

Next define the mapping, $L^+ : U \to P(\mathbf{R}^n_+)$, by

$$L^+(u) = L(u) + \mathbf{R}^n_+.$$

$L^+(u)$ is called the "free-disposal hull" by McFadden [9]. Färe [3] has shown that L^+ satisfies the conditions (L) as well as the free-disposal property, $x \in L^+(u)$ and $\hat{x} \geqq x$ implies $\hat{x} \in L^+(u)$. Define $D^+ : \Lambda^+ \to \mathbf{R}_+$ by

$$D^+(u,x) = \max\{\lambda | x/\lambda \in L^+(u) \wedge \lambda \in R_+\}$$

where

$$\Lambda^+ = \{(u,x) | u \in U \wedge x \in L^+(u)\}.$$

Given (L), D^+ is well defined and satisfies the positive-monotonicity property,

$$D^+(u,\hat{x}) \geq D^+(u,x) \ \forall \ ((u,x),(u,\hat{x})) \in \Lambda^+ \times \Lambda^+ | \hat{x} \geq x, \quad \text{(M)}$$

as well as the properties of D (principally homogeneity) (see [1] or [11]).

Finally, define

$$\mathrm{Isoq}^+(u) = \{x \in \mathbb{R}^n_+ | x \in L^+(u) \wedge \lambda x \notin L^+(u) \ \forall \ \lambda \in [0,1)\}$$

and

$$\mathrm{Eff}^+(u) = \{x \in \mathbb{R}^n_+ | (x \in L^+(u) \wedge \hat{x} \leq x) \Rightarrow \hat{x} \notin L^+(u)\}.$$

Although $\mathrm{Isoq}^+(u) \neq \mathrm{Isoq}(u)$, $\mathrm{Eff}^+(u) = \mathrm{Eff}(u)$ (see [3]).

3. Efficiency Measures

An efficiency measure is a mapping $E : U \times \hat{\mathbb{R}}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$, where $\hat{\mathbb{R}}^n_+ = \mathbb{R}^n_+/\{0^{(n)}\}$, with the property, $E(u,x) \leq 1$ if $x \in L(u)$ and $E(u,x) = +\infty$ if $x \notin L(u)$.

The desirable conditions for such a measure suggested by Färe and Lovell are

$$\left.\begin{array}{ll} E(u,x) = 1 \text{ if and only if } x \in \mathrm{Eff}(u) \ \forall \ (u,x) \in \Lambda. & \text{(FL1)} \\ E(u,\lambda x) = \lambda^{-1}E(u,x) \ \forall \ \lambda \in [D(u,x)^{-1},+\infty) \ \forall \ (u,x) \in \Lambda. & \text{(FL2)} \\ E(u,\hat{x}) < E(u,x) \text{ if } \hat{x} \geq x \ \forall \ (u,x,\hat{x}) \in \Lambda'. & \text{(FL3)} \end{array}\right\} \text{(FL)}$$

(Färe and Lovell also suggest a fourth condition -- that $E(u,x)$ "compares x to $\hat{x} \in \mathrm{Eff}(u)$" -- but this condition is not well defined for arbitrary efficiency measures and, I argue below, is redundant when well defined.)

The Debreu/Farrell measure is defined by

$$E_{DF}(u,x) = \begin{cases} D(u,x)^{-1} & \text{if } x \in L(u) \\ +\infty & \text{if } x \notin L(u). \end{cases}$$

It is easy to see that E_{DF} satisfies the homogeneity condition (FL2), since D satisfies (H), but does not satisfy (FL1) or (FL3) for production technologies with a pair $(\hat{x},x) \in R_+^{2n}$ satisfying $\hat{x} \geq x$, $x \in \text{Isoq}(u)$, and $\hat{x} \in \text{Isoq}(u)$. These facts are illustrated in Figure 1, where

$$E_{DF}(\hat{u},\overset{\circ}{x}) = D(\hat{u},\overset{\circ}{x})^{-1} = D(\hat{u},\bar{x})^{-1} = E_{DF}(\hat{u},\bar{x})$$

and $E_{DF}(\hat{u},\hat{x}) = 1$ although $\hat{x}$ is not efficient (in the sense of Koopmans [7]). (Moreover, E_{DF} "compares $\bar{x}$ to $\hat{x} \notin \text{Eff}(\hat{u})$.")

The Färe/Lovell measure is defined by

$$E_{FL}(u,x) = \begin{cases} \min_{\kappa}\{\sum_i \kappa_i / \sum_i \delta\ (x_i) \mid (\kappa_1 x_1, \ldots, \kappa_n x_n) \in L(u) \wedge \kappa_i \in [0,1] \\ \qquad \forall\ i\} \text{ if } x \in L^+(u) \\ \\ +\infty \text{ if } x \notin L^+(u) \end{cases}$$

where

$$\delta(x_i) = \begin{cases} 1 \text{ if } x_i > 0 \\ 0 \text{ if } x_i = 0 . \end{cases}$$

Färe and Lovell showed that E_{FL} satisfies (FL1). In Figure 1, $E_{FL}(\hat{u},\hat{x}) < 1$ (and E_{FL} compares $\bar{x}$ to $\overset{*}{x} \in \text{Eff}(\hat{u})$). As noted by Färe, Lovell, and Zeischang [5], however, E_{FL} does not satisfy (FL2), but instead satisfies the weaker property of "sub-homogeneity of degree minus one":

$$E_{FL}(u,\lambda x) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \lambda^{-1} E_{FL}(u,x) \text{ if } \lambda \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 1.$$

It appears, moreover, that E_{FL} does not satisfy the strict monotonicity condition (FL3):

Theorem 1: E_{FL} does not satisfy (FL3) for all technologies satisfying (L).

Proof: Define $x^{(-j)} \in \mathbf{R}_+^{n-1}$ as the vector $x \in \mathbf{R}_+^n$ with the j^{th} component deleted and consider the technology with the level set,

$$L(\bar{u}) = \{x \in \mathbf{R}_+^n | x^{(-j)} \geqq 0^{(n-1)} \wedge x_j \geq \bar{x}_j > 0\},$$

for some $\bar{u} \in U$. This technology is consistent with (L). As $u \notin U$, (L1) is vacuously satisfied. Since $x_j > 0 \ \forall \ x \in L(\bar{u})$, (L2) is satisfied. If $x \in L(\bar{u})$ and $\lambda \in [1,\infty)$, $\lambda x^{(-j)} \geqq 0^{(n-1)}$ and $\lambda x_j \geq \bar{x}_j$; hence $\lambda x \in L(\bar{u})$ and (L3) is satisfied. Finally, as the intersection of two closed sets, $L(\bar{u})$ is closed; hence, (L4) is satisfied.

Let $\overset{*}{x} \in L(\bar{u})$ and $\hat{x} \in L(\bar{u})$ satisfy $\hat{x}^{(-j)} \geq \overset{*}{x}^{(-j)} > 0^{(n-1)}$ and $\hat{x}_j = \overset{*}{x}_j = \bar{x}_j$. If $\overset{*}{\kappa}$ solves

$$\min_{\kappa} \sum_i \kappa_i / \sum_i \delta(\overset{*}{x}_i) | (\kappa_1 \overset{*}{x}_1, \ldots, \kappa_n \overset{*}{x}_n) \in L(\bar{u}) \wedge \kappa_i \in [0,1] \ \forall \ i\}$$

and $\hat{\kappa}$ solves

$$\min_{\kappa} \{\sum_i \kappa_i / \sum_i \delta(\hat{x}_i) | (\kappa_1 \hat{x}_1, \ldots, \kappa_n \hat{x}_n) \in L(\bar{u}) \wedge \kappa_i \in [0,1] \ \forall \ i\},$$

$\overset{*}{\kappa} = \hat{\kappa}$ (in particular, $\overset{*}{\kappa}_i = \bar{\kappa}_i = 0 \ \forall \ i \neq j$ and $\overset{*}{\kappa}_j = \hat{\kappa}_j = 1$). Consequently, $E_{FL}(u,\overset{*}{x}) = E_{FL}(u,\hat{x}) = 1/n.$ ‖

The Färe/Lovell condition does satisfy the weaker monotonicity condition,

$$E(u,x) \geq E(u,\hat{x}) \text{ if } \hat{x} \geq x \ \forall \ (u,x,\hat{x}) \in \Lambda', \qquad \text{(WFL3)}$$

as shown by the following theorem:

Theorem 2: E_{FL} satisfies (WFL3) for all technologies satisfying (L).

Proof: If $\overset{*}{\kappa}$ solves

$$\min_{\kappa}\{\sum_i \kappa_i / \sum_i \delta(x_i) \mid (\kappa_1 x_1, \ldots, \kappa_n x_n) \in L(u) \wedge \kappa_i \in [0,1]\ \forall\ i\}$$

and $\hat{x} \geq x$, then

$$(\overset{*}{\kappa}_1 \hat{x}_1, \ldots, \overset{*}{\kappa}_n \hat{x}_n) \in L^+(u),$$

since $(\overset{*}{\kappa}_1 \hat{x}_1, \ldots, \overset{*}{\kappa}_n \hat{x}_n) \geq (\overset{*}{\kappa}_1 x_1, \ldots, \overset{*}{\kappa}_n x_n)$. Consequently, there exists a $\hat{\kappa} \leqq \overset{*}{\kappa}$ such that $\hat{\kappa}_i \in [0,1]\ \forall i$ and

$$(\hat{\kappa}_1 \hat{x}_1, \ \ldots, \ \hat{\kappa}_n \hat{x}_n) \in L(u).$$

Thus, $E_{FL}(u,\hat{x}) \leq E_{FL}(u,x)$. ‖

These properties are illustrated in Figure 2. In this example,

$$E_{FL}(\hat{u},\hat{x}) = 1,$$

$$E_{FL}(\hat{u},\lambda\hat{x}) = \min\left\{\frac{\lambda^{-1}+\lambda^{-1}}{2}, \frac{0+1}{2}\right\} = \frac{1}{2},$$

and

$$E_{FL}(\hat{u},\tilde{x}) = \min\left\{\frac{1+\lambda^{-1}}{2}, \frac{0+1}{2}\right\} = \frac{1}{2},$$

since $\lambda > 1$. Thus, as $\lambda < 2$,

$$\frac{1}{2} = E_{FL}(\hat{u},\lambda\hat{x}) < \lambda^{-1} E_{FL}(\hat{u},\hat{x}) = \lambda^{-1},$$

violating homogeneity (but satisfying sub-homogeneity), and $E_{FL}(\hat{u},\tilde{x}) = E_{FL}(\hat{u},\lambda\hat{x})$, violating strict monotonicity (but satisfying weak monotonicity).

4. A Complete Characterization of the Färe/Lovell Conditions

I next prove necessary and sufficient conditions for (FL) to hold. The characterization requires a new concept. Consider a function, $\phi : \Gamma \rightarrow \mathbf{R}_+$, where

$$\Gamma = \{(u,x) \in \Lambda \mid x \in \mathrm{Isoq}^+(u)\}$$

and define ϕ^j by

$$\phi^j(u,x_j,x^{(-j)}) = \phi(u,x).$$

Also define the set-valued mapping, $\gamma^j : U \times \mathbf{R}_+^{n-1} \rightarrow \mathbf{R}_+$, by

$$\gamma^j(u,x^{(-j)}) = \{x_j \mid x \in \mathrm{Isoq}^+(u)\}$$

and the set $\Gamma^{(x_j > 0)}$ by

$$\Gamma^{(x_j > 0)} = \{(u,x) \mid (u,x) \in \Gamma \wedge x_j > 0\}.$$

The condition we need is

$$\phi^j(u,x_j,\lambda\hat{x}^{(-j)}) > \lambda^{-1}\phi(u,\hat{x}) \;\forall\; x_j \in \gamma^j(u,\hat{x}^{(-j)}),\; \forall\; \lambda \in (1,+\infty),$$

$$\forall\; (u,\hat{x}) \in \Gamma^{(\hat{x}_j > 0)},\; \forall\; j. \qquad \text{(SSH)}$$

If $\gamma^j(u,\lambda\hat{x}^{(-j)})$ were a singleton, we could write

$$\tilde{\phi}^j(u,\lambda\hat{x}^{(-j)}) > \lambda^{-1}\tilde{\phi}^j(u,\hat{x}^{(-j)}) \ \forall \ \lambda \in (1,+\infty)$$

where

$$\tilde{\phi}^j(u,x) = \phi^j(u,\gamma^j(u,x^{(-j)}), \ x^{(-j)}).$$

This is equivalent to

$$\tilde{\phi}^j(u,\lambda\hat{x}^{(-j)}) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \lambda^{-1}\tilde{\phi}^j(u,\hat{x}^{(-j)}) \quad \text{if } \lambda \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 1, \qquad \text{(SSH)}'$$

as seen by the following equivalences:

$$\tilde{\phi}^j(u,\lambda\hat{x}^{(-j)}) > \lambda^{-1}\tilde{\phi}^j(u,\hat{x}^{(-j)}), \ \lambda > 1$$

$$\iff \lambda\tilde{\phi}^j(u,x^{(-j)}) > \tilde{\phi}^j(u,\lambda^{-1}x^{(-j)}), \ \lambda > 1$$

$$\iff \tilde{\phi}^j(u,\overset{*}{\lambda}x^{(-j)}) < \overset{*}{\lambda}{}^{-1}\tilde{\phi}^j(u,x^{(-j)}), \ \overset{*}{\lambda} < 1 \ ,$$

where $x^{(-j)} = \lambda\hat{x}^{(-j)}$ and $\lambda^* = \lambda^{-1}$. The condition (SSH)' is "super-homogeneity of degree minus one." In this vein, one might refer (cumbersomely) to (SSH) as "<u>subspace</u> superhomogeneity of degree minus one" (since only the $(n-1)$-dimensional subvector $x^{(-j)}$ is expanded by the scalar λ). I prefer to let "(SSH)" speak for itself.

The following lemma is used in the proof of the principal theorem.

<u>Lemma</u>: Suppose $\hat{x} \in \text{Isoq}^+(u)$. If $x_j \in \gamma^j(u,\lambda\hat{x}^{(-j)})$ for $\lambda \in (1,+\infty)$, then $\hat{x}_j \geq x_j$.

<u>Proof</u>: Suppose not: $\hat{x}_j < x_j$. Then $x \in \{\hat{x}\} + \mathbf{R}^n_{++}$ (where $\mathbf{R}^n_{++}$ is the strictly positive Euclidean n-orthant and $x^{(-j)} = \lambda\hat{x}^{(-j)}$). As $L^+(u) = L(u) + R^n_+$, $\{\hat{x}\} + \mathbf{R}^n_{++} \subseteq \text{Int}(L^+(u))$ and x is an interior point

of $L^+(u)$. Consequently, there exists an $\varepsilon \in (0,+\infty)$ and an ε-neighborhood $N(\varepsilon,x)$ such that

$$N(\varepsilon,x) \subseteq \text{Int}(L^+(u)).$$

Choose $\lambda \in (0,1)$ such that $|x - \lambda x| \leq \varepsilon$. Then $\lambda x \in N(\varepsilon,x) \subseteq \text{Int}(L^+(u))$, contradicting the requirement that $x \in \text{Isoq}^+(u)$. ‖

We are now ready to state and prove the principal result.

Theorem 3: An efficiency measure, E, satisfies conditions (FL) for all technologies satisfying (L) if and only if

$$E(u,x) = \phi(u,x/D^+(u,x))/D^+(u,x) \quad \forall \ (u,x) \in \Lambda, \tag{1}$$

where ϕ satisfies (SSH), $\phi(u,x) = 1$ for all $(u,x) \in \Gamma$ such that $x \in \text{Eff}(u)$, and $\phi(u,\hat{x}) < \phi(u,x)$ if $\hat{x} \geq x$ for all $(u,x) \in \Gamma$.

Proof: Proof of sufficiency ((1) => (FL)) is in three parts:

(1) => (FL1): As $\text{Eff}(L(u)) \subseteq \text{Isoq}^+(u)$, $D^+(u,x) = 1 \ \forall \ x \in \text{Eff}(L(u))$; hence, $\phi(u,x/D^+(u,x))/D^+(u,x) = \phi(u,x) = 1 \ \forall \ x \in \text{Eff}(u)$. If $x \notin \text{Eff}(u)$, there exists an $\hat{x} \in \text{Eff}(u)$ such that $x \geq \hat{x}$. If $x \in \text{Isoq}^+(u)$, $D^+(u,x) = 1$ and, hence, $\phi(u,x/D^+(u,x))/D^+(u,x) < 1$. Since $x \in L(u)$, $x \notin \text{Isoq}^+(u)$ implies $D^+(u,x) > 1$ and, as $\phi(u,x/D^+(u,x)) \leq 1$, again, $\phi(u,x/D^+(u,x))/D^+(u,x) < 1$.

(1) => (FL2): This follows immediately from the positive linear homogeneity of D^+:

$$\begin{aligned}\phi(u,\lambda x/D^+(u,\lambda x))/D^+(u,\lambda x) &= \phi(u,\lambda x/\lambda D^+(u,x))/\lambda D^+(u,x) \\ &= \lambda^{-1}\phi(u,x/D^+(u,x))/D^+(u,x) \\ &\quad \forall \ \lambda \in [D^+(u,x)^{-1},+\infty).\end{aligned}$$

(1) => (FL3): Consider two input vectors $(x,\hat{x}) \in \hat{\mathbb{R}}^n_+ \times \hat{\mathbb{R}}^n_+$ satisfying $x_i = \hat{x}_i \ \forall\, i \neq j$ and $x_j < \hat{x}_j$. Let $z = x/D^+(u,x)$ and $\hat{z} = \hat{x}/D^+(u,\hat{x})$ so that

$$D^+(u,x) \cdot z_i = x_i = \hat{x}_i = D^+(u,\hat{x}) \cdot \hat{z}_i \quad \forall\, i \neq j.$$

Consequently,

$$z_i = \frac{D^+(u,\hat{x})}{D^+(u,x)}\, \hat{z}_i \quad \forall\, i \neq j. \tag{2}$$

Moreover, as $\hat{x}_j > x_j \geq 0$, $\hat{z}_j = \hat{x}_j/D(u,\hat{x}) > 0$. Substitution of (2) into $\phi(u,z)$ yields

$$\phi(u,z) = \phi^j\left(u, z_j, \frac{D^+(u,\hat{x})}{D^+(u,x)}\, \hat{z}^{(-j)}\right).$$

By monotonicity of $D^+(u,\cdot)$, together with $\hat{x} \geq x$, either

$$\frac{D^+(u,\hat{x})}{D^+(u,x)} > 1 \tag{3}$$

or

$$\frac{D^+(u,\hat{x})}{D^+(u,x)} = 1. \tag{4}$$

Suppose (3) holds. Then (SSH) implies

$$\phi^j\left(u, z_j, \frac{D^+(u,\hat{x})}{D^+(u,x)}\, \hat{z}^{(-j)}\right) > \frac{D^+(u,x)}{D^+(u,\hat{x})}\, \phi(u,\hat{z}),$$

and

$$\frac{\phi(u,z)}{D^+(u,x)} > \frac{\phi(u,\hat{z})}{D^+(u,\hat{x})},$$

as was to be shown.

If, on the other hand, (4) holds, then, from (2), $z_i = \hat{z}_i \ \forall\, i \neq j$. Moreover, since $\hat{x}_j > x_j$

$$\hat{z}_j = \hat{x}_j/D^+(u,\hat{x}) > x_j/D^+(u,x) = z_j.$$

By the strict monotonicity of $\phi(u,\cdot)$, therefore,

$$\frac{\phi(u,z)}{D^+(u,x)} > \frac{\phi(u,\hat{z})}{D^+(u,\hat{x})},$$

as was to be shown.

To prove necessity ((FL) => (1)), let ϕ be the restriction of E to $\Gamma = \{(u,x) \mid u \in U \wedge x \in \mathrm{Isoq}^+(u)\}$. (FL3) and (FL1) imply that ϕ is decreasing in x and

$$\phi(u,x) = 1 \quad \text{iff } x \in \mathrm{Eff}(u).$$

For any $x \in L(u)$, (FL2) implies

$$E(u,x/D^+(u,x)) = D^+(u,x) \cdot E(u,x).$$

But, as $x/D(u,x) \in \mathrm{Isoq}^+(u)$,

$$E(u,x/D^+(u,x)) = \phi(u,x/D^+(u,x)).$$

Consequently,

$$E(u,x) = \phi(u,x/D^+(u,x))/D^+(u,x).$$

To prove that ϕ must satisfy (SSH), suppose the contrary; i.e., suppose there exist a j, $(u,\hat{z}) \in \Gamma^{(z_j > 0)}$, and $\lambda \in (1,+\infty)$ such that

$$\phi^j(u,z_j,\lambda\hat{z}^{(-j)}) \leq \lambda^{-1}\phi(u,\hat{z}), \tag{5}$$

for some $z_j \in \gamma^j(u,\lambda\hat{z}^{(-j)})$. Let $z^{(-j)} = \lambda\hat{z}^{(-j)}$ and choose $x = \delta z$ and $\hat{x} = \delta\lambda\hat{z}$, where $\delta \in (1,+\infty)$. Evidently, $x_i = \hat{x}_i \;\forall\; i \neq j$, since

$$x_i = \delta z_i = \delta\lambda\hat{z}_i = \hat{x}_i \quad \forall\; i \neq j.$$

Moreover, $\hat{x}_j > x_j$, since $x_j = \delta z_j$, $\hat{x}_j = \delta\lambda\hat{z}_j$, $\delta > 1$, $\lambda > 1$, $\hat{z}_j > 0$, and $\hat{z}_j \geq z_j$ (by the Lemma).

Finally, $D^+(u,x) = \delta$ and $D^+(u,\hat{x}) = \delta\lambda$, so that

$$\frac{D^+(u,x)}{D^+(u,\hat{x})} = \lambda^{-1}.$$

Substitution into (5) yields

$$\phi(u,z) \leq \frac{D^+(u,x)}{D^+(u,\hat{x})}\,\phi(u,\hat{z})$$

or

$$\frac{\phi(u,z)}{D^+(u,x)} \leq \frac{\phi(u,\hat{z})}{D^+(u,\hat{x})},$$

contradicting (FL3). ‖

The Färe/Lovell conditions with the weaker monotonicity condition (WFL3) substituted for (FL3) can be characterized by a weakening of (SSH), obtained by converting the strict inequality in (SSH) to a weak inequality. Call the weak condition (WSSH). Proof of the following corollary is easily adapted from the proof of Theorem 3:

Corollary: An efficiency measure, E, satisfies conditions (FL1), (FL2), and (WFL3) for all technologies satisfying (L) if and only if

$$E(u,x) = \phi(u,x/D^{+}(u,x))/D^{+}(u,x)\ \forall\ (u,x) \in \Lambda,$$

where ϕ satisfies (WSSH), $\phi(u,x) = 1$ for all $(u,x) \in \Gamma$ such that $x \in \mathrm{Eff}(u)$, and $\phi(u,\hat{x}) < \phi(u,x)$ if $\hat{x} \geq x$ for all $(u,x) \in \Gamma$.

Zieschang [14] has proposed a measure that is a special case of (1), with $\phi(u,\cdot)$ given by the restriction of $E_{FL}(u,\cdot)$ to $\mathrm{Isoq}^{+}(u)$. As Zieschang notes, his measure satisfies (FL1) and (FL2) but fails to satisfy (FL3) for all technologies satisfying (L). In fact, it is possible to show that any measure based on proportional shrinkage along isoquants must fail to satisfy (WFL3), and a fortiori (FL3), for some technology satisfying (L).

Consider the class of technical efficiency measures $\{E_\alpha\}$ defined by

$$E_\alpha(u,x) = \phi_\alpha(u,x/D^{+}(u,x))/D^{+}(u,x)\ \forall\ (u,x) \in \Lambda,$$

where

$$\phi_\alpha(u,x) = \min_x\{\alpha(\kappa,\Delta(x))\,|\,(\kappa_1 x_1,\ldots,\kappa_n x_n) \in L(u) \wedge \kappa_i \in [0,1]\ \forall\ i\};$$

$$\Delta(x) = (\delta(x_1),\ldots,\delta(x_n)),$$

and $\alpha(\cdot,\Delta(x))$ is positively monotonic. The Zieschang measure is a special (linear) case of ϕ_α.

Theorem 4: There does not exist a member of $\{E_\alpha\}$ satisfying (FL1), (FL2), and (WFL3) for all technologies satisfying (L).

Proof: Consider the technology with the level set

$$L(\bar{u}) = \{x \in \mathbb{R}^n_+ | (x^{(-1)} \geqq \hat{x}^{(-1)} > 0^{(n-1)} \ \forall\ x \mid x_1 \geq \hat{x}_1)$$
$$\wedge\ (x^{(-1)} \geqq \lambda\hat{x}^{(-1)},\ \lambda \in (1,n),\ \forall\ x \mid x_1 < \hat{x}_1),\ \hat{x}_1 > 0\}.$$

Proof that this technology (illustrated in Figure 2 above) is consistent with (L) is virtually identical to its counterpart in the proof of Theorem 1 above. Note that $\tilde{x} = (\hat{x}_1, \lambda\hat{x}^{(-1)})$ is not in $\mathrm{Eff}^+(\bar{u})$ and calculate

$$E_\alpha(u,\hat{x}) = 1$$

and

$$E_\alpha(u,\hat{x}_1,\lambda\hat{x}^{(-1)}) = \min\{\alpha(1,\lambda^{-1},\ldots,\lambda^{-1},\Delta(\tilde{x})),\ \alpha(0,1,\ldots,1,\Delta(\tilde{x}))\ .$$

Regardless of which term in the brackets is minimal, weak monotonicity (WFL3) requires (by the Corollary) that

$$\alpha(0,1,\ldots,1,\Delta(\tilde{x})) \geq \lambda^{-1}E_\alpha(u,\hat{x}) = \lambda^{-1}\ \forall\ \lambda \in (1,+\infty).$$

Letting $\lambda \to 1$ from above, we find that

$$E_\alpha(\bar{u},\tilde{x}) = \alpha(0,1,\ldots,1,\Delta(\tilde{x})) = 1,$$

a contradiction of (FL1). ‖

5. The "Compares to" Condition

The foregoing discussion revolves around three of the four conditions proposed by Färe and Lovell. The fourth condition is that "E(u,x) should compare x to an $\overset{*}{x}$ in Eff(u)." As noted earlier, while this condition is well defined for the particular measures investigated by Färe and Lovell (and in subsequent papers, cited above), it is not generally well defined (given only the definition of an efficiency measure). That is, "compares to" is not a general mathematical concept.

It seems clear, however, that what Färe and Lovell have in mind by this condition is an efficiency measure that is defined in terms of some notion of distance between input vectors. Given such a distance metric, one might then characterize the efficiency measure as a distance between a point and a set (namely, (Eff(u)). The purpose of this section is to make these notions somewhat rigorous and to show that, given a rigorous notion of rationalization of an efficiency measure by a distance metric, the "compares to" condition of Färe and Lovell is implied by their other three conditions.

To these ends, define

$$\chi = \{(u,x,\hat{x}) \mid u \in U \wedge x \in L(u) \wedge \hat{x} \in L(u) \wedge \hat{x} \geqq x\}.$$

A "proportional distance metric" is a mapping $d : \chi \to [0,+\infty)$ that satisfies

$$d(u,x,\hat{x}) = 1 \quad \text{if } x = \hat{x}$$

and

$$d(u,x,\lambda\hat{x}) = \lambda^{-1} d(u,x,\hat{x}).$$

An efficiency measure, E, is rationalized by a proportional distance metric, d, if

$$E(u,\hat{x}) = \min_x \{d(u,x,\hat{x}) \mid x \in L(u)\}.$$

The Debreu/Farrell measure is rationalized by

$$d_{DF}(u,x,\hat{x}) = \frac{D(u,x)}{D(u,\hat{x})},$$

since

$$\min_x \{D(u,x)/D(u,\hat{x}) \mid x \in L(u)\} = D(u,\overset{*}{x})/D(u,\hat{x}) = D(u,\hat{x})^{-1},$$

where $\overset{*}{x} \in \text{Isoq}(u)$. (Thus, the proportional distance metric underlying the Debreu/Farrell measure treats two vectors as being "u-close" to one another if they are approximately equal radial distances from Isoq(u).) Similarly, the Färe/Lovell measure is rationalized by

$$d_{FL}(u,x,\hat{x}) = \sum_i \frac{x_i}{\hat{x}_i} / \sum_i \delta(x_i).$$

(Thus, the proportional distance metric underlying the Färe/Lovell efficiency measure is analogous to the usual grid metric.) Finally, the Zieschang measure is rationalized by

$$d_z(u,x,\hat{x}) = d_{FL}(u,x,\hat{x}/D^+(u,\hat{x}))/D^+(u,\hat{x}).$$

We say that E(u,x) compares x to $\overset{*}{x}$ if E is rationalized by a proportional distance function, d, and

$$E(u,x) = d(u,\overset{*}{x},x).$$

The fourth Färe/Lovell condition can now be stated formally:

$$E(u,x) \text{ compares } x \text{ to } \overset{*}{x} \in \mathrm{Eff}(u) \ \forall\ (u,x) \in \Lambda \qquad \text{(FL4)}$$

Clearly, $E_{FL}(u,x)$ compares x to an $\overset{*}{x} \in \mathrm{Eff}(u)$, whereas $E_{DF}(u,x)$ compares x to an $\overset{*}{x} \in \mathrm{Isoq}(u)$, but not necessarily to an $\overset{*}{x} \in \mathrm{Eff}(u)$.

I now state formally the principal result of this section.

Theorem 5: Suppose that E satisfies (FL) for all technologies satisfying (L). Then E satisfies (FL4) for all technologies satisfying (L).

Proof: By Theorem 3,

$$E(u,x) = \frac{\phi(u,x/D^{+}(u,x))}{D^{+}(u,x)} \ \forall\ (u,x) \in \Lambda,$$

where ϕ satisfies strict negative monotonicity and (SSH). Define the distance metric, d, by

$$d(u,x,\hat{x}) = \frac{D^{+}(u,x)}{D^{+}(u,\hat{x})}\,\frac{\phi(u,\hat{x}/D(u,\hat{x}))}{\phi(u,x/D(u,x))}\ .$$

Then

$$\min_{x}\{d(u,x,\hat{x}) \mid x \in L(u)\} = \frac{D^{+}(u,\overset{*}{x})}{D^{+}(u,\hat{x})}\,\frac{\phi(u,\hat{x}/D^{+}(u,\hat{x})}{\phi(u,\overset{*}{x}/D^{+}(u,\overset{*}{x}))}$$

$$= \frac{\phi(u,\hat{x}/D^{+}(u,\hat{x}))}{D^{+}(u,\hat{x})}$$

$$= E(u,x), \qquad \forall\ (u,x) \in \Lambda\ ,$$

since $D^+(u,x)$ reaches a minimum on $L(u)$ at $D^+(u,\tilde{x}) = 1$ (where $\tilde{x} \in \text{Isoq}^+(u)$) and $\phi(u,\tilde{x}/D^+(u,\tilde{x})) = \phi(u,\tilde{x})$ reaches a maximum on $\text{Isoq}^+(u)$ at $\phi(u,\overset{*}{x}) = 1$ where $\overset{*}{x} \in \text{Eff}(u)$. Consequently, $E(u,x)$ compares x to an $\overset{*}{x} \in \text{Eff}(u)$. ‖

The next result follows immediately from Theorems 3 and 5.

Corollary: An efficiency measure, E, satisfies conditions (FL) and (FL4) for all technologies satisfying (L) if and only if

$$E(u,x) = \phi(u,x/D^+(u,x))/D^+(u,x) \;\; \forall \; (u,x) \in \Lambda,$$

where ϕ satisfies (SSH), $\phi(u,x) = 1$ for all $(u,x) \in \Gamma$ such that $x \in \text{Eff}(u)$, and $\phi(u,\hat{x}) < \phi(u,x)$ if $\hat{x} \geq x$ for all $(u,x) \in \Gamma$.

6. Concluding Remarks: Prices and Economic Efficiency

The foregoing discussion is entirely in the spirit of much of the recent literature on measures of technical efficiency, treating them as if they were completely independent of prices and notions of economic efficiency. Even if market prices are not known, however, economic efficiency is not irrelevant to the analysis of technical efficiency. (Both Debreu [2] and Farrell [6] emphasized the relationships between technical and economic efficiency.) In fact, even if market prices do not exist, notions of economic efficiency are not irrelevant to the analysis of technical efficiency: shadow prices, implicit in all production technologies, are relevant. In this concluding section, using some well-known duality relationships (Shephard [11] and Blackorby, Primont, and Russell [1, ch. 2]), I sketch briefly some of the relationships between economic and technical efficiency and, in the process, provide some support for resurrecting the Debreu/Farrell measure of technical efficiency as the appropriate one from an economic perspective.[1]

The natural measure of economic efficiency is the ratio of minimal to actual costs of inputs:

$$E(u,x,p) = C(u,p)/p\cdot x,$$

where $p \in \mathbb{R}^n_{++}$ is a price vector and C is the usual cost function, defined by

$$C(u,p) = \min_x \{p\cdot x \mid x \in L(u)\}.$$

The cost and distance functions are related by

$$C(u,p)\cdot D(u,x) \leq p\cdot x \quad \forall\ (u,p,x),$$

which can be written

$$\frac{p\cdot x}{C(u,p)} \leq \frac{1}{D(u,x)} \quad \forall\ (u,p,x),$$

or

$$E(u,x,p) \leq E_{DF}(u,x) \quad \forall\ (u,p,x).$$

Thus, the Debreu/Farrell measure of technical efficiency is an upper bound on the measure of economic efficiency, a point that was stressed by Debreu [2].[2]

If we normalize prices to satisfy $p\cdot x = 1$ (alternatively, interpret p as normalized prices $p/p\cdot x$) the measure of economic efficiency is $C(u,p)$, the cost-function analogue of the Debreu/Farrell measure of technical efficiency, $D(u,x)^{-1}$. In fact, the measure of economic efficiency can be expressed as a distance function in price space:

$$C(u,p) = \max\{\lambda \in \mathbb{R}_+ | p/\lambda \in L^*(u)\},$$

where $L^*(u) \subseteq \mathbb{R}^n_+$ is a lower indirect level set (the set of prices at which affordable input bundles can produce no more than u). Thus, C(u,p) is the maximal proportionate amount by which prices can be increased and still produce output u by performing more efficiently. Thus, the natural measure of economic efficiency and the Debreu/Farrell measure of technical efficiency are dual to one another.

The duality between distance functions and cost functions provides the Debreu/Farrell technical efficiency measure with an evocative economic interpretation even when market prices do not exist. The distance function, D, can alternatively be defined (or derived) as the minimum imputed value of a given input vector, subject to the constraint that the chosen shadow-price vector be contained in the lower indirect level set:

$$D(u,x) = \min_p\{p \cdot x | p \in L^*(u)\} = \pi(u,x) \cdot x,$$

where $\pi(u,x)$ is the (minimizing) shadow-price vector. The shadow-price vector used to evaluate the input vector, $\bar{x}$, is given by the normal of the plane that supports L(u) at x/D(u,x). Thus,

$$E_{DF}(u,x) = D(u,x)^{-1} = \frac{\pi(u,x) \cdot \frac{x}{D(u,x)}}{\pi(u,x) \cdot x}$$

$$= \frac{C(u,\pi(u,x))}{\pi(u,x) \cdot x} .$$

That is, the Debreu/Farrell index can be characterized as the ratio of minimal to actual expenditures on inputs at the shadow price vector

$\pi(u,x)$. Of course, if $x/D(u,x)$ is not a technically efficient point (in the sense of Koopmans [8]), as in Figure 1 above, one or more of the shadow prices will be equal to zero. This is natural, for, at these prices, the input vector $\overset{*}{x}$ is economically (and, a fortiori, technically) efficient.

To summarize, the Debreu/Farrell measure of technical efficiency has two attractive properties:

(1) It is the natural (quantity based) dual to the usual measure of economic efficiency:

- E is a distance function in (normalized) price space; E_{DF} is a distance function in quantity space.
- E is a ratio of actual to minimal cost at market prices; E_{DF} is a ratio of actual to minimal cost at shadow prices (determined at output u and input direction $|\mathbf{x}|$).

(2) It is an upper bound on the usual measure of economic efficiency (using market prices).

These properties are not shared by measures of technical efficiency satisfying the Färe/Lovell conditions.

Footnotes

1. The arguments of this section are casually stated; to make them rigorous, we would have to posit convexity and free-disposal assumptions.

2. This point is related to the decomposition of economic inefficiency into technical and allocative inefficiency emphasized by Farrell [6] and revisited recently by Kopp and Diewert [7] and Zieschang [13].

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LIST OF SYMBOLS

$\mathbf{R}^n_+$ and $\mathbf{R}^n_{++}$:	bold-face "are" with superscript lc "en" and subscripts plus sign
x:	lc "ex"
$>$, $\geq$, and $\geqq$:	inequality signs
$0^{(n)}$ and $0^{(m)}$:	zero with superscripts lc "en" and lc "em" in parentheses
L:	uc "el"
u:	lc "you"
$\mathcal{P}$:	script uc "pea"
$\in$:	set membership notation
ε:	lc epsilon (Greek)
$\forall$:	"for all" sign
U:	uc "you"
$\wedge$:	conjunction sign ("and")
Λ:	uc lambda (Greek)
$\rightarrow$:	mapping sign
D:	uc "dee"
$\times$:	multiplication sign
κ:	lc kappa (Greek)
∞:	infinity
$=>$:	"implies" sign
$\subseteq$:	set containment sign
+:	plus sign, frequently appears as a superscript
λ:	lc labmda (Greek)
E:	uc "ee"

δ:	lc delta (Greek)
$\sum$:	summation sign
Γ:	uc gamma (Greek)
ϕ:	lc phi (Greek)
γ:	lc gamma (Greek)
Δ:	uc delta (Greek)
α:	lc alpha (Greek)
E:	uc script "ee"
p:	lc "pea"
π:	lc pi (Greek)
[ ]:	"closure" brackets
( ):	"open" brackets
{ }:	"set definition" brackets
d:	lc "dee"