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STOCHASTIC EQUILIBRIUM CYCLES

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Introduction

Since the seminal works of Lucas [1972], [1975], there has been an increasing interest in models of competitive economies that can explain or give some insight into business cycles. Recently, in the context of overlapping generations models, conditions for the existence of equilibrium cycles have been given by Grandmont [1983] and also by Benhabib and Day [1982]. The relation between models of sunspot equilibria developed by Cass and Shell [1983], and equilibrium cycles in overlapping generation models has been studied by Azariadis and Guesnerie [1983]. More recently Kydland and Prescott [1982] and Long and Plosser [1983] studied stochastic models with a single infinitely lined representative agent and, using simulation methods, generated trajectories for economic variables that closely resemble the behavior of these variables during a typical business cycle.

This paper is also an attempt to gain insight into the fluctuations of prices, outputs, employment and capital stocks that may arise in an equilibrium model of a real neoclassical economy with a stochastic production technology. Our approach is to consider a single two-sector neoclassical model with an investment and a consumption good and, like in growth theory, assume that an infinitely-lived representative household maximizes an additively separable utility function of consumption and leisure, subject to the technological constraints. We assume that the output of the investment good is subject to a stochastic shock. It is well known that the solution of this model can be given a decentralized market interpretation. Our results in this paper give conditions for a general neoclassical production technology, expressed as relative capital-intensity differences between sectors, that lead to fluctuations in the distributions of the outputs, stocks, prices and employment, so that a single shock generates an oscillatory time path for these variables. Furthermore, under capital-intensity assumptions, if capital and labor are economy-wide complements (as defined later) employment, consumption and the capital stock will be positively correlated (see Theorem 3).

One of the undesirable features of our model is that it has only two sectors. Thus if the capital stock is to fluctuate, an increase in its level will be accompanied by a fall in the net output of investment so that capital and net investment are negatively correlated. In a more general multi-sector model more complicated cycles would arise between different capital goods, so that while the output of one investment good increases the output of another countercyclical good may decline.

As is well known, even if the deterministic part of the model (which can be obtained by making the exogenous random shocks degenerate) gives rise to attracting periodic cycles (as in Benhabib and Nishimura [1984]) or to other more complicated attracting sets, the stochastic model can have a unique stationary invariant distribution if the random shocks are "large" and provide sufficient "mixing." However, the distributions of variables like employment, prices, outputs or the capital stock in the periods ahead, conditional on the value of the capital stock this period, can oscillate through time.[1] We will characterize the neoclassical

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[1] This points out, as is well known, that convergence to a stationary invariant distribution in a stochastic model is not a generalization of a unique attracting steady state in a deterministic model.

technologies (in terms of factor-intensity differences) that give rise to such oscillations through time and study the covariances between variables of the model.

In the next section we give a diagrammatic exposition of the fluctuations that occur in a very simplified deterministic version of our model with an inelastic labor supply and full capital depreciation. In Section III we present our general model and derive our results in Theorems 1 and 2.

Section IV discusses an alternative and stronger concept of oscillations which can be viewed as a direct generalization of periodic cycles in deterministic models. Instead of a stationary invariant distribution, we obtain conditions for the existence of cyclic sets (see Section IV for a definition). We feel, however, that oscillations in conditional distributions as given in Theorems 1 and 2 are more relevant for explaining economic data than the cyclic sets in Section IV.

II. A Diagrammatic Exposition

The basic reason for the oscillation of the economic variables in our model is that the accumulation of capital goods can change the tradeoffs in the production of different goods. Basically, if the accumulation of a capital good changes the slope of the production possibility frontier in favor of other goods, that capital good may then be decumulated. In a deterministic context, the persistence of periodic cycles requires some discounting to allow relative prices to oscillate without generating unexploited intertemporal arbitrage possibilities.[2]

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[2] The existence of cycles or oscillations does not depend on time being discrete. For robust examples in continuous time see Benhabib and Nishimura [1979]. We should point out that cycles arise in our continuous time model for essentially the same technological reasons as they do in discrete time, although in continuous time we need at least two capital goods.

In fact it can be shown that the amplitude of the periodic cycles depends on the discount rate (see Benhabib and Nishimura (1984)). In this paper, however, we will derive sufficient conditions for the existence of stochastic oscillations rather than of periodic cycles.

It will be useful to contrast our two-sector model giving rise to oscillations with a one-sector model that gives rise to the monotonic behavior of the economic variables. Consider a representative individual maximizing $\sum_{t=0}^{\infty} U(c_t)\beta^t$ where $0 < \beta < 1$ and $U(c_t)$ is the utility of consumption. Suppose accumulation is governed by $k_{t+1} = f(k_t) + (1 - g)k_t - c_t$ with k_0 given. Here $f(k_t)$ is per capita output and k_t is the capital stock at t and g is the depreciation rate. In a dynamic programming framework the problem becomes $V(k_0) = \underset{c}{\text{Max}}\ U(c) + \beta V(k_1)$ where $k_1 = f(k_0) + (1 - g)k_0 - c$ and where $V(k_0)$ is the value function. If $U(\cdot)$ and $f(\cdot)$ are concave so is $V(\cdot)$. Note that we can treat $U(c) + \beta V(k_1)$ as a separable and concave utility function in c and k_1. Given k_0, and for simplicity assuming full depreciation so that $g = 1$, we have the linear budget constraint $f(k_0) = k_1 + c$. Separability implies that the two "goods" k_1 and c are normal so that an increase in k_0 leads to choices of higher k_1 and c. In Figure 1 suppose that the optimal choice of k_1 is greater than k_0. Then k_2 must be greater than k_1, which implies the monotonicity of the capital stock (at least on an interior path).

Now consider a two-sector model where the per capita output of the consumption good c is related to the per capita output of the investment good y and the capital-labor ratio k by the function $c = T(y,k)$. For fixed k this function represents the production possibility frontier. Let subscripts denote time periods. In Figure 2 assume that y_0, which equals k_1 since depreciation is full, is greater than k_0. If $k_0 < k_1$ the production frontier will shift out but not in a parallel fashion. If the consumption good is capital intensive the shift will favor the output of the consumption good, as shown in Figure 2.[3] Then the choice of k_2 may be less than k_1, implying oscillations in the capital stock. We should point out that in a multi-sector model the production of goods other than the consumption good may increase with an increase in the stock of a capital good, while the production of that particular capital good may decline. Thus the consumption good need not necessarily be intensive in a capital good for oscillations to occur in a model of a multi-sector economy. In the next section we will explore precise conditions under which such oscillations occur in a more complex stochastic model where the labor supply is endogenous and we will study the covariances between the oscillating variables.

III. Stochastic Cycles

We will consider a stochastic neoclassical two-sector growth model which, as is well known, can also be interpreted as a decentralized model of intertemporal equlibrium with a representative consumer and profit maximizing firm operating under perfect competition (see, for example, Becker [1980]). We assume that the consumption good c and the investment good y are produced under constant returns to scale with neoclassical, concave production functions $c = c(k_c, \ell_c)$ and $y = (k_y, \ell_y)$ which are twice

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[3] The empirical evidence that we are aware of suggests that the capital-goods sector, taken as a whole, is more labor-intensive (and less capital-intensive) than the consumption goods sector. See R. A. Gordon [1961], p. 948.

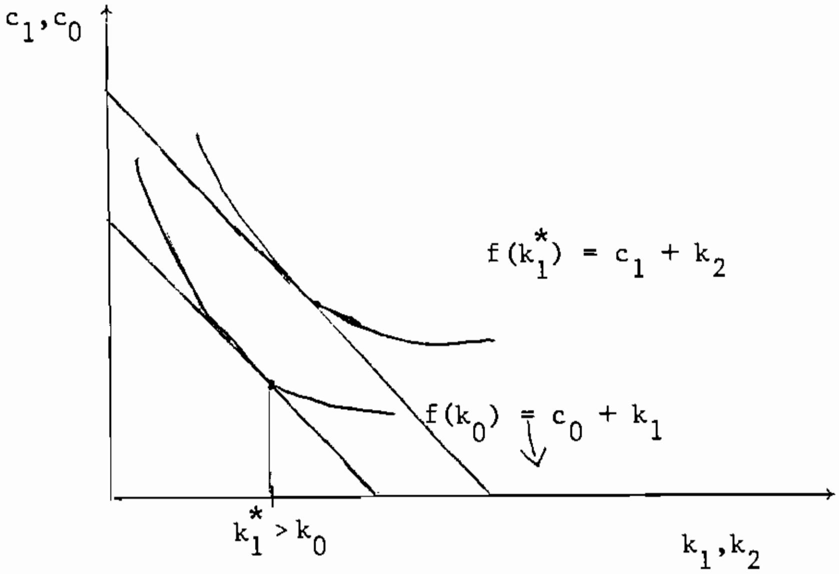

Figure 1

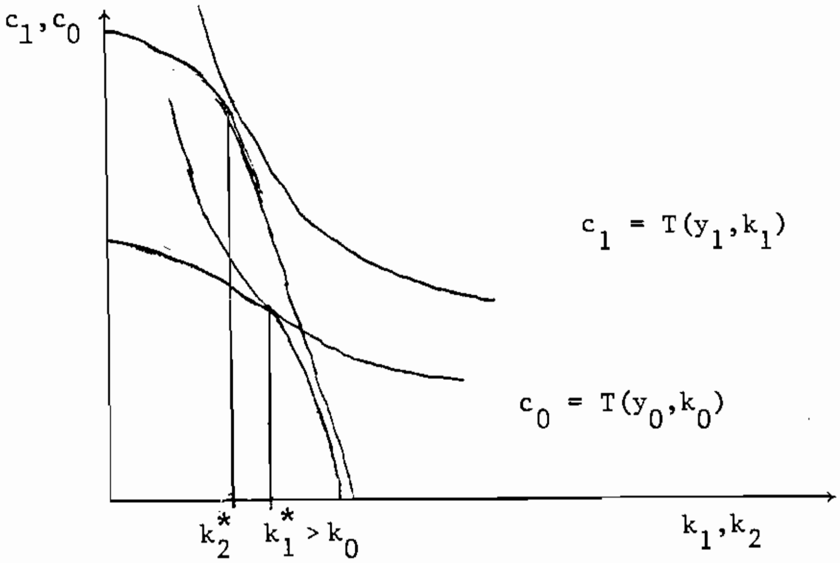

Figure 2

differentiable and are increasing in their arguments. Efficient production can be described by the solution to the following problem:

$$\text{Max } c(k_c, \ell_c)$$

$$\text{Subject to} \quad \bar{y} \leq y(k_y, \ell_y)$$

$$k \geq k_y + k_c$$

$$\ell \geq \ell_y + \ell_c$$

$$k_y,\ k_c,\ \ell_y,\ \ell_c \geq 0$$

It is easily shown that the level of consumption that can be attained by the solution to this problem can be expressed as a function relating the output c of the consumption good to the inputs k, ℓ, and to the output of the investment good y. We define this function as $c = T(y,k,\ell)$. The function T can be shown to be homogeneous of degree one and twice differentiable for c, y, k, $\ell > 0$. For later use we define $\bar{\bar{y}}(k,\ell)$ for all $k,\ell > 0$ to be the value of y which satisfies $T(y,k,\ell) = 0$. Thus $\bar{\bar{y}}(k,\ell)$ is the maximum amount of the investment good that can be produced with inputs (k,ℓ).

The representative consumer solves the following intertemporal problem:[4]

$$\text{(P)} \qquad \underset{\{y_t, \ell_t\}}{\text{Max}} \ E \sum_{t=0}^{\infty} (c_t + R(\bar{\ell} - \ell))\beta^t$$

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[4] This problem is a generalization of the Brock-Mirman [1972] stochastic growth model to a two-sector model with an endogenous labor supply. For the deterministic problem see McKenzie [1976] or Brock and Scheinkman [1976].

Subject to $c_t = T(y_t, k_t, \ell_t) \geq 0$

$$k_{t+1} = (y_t + \delta k_t) z_t$$

$$0 \leq \ell \leq \bar{\ell},\ k_0 \text{ given.}$$

R is a strictly concave function expressing the utility of leisure and $\bar{\ell}$ is the maximum work possible per period. β is the discount factor such that $0 < \beta < 1$. z_t is an i.i.d. random variable taking values on $(\underline{z}, \bar{z})$ with $E(z_t) = 1$ for all t and δ is one minus the depreciation rate. Thus the capital stock k_{t+1} is a random variable equal to the output of the investment good plus the capital carried over from t, but subjected to a shock z_t. We will assume that there exists a $\hat{k}$ such that for y satisfying $T(y,k,\bar{\ell}) = 0$, $(y + \delta z)\bar{z} < k$ if $k > \hat{k}$ and $(y + \delta k)\bar{z}_t > k$ if $k < \hat{k}$. Thus any k greater than $\hat{k}$ is not sustainable. For simplicity we will also assume that R satisfies Inada type conditions so that the optimal choice of ℓ is interior, that is, $0 < \ell < \bar{\ell}$. In addition, we assume that $k_0 < \hat{k}$.

The consumer's problem can then be written as a dynamic programming problem as follows:

$$(P^*) \qquad W(k_0) = \max_{(\ell_0, y_0)} [T(y_0, k_0, \ell_0) + R(\bar{\ell} - \ell_0) + \beta E W((y_0 + \delta k_0) z_0)]$$

where $0 \leq \ell \leq \bar{\ell}$ and $0 \leq y_0 \leq \bar{\bar{y}}(k_0, \ell_0)$. Our purpose is to derive sufficient conditions, in terms of the relative capital intensities of the two sectors, under which the optimal path for outputs, stocks and employment takes the form of persistent stochastic oscillations. Note that the solution (ℓ_0, y_0) to (P^*) depends on k_0. Since the value function

W is concave if T and R are concave and since we will assume that R is strictly concave in $(\ell - \ell_0)$, it follows that the optimizing values of ℓ_0 and y_0 are unique.[5] Thus we can express these optimal values as continuous functions $\ell_0(k_0)$ and $y_0(k_0)$. For later use we define $\bar{y}(k)$ as $\bar{\bar{y}}(k,\ell(k)) = \bar{y}(k)$. Note that if $y = \bar{y}(k)$, the output of c becomes zero.

Before stating our main result we will have to develop further certain relationships implicit in the technology described by $T(y,k,\ell)$. Note that for given (k,ℓ), T describes the production possibility frontier between c and y. $T_1 \equiv \frac{\partial T(y,k,\ell)}{\partial y}$ is the slope of this frontier and can be interpreted as the negative of the relative price of the investment good in terms of the consumption good. We define $T_1 = -p$. Similarly, $T_2 = \frac{\partial T(y,k,\ell)}{\partial k}$ is the marginal product of capital and $T_3 = \frac{\partial T(y,k,\ell)}{\partial \ell}$ is the marginal product of labor in terms of the price of the consumption good. We define $T_2 \equiv r$ and $T_3 \equiv w$.

Under constant returns to scale, we can express output prices as functions of relative input prices alone. We can obtain $p = p(w,r)$ where $(1,p) = (w,r)A$ and A is the cost-minimizing input coefficient matrix for the factor prices w. We define a_{11}, a_{21} and a_{12}, a_{22} as the labor and capital requirements for the consumption and investment good respectively. From Sheperd's Lemma we have $(0,dp) = (dw,dr)A$. Eliminating dw we obtain $b \equiv dp/dr = a_{12}\left(\frac{a_{22}}{a_{12}} - \frac{a_{21}}{a_{11}}\right)$. The sign of b

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[5] It can be shown that strictly diminishing marginal products with constant returns to scale production functions imply that $T(y,k,\ell)$ is jointly strictly concave in (k,ℓ) and that it is strictly concave in y alone if production functions for c and y are not identical. If production functions in the two industries are identical factor-intensities would also be identical. This is ruled out by assumption A below.

depends on whether the investment good or the consumption good is more capital intensive. It is easily shown that the value of b is independent of the definitions of the units of goods. Note that since input coefficients are functions of w, where $w = T_3(y,k,\ell)$ and r, where $r = T_2(y,k,\ell)$, b is a function of (y,k,ℓ). Furthermore, along an optimal path $y = y(k)$ and $\ell = \ell(k)$ so that b is a function of k alone. We write it as $b = b(k)$.

We can now state our results which give sufficient conditions on technology in terms of relative capital intensities that lead to stochastic oscillations in outputs, stocks and employment.

We first note that since $E(z) = 1$, $E(k_1|k_0) = y(k_0) + \delta k_0$, where $y(k_0)$ is the optimal choice for the output of the investment good. We make the following assumptions:

(AI) There exists an interval $(\underline{k},\bar{k})$ of the capital stock such that $0 < y(k) < \bar{y}(k)$ for all $k \in (\underline{k},\bar{k})$

(AI) requires the existence of an interval of the capital stock such that the optimal choice of investment output does not lead to specialization in either the consumption or the investment good.

(AII) $-\delta^{-1} < b(k) < 0$ for $k \in (\underline{k},\bar{k})$.

(AII) is a relative capital intensity condition requiring that the consumption good be more capital intensive than the investment good ($b < 0$) but not too strongly so ($b > -\delta^{-1}$). Note that with full depreciation $\delta = 0$ and (ii) simply requires that the consumption good be capital intensive, that is, $b < 0$. For later use we also state the converse of assumption (AII):

(AII)' $b(k) \notin [-\delta^{-1}, 0]$ for all $k \in (\underline{k}, \bar{k})$

The theorem below shows how the changes in the capital stock from one period to the other are inversely related to the changes in the expected values of the capital stock in the subsequent periods.

Theorem 1: Let (A) hold and let $k_0 \in (\underline{k}, \bar{k})$. If the realization of k_1 is higher (lower) than k_0, then the expected value of k_2 given k_1 is lower (higher) than the expected value of k_1 given k_0: $(k_1 - k_0)(E(k_2|k_1) - E(k_1|k_0)) < 0$ for $k_1 \neq k_0$.

Remark 1: It is important to note that the above Theorem holds irrespective of whether k_0 is less or greater than $E(k_1|k_0)$. For example, if k_0 is not expected to change so that $k_0 = E(k_1|k_0)$, a realization of k_1 above (below) k_0 implies that the expected value of k_2 will be below (above) k_0; that is, for $k_1 \neq k_0$, $(k_1 - k_0)(E(k_2|k_1) - k_0) < 0$. In particular, if for all initial stocks the distribution of the capital stock converges to a stationary distribution with mean $\bar{k}$, then for large t, a realization of k_{t-1} at $\bar{k}$ followed by a realization of k_t above (below) $\bar{k}$ will imply that the expected value of k_{t+1} is below (above) $\bar{k}$; that is, for $k_t \neq \bar{k}$, $(k_t - \bar{k})(E(k_{t+1}|k_t) - \bar{k}) < 0$.[6] In the long run we expect an irregular but oscillatory pattern over time for the capital stock (see also Section IV).

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[6] With some additional technical assumptions convergence to a unique stationary distribution will occur if there is a neighborhood of some k from which any other neighborhood in the domain can be reached in a finite number of steps, independently of the neighborhood. This can be assured if the lower bound of $z, \underline{z}$ is zero. A technical proof, however, is beyond the scope of this paper. For the case where convergence to a unique stationary distribution fails and cyclic sets develop, see Section IV.

Remark 2: If we adopt the converse of the capital intensity assumption (AII) and replace it with (AII)' the inequality in Theorem 1 is reversed so that $(k_1 - k_0)(E(k_2|k_1) - E(k_1|k_0)) > 0$. This follows since (AII)' implies that $G(k)$ in Lemma 1 is positive. Thus under (AII)' an increase in the capital stock leads to an increase in the expected value of the stock in the subsequent period.

Proof of Theorem 1: Consider first problem P^* above. Given Inada type conditions on R, the maximizing choice of ℓ_0 must satisfy $T_3(y_0,k_0,\ell_0) = R'(\bar{\ell} - \ell_0)$. Using the implicit function theorem and the strict concavity of R, we can obtain a differentiable function $\ell_1 = \ell(y_0,k_0)$. Substituting this into the first two terms of the right-hand side of P^*, we obtain the expression $T(y_0,k_0,\ell(y_0,k_0)) + R(\ell - \ell(y_0,k_0))$. This expression can be shown to be concave provided $T(y,k,\ell)$ and $R(\bar{\ell} - \ell)$ are concave. Thus if we define $V(k,y) = T(y,k,\ell(y,k)) + R(\bar{\ell} - \ell(y,k))$, problem (P^*) above can be written as $W(k_0) = \underset{y_0}{\text{Max}}\ V(k_0,y_0) + \beta EW((y_0 + \delta k_0)z)$, where $0 \leq y_0 \leq \bar{y}(k_0)$. Since we assumed before that $T(y,k,\ell)$ was strictly concave in y, the optimal choice of y_0 is unique given k_0, as we argued previously following the statement of problem P^*. We can then define the function $G(k) = V_{12}(k,y(k)) - \delta V_{22}(k,y(k))$.

Lemma 1: Let $G(k) < 0$ and $0 < y(k) < \bar{y}(k)$ for all k in an interval $(\underline{k},\bar{k})$. Then $k_1 = (y(k_0) + \delta k_0)z_0 \equiv h(k_0,z_0)$ is decreasing in k_0 over $(\underline{k},\bar{k})$.[7]

Proof: See Appendix.

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[7] If $G(k) > 0$ then $h(k_0,z_0)$ is increasing in k_0.

Given Lemma 1 it is clear that $E(k_{t+1}|k_t)$ is decreasing in k_t. Theorem 1 immediately follows from Lemma 1. To complete the proof we must show that the hypothesis $G(k) < 0$ of Lemma 1 corresponds to assumption (AII).

From the definition above we have $V(k,y) = T(y,k,\ell(y,k)) + R(\bar{\ell} - \ell(y,k))$ and $G(k) = V_{12}(k,y(k)) - V_{22}(k,y(k))$. We have to show that $G(k) < 0$ under suitable assumptions. Calculating $G(k)$ in terms of the derivatives of the function T and making use of the condition $T_3 = R'$, we obtain

$$G(k) = V_{12} - \delta V_{22} = (T_{12} - \delta T_{11}) + (T_{32} - \delta T_{31})\ell_1 + (\ell_2 - \delta\ell_1)(T_{13} + (T_{33} + R'')\ell_1)$$

where $\ell_1 = \frac{d\ell(y,k)}{dy}$ and $\ell_2 = \frac{d\ell(y,k)}{dk}$. Using $T_3 = R'$ we can obtain $\ell_2 - \delta\ell_1 = (T_{32} - \delta T_{31})/(-R'' - T_{33})$. Substituting this into $G(k)$, we obtain $G(k) = (T_{12} - \delta T_{11}) + (T_{32} - \delta T_{31})(T_{13}/(-R'' - T_{33}))$

At this point we exploit certain properties of the function T. As noted earlier when prices equal cost we have $(1,p) = (w,r)A$. Eliminating w, we have $p(r) = \frac{a_{12}}{a_{11}} + \left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right)r$. Furthermore as discussed earlier, using Shepard's Lemma we have $(0,dp) = (dw,dr)A$ which yields $dp/dr = a_{22} - \frac{a_{12}a_{21}}{a_{11}} \equiv b$. Consider now the following terms: $T_{11} = \frac{-\partial p}{\partial y}$, $T_{12} = \frac{-\partial p}{\partial k}$, $T_{13} = \frac{-\partial p}{\partial \ell}$, $T_{22} = \frac{\partial r}{\partial k}$, $T_{23} = \frac{\partial r}{\partial \ell}$ and $T_{33} = \frac{\partial w}{\partial \ell}$. Using $p = p(r)$, we obtain $T_{11} = \frac{-\partial p}{\partial y} = \frac{-\partial p}{\partial r}\frac{\partial r}{\partial y} = \frac{-\partial p}{\partial r} T_{21} = \frac{-\partial p}{\partial r} T_{12}$. Also, $T_{12} = \frac{-\partial p}{\partial k} = \frac{-\partial p}{\partial r}\frac{\partial r}{\partial k} = \frac{-\partial p}{\partial r} T_{22}$.

Thus $T_{11} = \left(\frac{\partial p}{\partial r}\right)^2 T_{22}$. Thus, we obtain $T_{12} - \delta T_{11} = \left(\frac{-\partial p}{\partial r}\right)\left(T_{22}\right)\left(1 + \delta \frac{\partial p}{\partial r}\right)$ $= -b(1 + \delta b)T_{22}$. Similarly we have $T_{13} = \frac{-\partial p}{\partial \ell} = \frac{-\partial p}{\partial r}\frac{\partial r}{\partial \ell} = \frac{-\partial p}{\partial r} T_{23} = \frac{-\partial p}{\partial r} T_{32}$. We then obtain $(T_{32} - \delta T_{31}) = T_{32}(1 + \delta b)$. Substituting into the expression for $G(k)$ we obtain

$$G(k) = -b[1 + \delta b][T_{22} + T_{32}^{\,2}/(-R'' - T_{33})]$$

$$= -b(1 + \delta b)[(-T_{22}R'' - (T_{22}T_{33} - T_{23}^{\,2})/(-R'' - T_{33})]$$

Assuming that $T(y,k,\ell)$ is jointly strictly concave in (k,ℓ),[8] $(T_{22}T_{33} - T_{23})^2 > 0$ and the expression in square brackets will be negative. Thus $G(k) < 0$ if $b(1 + \delta b) < 0$, which requires that $-\delta^{-1} < b < 0$. Thus the requirement that $G(k) < 0$ is the same as assumption (AII).[9] Q.E.D.

The results of Theorem 1 are given in terms of the realization of k_1 relative to k_0. It may also be of interest to derive results on the expected values of the capital stocks in the next and subsequent periods for any given values of the current capital stock. However, for given k_0, $k_2 = [y((y(k_0) + \delta k_0)z_0) + \delta(y(k_0) + \delta k_0)z_0]z_1$, so that k_2 depends on z_0 in a non-linear way. Since $y(k)$ is not necessarily a concave function it is not possible to establish the sign of $E(k_2|k_0) - E(k_1|k_0)$ from the sign of $E(k_1|k_0) - k_0$. Nevertheless, it is possible to show that the probability of a realization of k_1 which gives rise to an oscillation

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[8] $T(y,k,\ell)$ is jointly strictly concave in (k,ℓ). See footnote 4.

[9] In a one-sector model like Brock and Mirman [1972] relative capital intensities cannot be an issue and $G(k)$ is always positive, even if one allows for an endogenous labor supply. Thus Remark 2 above applies.

in the distribution of the capital stock is high. The following symmetry condition on the distribution of the exogenous random variable z_t is useful for stating Theorem 2. Note that for convenience we had assumed that the mean of z_t is equal to one. Let $\pi(z_t)$ be the distribution function of z_t where $z_t \in (\underline{z}, \bar{z})$.

Condition (S): $\pi(1) = \frac{1}{2}$.

Under (A) and (S) Theorem 2 below states that when the expected value of k_1 given k_0 is greater (less) than k_0, the probability of a realization of k_1 such that the expected value of k_2 given k_1 is less (greater) than k_1 will be larger than $\frac{1}{2}$. Similarly, the probability of a realization of k_1 such that the expected value of k_2 given k_1 is less (greater) than the expected value of k_1 given k_0 will also be larger than $\frac{1}{2}$.

Theorem 2: Let assumption (A) and condition (S) hold and let $k_0 \in (\underline{k}, \bar{k})$.

(I) If $E(k_1|k_0) > k_0$

(i) $\Pr(k_1|E(k_2|k_1) < k_1) = 1 - \pi(z_\ell) > \frac{1}{2}$

(ii) $\Pr(k_1|E(k_2|k_1) < E(k_1|k_0)) = 1 - \pi(\hat{z}_\ell) > \frac{1}{2}$

(II) If $E(k_1/k_0) < k_0$

(i) $\Pr(k_1|E(k_2/k_1) > k_1) = \pi(z_u) > \frac{1}{2}$

(ii) $\Pr(k_1|E(k_2|k_1) > E(k_1|k_0)) = \pi(\hat{z}_u) > \frac{1}{2}$

where z_u, $\hat{z}_u$, z_ℓ, $\hat{z}_\ell$ are defined in the proof below.

Proof of Theorem 2: We will first prove case I. Since $(y(k_0) + \delta k_0) = E(k_1|k_0) > k_0$ we have (from Lemma 1) that

$$y(y(k_0) + \delta k_0) + \delta(y(k_0) + \delta k_0) < y(k_0) + \delta k_0 \quad (1)$$

This follows since Lemma 1 in the proof of Theorem 1 shows that $(y(k) + \delta k)$ is decreasing in k. Now consider

$$E(k_2|k_1) = y[(y(k_0) + \delta k_0)z_0] + \delta[(y(k_0) + \delta k_0)z_0]. \tag{2}$$

For $z_0 = 1$, this corresponds to the left side of (1). Since $y(k) + \delta k$ is decreasing in k and $k_1 = (y(k_0) + \delta k_0)z_0$ is increasing in z_0, either there will be a $z_\ell \in [\underline{z},1)$ such that

$$y[(y(k_0) + \delta k_0)z_\ell] + \delta[(y(k_0) + \delta k_0)z_\ell] = (y(k_0) + \delta k_0)z_\ell \tag{3}$$

or for all $z_0 \in [\underline{z},1]$ the right side of (2) is less than $(y(k_0) + \delta k_0)z_0$. In case of the latter we let $z_\ell = \underline{z}$. Thus for all $z_0 \in (z_\ell,\bar{z}]$, the left side of (3) is less than the right side of (3) and thus $E(k_2|k_1) < k_1$. But the probability that $z_0 \in (z_\ell,\bar{z}]$ is $1 - \pi(z_\ell)$ which is greater than $\frac{1}{2}$ by condition (S) since $z_\ell < 1$. To prove part (ii) of (I) we simply replace $k_1 = (y(k_0) + \delta k_0)z_0$ by $E(k_1|k_0) = y(k_0) + \delta k_0$ on the right side of (3). Now we can find a $\hat{z}_\ell \leq z_\ell$ to equate both sides of (3) since the right side is now simply $y(k_0) + \delta k_0$ and does not increase with z_0. As before if there is no z that equates both sides of (3), we set $\hat{z}_\ell = \underline{z}$. Then the proof is the same as for (i). To prove part II of Theorem 2, we simply note that the inequality (1) will be reversed and to restore the equality as in (3) we have to choose $z_u \in (1,\bar{z}]$. As in the proof of (I), if this cannot be done we set $z_u = \underline{z}$. Thus for all $z \in [\underline{z},z_u)$ we have $E(k_2|k_1) > k_1$ and since $z_u > 1$ the result follows under (S). The proof of part (ii) of II is analogous to part (ii) of I, where now $\hat{z}_u > z_u$. Q.E.D.

Broadly speaking, Theorem 2 states that whenever the capital stock is expected to increase (decrease) from this period to the next, the odds are better than even that next period the capital stock will be expected to decrease (increase) in the subsequent period.

Remark 3: If we replace (AII) with (AII)', Theorem II will be reversed so that parts I(i) and I(ii) will hold if $E(k_1|k_0) < k_0$ and parts II(i) and II(ii) will hold if $E(k_1|k_0) > k_0$. This follows because (AII)' implies $G(k) > 0$, which reverses the results of Lemma 1.

We now turn to study how employment, consumption and investment are correlated with the capital stock. An oscillatory path for the capital stock implies that the path of investment must also oscillate. Using Lemma 1 we can easily show that an increase in the capital stock implies that the output of the investment good declines. In other words, $k_1 > k_0$ implies $y(k_0) > y(k_1)$ since from Lemma 1 we have $y(k_1) - y(k_0) < \delta(k_0 - k_1) < 0$. It may also follow that if k_1 exceeds k_0 and the output of the investment good falls, the output of the consumption good will rise on account of the fall in the output of the investment as well as the rise in the capital stock. This, however, is not certain since the behavior of employment is ambiguous. We can study how employment changes when the capital stock increases by considering the relation $T_3(y(k),k,\ell) = R'(\ell)$. We obtain $\frac{d\ell}{dk} = (T_{31} \cdot \frac{dy}{dk} + T_{32})/(-R'' - T_{33})$. However, by Lemma 1 $k' > k$ implies $y(k') + \delta k' < y(k) + \delta k$ or $y(k) - y(k') > \delta(k' - k)$. This implies that $\frac{-dy}{dk} \geqq \delta$. On the other hand, $T_{31} = \frac{\partial p}{\partial r} \cdot T_{32} = -bT_{32}$ and $T_{32} = \frac{\partial w}{\partial k}$. Therefore, $T_{31} \cdot \frac{dy}{dk} + T_{32} = \frac{\partial w}{\partial k}\,[1 - \frac{dy}{dk} \cdot b]$. From our assumption on relative capital intensities we have $-\delta^{-1} < b < 0$ which, together with $\frac{-dy}{dk} \geqq \delta$, implies that $(1 - b \cdot \frac{dy}{dk}) > 0$. Therefore, since $(-R'' - T_{33})$ is positive, $\frac{d\ell}{dk}$ is of the same as the sign of $\frac{\partial w}{\partial k}$. Whether employment increases or not with a rise in the capital stock, therefore, depends on whether capital and labor are substitutes or complements at the economy-wide level. If labor and capital are complements, so that $\frac{\partial w}{\partial k} > 0$, a rise in the capital stock will imply that employment also increases. Since investment also decreases, consumption must rise. Thus consumption, employment and the capital stock will rise or fall simultaneously while the output of the investment good will move in the opposite direction of the capital stock. We can express this as a theorem:

Theorem 3: Let $\frac{\partial w(y(k),k,\ell(k))}{\partial k} > 0$ for $k \in (\underline{k},\bar{k})$ and let the hypotheses of Theorem 1 hold. Then for $k_1 \neq k_0$, $(k_1 - k_0)(\ell(k_1) - \ell(k_0)) > 0$, $(k_1 - k_0)(c(k_1) - c(k_0)) > 0$ and $(k_1 - k_0)(y(k_1) - y(k_0)) < 0$.

The corollary below, which follows immediately from Theorem 3, gives the covariance of employment, consumption and investment with the capital stock.

Corollary: Under the hypothesis of Theorem 3 the following hold when $k_1 \neq E(k_1|k_0)$; $(k_1 - E(k_1|k_0))(\ell(k_1) - \ell(E(k_1|k_0)) > 0$, $(k_1 - E(k_1|k_0))(c(k_1) - c(E(k_1|k_0)) > 0$ and $(k_1 - E(k_1|k_0))(y(k_1) - y(E(k_1|k_0))$ < 0.

IV. Cyclic Sets

An alternative and stronger concept of oscillations in a stochastic context is that of cyclic sets. The results of the previous sections do not depend on the asymptotic convergence of the distribution of the capital stock to a unique stationary distribution. In fact if the support of the distribution of the random variable is small so that the shocks are not sufficiently mixing, we can have cyclic sets, as defined below.

Definition: Two disjoint intervals $[\underline{k}_A,\bar{k}_A]$ and $[\underline{k}_B,\bar{k}_B]$ are period-two cyclic sets if for any realization of the random variable z_t, $k_t \in [\underline{k}_A,\bar{k}_A]$ implies $k_{t+1} = h(k_t,z_t) \in [\underline{k}_B,\bar{k}_B]$ and $k_t \in [\underline{k}_B,\bar{k}_B]$ implies $k_{t+1} = h(k_t,z_t) \in [\underline{k}_B,\bar{k}_B]$.

For our problem we let $h(k,z) = g(k)z$ and $g(k) = y(k) + \delta k$. In Figure 3, $g(k_t)\bar{z}$ denotes the highest and $g(k_t)\underline{z}$ the lowest possible realization of k_{t+1} for each k_t. It is easily seen by inspection that $[\underline{k}_A,\bar{k}_A]$ and $[\underline{k}_B,\bar{k}_B]$ form period-two cyclic sets. We will show that if a deterministic economy where z is constant has a stable period-two cycle, then we can construct a stochastic economy by choosing an appropriate support $[\underline{z},\bar{z}]$ for the random variable z and obtain cyclic sets.

Consider the deterministic system obtained by setting $z = \hat{z}$. We have shown in Benhabib and Nishimura (1984) that if there exists a discount factor β^* such that at the steady state corresponding to β^*, $b(\beta^*) \in (-(1+\delta)^{-1}, -\beta^*(1-\beta^*\delta)^{-1})$ (where b is as in section III above),[10] then the economy will have a period-two cycle $(\underline{k},\bar{k})$ so that $\bar{k} = g(g(\bar{k})z)z$ and $g(\bar{k})z = \underline{k}$. Also, readers can find the stability arguments of period-two

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[10] Compare with assumption II above.

cycles in corollary 1 of Benhabib and Nishimura (1984). We shall assume the differentiability of g(k) (in turn of y(k)) in proving the following theorem.

Theorem 4: Suppose that the deterministic problem obtained by setting $z = \hat{z}$ in the stochastic problem (P) has a cycle of period two that is stable.[11] Then there is a neighborhood N of $(\hat{z},\hat{z})$ such that the stochastic problem with a random variable z with any support $(\underline{z},\bar{z}) \in N$ has period-two cyclic sets.

Proof: By hypothesis there exists $\underline{k}$ and $\bar{k}$ such that

$$g(\bar{k})\hat{z} = \underline{k} < \bar{k} \tag{4}$$

and

$$g(g(\bar{k})\hat{z})\hat{z} = \bar{k} \tag{5}$$

Since the deterministic period-two cycle is stable,

$$\frac{d(g(g(\bar{k})\hat{z})\hat{z})}{d\bar{k}} < 1$$

so that we can apply the implicit function theorem to (5) and obtain the locally differentiable function $k = k(z_1,z_2)$ defined on a neighborhood N of $(\hat{z},\hat{z})$ that satisfies

$$g(g(k)z_1)z_2 = k \tag{6}$$

Differentiating, we note that

$$k_1 = \frac{\partial k(z_1,z_2)}{\partial z_1} < 0 \quad \text{and} \quad k_2 = \frac{\partial k(z_1,z_2)}{\partial z_2} > 0 \tag{7}$$

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[11] That is, optimal paths starting in a neighborhood of the cycle converges to the cycle.

evaluated at $(\hat{z},\hat{z})$. We can assume that (7) and $g'(k) < 0$ holds on a neighborhood N. Without loss of generality choose $(\underline{z},\bar{z})$ in N. Then $\bar{k}_B = k(\underline{z},\bar{z})$ satisfies

$$g(g(\bar{k}_B)\underline{z})\bar{z} = \bar{k}_B \tag{8}$$

Define $\underline{k}_A = g(\bar{k}_B)\underline{z}$; then it satisfies

$$g(\underline{k}_A)\bar{z} = \bar{k}_A \quad \text{and} \quad g(g(\underline{k}_A)\bar{z})\underline{z} = \underline{k}_A \tag{9}$$

Similarly, $\underline{k}_B = k(\bar{z},\underline{z})$ and $\bar{k}_A = g(\underline{k}_B)\bar{z}$ satisfies

$$g(g(\underline{k}_B)\bar{z})\underline{z} = \underline{k}_B \quad \text{and} \quad g(g(\bar{k}_A)\underline{z})\bar{z} = \bar{k}_A \tag{10}$$

Since $k_1 < 0$ and $k_2 > 0$ on N, $\bar{z} > \underline{z}$ implies that $\bar{k}_B > \underline{k}_B$. Also, $g'(k) < 0$ (as shown in lemma 1) and $\bar{z} > \underline{z}$ implies $\bar{k}_A > \underline{k}_A$.

Since $k(z_1,z_2)$ is continuous, we can choose a neighborhood N of $(\hat{z},\hat{z})$ small enough so that $(\underline{k}_B,\bar{k}_B)$ and $(\underline{k}_A,\bar{k}_A)$ are sufficiently close to $(\bar{k},\bar{k})$ and $(\underline{k},\underline{k})$, and they are disjoint. Then $\bar{k} > \underline{k}$ implies $\underline{k}_B > \bar{k}_A$. We have so far shown that

$$\begin{aligned} g(\underline{k}_A)\bar{z} &= \bar{k}_B, \quad g(\bar{k}_A)\underline{z} = \underline{k}_B \\ g(\underline{k}_B)\bar{z} &= \bar{k}_A, \quad g(\bar{k}_B)\underline{z} = \underline{k}_A \end{aligned} \tag{11}$$

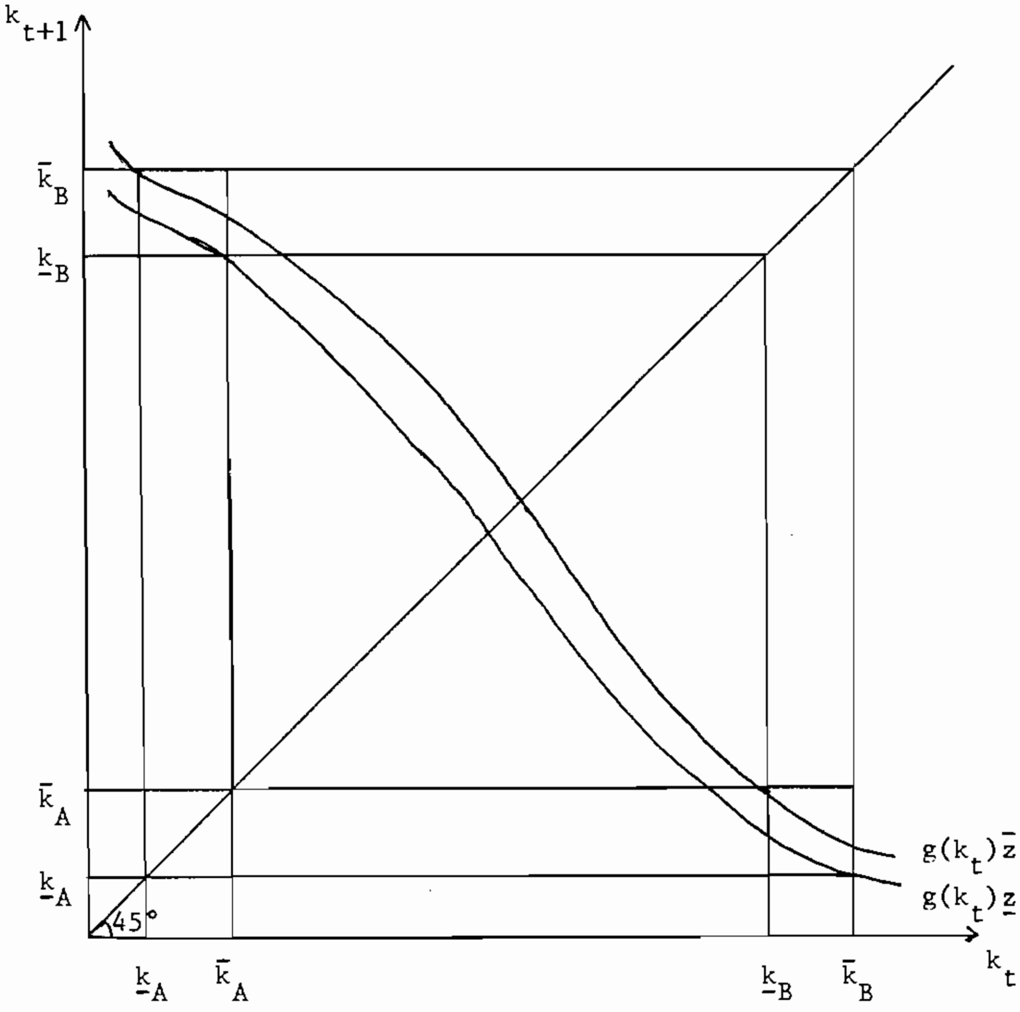
where $\underline{k}_A < \bar{k}_A < \underline{k}_B < \bar{k}_B$ and $\underline{z} < \bar{z}$. Then $\underline{z} < \bar{z}$ and $g'(k) < 0$ implies that for $z \in [\underline{z},\bar{z}]$

$$g(k)z \in [\underline{k}_B,\bar{k}_B] \text{ for } k \in [\underline{k}_A,\bar{k}_A]$$

and

$$g[k]z \in [\underline{k}_A,\bar{k}_A] \text{ for } k \in [\underline{k}_B,\bar{k}_B]$$

Furthermore, since the deterministic cycle is locally stable, that is, $\left|\frac{d(g(g(\bar{k})z_1),z_2}{d\bar{k}}\right| < 1$ for $z_1 = z_2 = \hat{z}$, for z_1 and z_2 sufficiently close to $\hat{z}$, the sets $(\underline{k}_A,\bar{k}_A)$ and $(\underline{k}_B,\bar{k}_B)$ will also be locally attracting. Q.E.D.


k_{t+1}
$\bar{k}_B$
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$\underline{k}_B$
$\bar{k}_B$
k_t
$g(k_t)\bar{z}$
$g(k_t)\underline{z}$

Figure 3

APPENDIX

Proof of Lemma 1: We first prove that $h(k_0,r)$ is decreasing in a sufficiently small neighborhood of k_0 where $k_0 \in (\underline{k},\bar{k})$. Define

$$\begin{aligned} a &= y(k_0') + \delta(k_0' - k_0) \\ b &= y(k_0) + \delta(k_0 - k_0') \end{aligned} \tag{1}$$

Since $y(k_0)$ is a continuous function, $y(k_0) < \bar{y}(k_0)$ implies that $a = y(k_0') + \delta(k_0' - k_0) < \bar{y}(k_0)$ for k_0' sufficiently close to k_0. This means that a is producible from k_0. Next we note that $0 < \delta(k_0' - k_0) < y(k_0)$ holds for k_0' sufficiently close to k_0 since $y(k) > 0$. Then $0 < b = y(k_0) + \delta(k_0 - k_0') < y(k_0)$, which means that b is producible from k_0' since $k_0' > k_0$. We use the fact that a is producible from k_0 and that b is producible from k_0' in deriving the inequality below.

By the principle of optimality we have

$$\begin{aligned} V(k_0,y(k_0)) + BEW((y(k_0) + \delta k_0)z_0) &\geq \\ V(k_0,a) + BEW(a + \delta k_0)z_0) & \end{aligned} \tag{2}$$

and

$$\begin{aligned} V(k_0',y(k_0')) + \beta EW(y(k_0') + \delta k_0')z_0) &\geq \\ V(k_0',b) + \beta EW(b + \delta k_0')z_0) & \end{aligned} \tag{3}$$

Adding up (2) and (3), we get

$$V(k_0,y(k_0)) + V(k_0',y(k_0') \geq V(k_0,a) + V(k_0',b) \tag{4}$$

Hence

$$V(k_0,y(k_0)) - V(k_0,a) + V(k_0',y(k_0')) - V(k_0',b) \geq 0 \tag{5}$$

Let $s = y + \delta k_0$ and $U(k_0,s) \equiv V(k_0,s - \delta k_0)$. Then the left hand side of inequality (5) is transformed into the following:

$$L = \int_{a+\delta k_0}^{y(k_0)+\delta k_0} \frac{\partial U(k_0,s)}{\partial s}\, ds + \int_{b+\delta k_0'}^{y(k_0')+\delta k_0'} \frac{\partial U(k_0',s)}{\partial s}\, ds$$

$$= \int_{y(k_0)+\delta k_0}^{y(k_0')+\delta k_0'} \left[\frac{\partial U(k_0',s)}{\partial s} - \frac{\partial U(k_0,s)}{\partial s}\right] ds \geq 0 \tag{6}$$

We used $a + \delta k_0 = y(k_0') + \delta k_0'$ and $b + \delta k_0' = y(k_0) + \delta k_0$ to obtain the integral in (6). Let k_0' be close enough to k_0 so that $\frac{\partial^2 U(k,y)}{\partial k \partial y} = V_{12}(k,y) - \delta V_{22}(k,y) < 0$ holds on the rectangular region $D = [k_0,k_0'] \times [y(k_0'),y(k_0)]$, as required by the hypothesis $G(k) < 0$ in Lemma 1. Note that any $(k,y) \in D$ is feasible in the sense that y can be produced from k. This follows because both $y(k_0)$ and $y(k_0')$ are less than $\bar{y}(k_0)$ and because k_0 is close to k_0'.

Since $\frac{\partial^2 U(k,y)}{\partial k \partial y} < 0$, it is clear from (6) that if $k_0' > k_0$ and $y(k_0') + \delta k_0' > y(k_0) + \delta k_0$ the value of the integral of (6) is negative. This is a contradiction to (5). Hence $y(k_0') + \delta k_0' \leq y(k_0) + \delta k_0$ for $k_0' > k_0$. This means that $k_1 \equiv h(k_0,z)$ is locally non-increasing. But the choice of k_0 was arbitrary on $[\underline{k},\bar{k}]$ and, hence, $h(k_0,r)$ is non-increasing on the whole interval $[\underline{k},\bar{k}]$.

Next we prove that $h(k_0,z)$ is strictly decreasing on $[\underline{k},\bar{k}]$. Suppose that $k_0' > k_0$ and $s = y(k_0') + \delta k_0' = y(k) + \delta k_0$. We note that since W(k) is concave, it is differentiable almost everywhere on $[\underline{k},\bar{k}]$. Therefore, $\int_{\underline{z}}^{\bar{z}} W'((y + \delta k_0)z)dz \equiv EW'((y + \delta k)z)$ exists for all $y \in (0,\bar{y}(k_0))$. It then follows that the optimal choice of $y = y(k_0)$ must satisfy

$$V_2(k_0,y(k_0)) + \beta\, EW'((y(k_0) + \delta k_0)z) = 0 \tag{7}$$

since $y(k_0) \in (0,\bar{y}(k_0))$ by hypothesis.

Using s, (7) may be stated as

$$V_2(k_0,s - \delta k_0) + \beta\, E Z W'[sz] = 0 \tag{8}$$

But $V_{12} - \delta V_{22} \neq 0$ implies that

$$V_2(k_0,s - \delta k_0) \neq V_2(k_0',s - \delta k_0')$$

for k_0' close enough to k_0. So equation (8) cannot be satisfied for both k_0 and k_0'. Therefore,

$$y(k_0) + \delta k_0 > y(k_0') + \delta k_0'$$

for $k_0' (> k_0)$ sufficiently close to k_0, which implies that $h(k_0,z) < h(k_0',z)$ for a given realization of z. This holds for all $k_0 \in [\underline{k},\bar{k})$. hence for all $k_0 \in [\underline{k},\bar{k}]$. Q.E.D.

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