ECONOMIC RESEARCH REPORTS

GENERAL EQUILIBRIUM WITH FUTURES TRADING

by

Manfred Nermuth

R.R. #85-03 January 1985

C. V. STARR CENTER
FOR APPLIED ECONOMICS

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

GENERAL EQUILIBRIUM WITH FUTURES TRADING

by

Manfred Nermuth*

December 1984

This paper is to a large extent the result of extensive discussions with M. Magill, to whom I am deeply indebted. Remaining errors are mine alone. I would also like to acknowledge support received from the C.V. Starr Center for Applied Economics in the preparation of this manuscript.

* Department of Economics
New York University
269 Mercer Street
New York, NY 10003

and

Institut fuer Wirtschaftswissenschaften
Universitaet Wien
Liechtensteinstr. 13/11
A-1090 Vienna, Austria

Abstract

We study an incomplete sequential market in which state-contingent contracts cannot be made, but futures trading (for uncontingent delivery) is possible. We show that the absence of arbitrage opportunities in the futures market is equivalent to the existence of a probability measure relative to which futures prices form a martingale. Moreover, the futures market has the same (Pareto-efficient) equilibria as an Arrow-Debreu complete market, provided there is *sufficient price variability* in the following sense: for each date-event pair in the event tree, the price vectors corresponding to its possible "successors" at the next date are linearly independent. Essentially, the number of commodities traded on the futures market must be at least as large as the "branching number" of the information structure.

1. Introduction

In this paper we study an incomplete sequential market model in the sense of RADNER '72, with consumption taking place at every date, and distinguished by the following feature: state-contingent contracts à la Arrow-Debreu cannot be made, but futures trading (for uncontingent delivery) is possible. First we show that the absence of arbitrage opportunities in the futures market is equivalent to the existence of a probability measure relative to which futures prices form a martingale (Theorem 3.1, Lemma 3.1). Next we ask under which conditions our incomplete market will be equivalent to an Arrow-Debreu complete market system, in the sense that it has the same (Pareto-efficient) equilibrium allocations.

We show that this is the case if there is sufficient price variability in the following sense: for each date-event pair (i.e., each "market"), the price vectors corresponding to its possible "successors" at the next date are linearly independent (Theorem 4.1). The number of possible successors of a given market in an event-tree is a crude measure of the amount of information revealed, and called the "branching number" of the information structure at that date-event.

In particular, if futures trading is short-term (one period ahead) only, then a necessary condition for sufficient price variability is that at any date-event, the number of commodities traded on the futures market be at least as large as the branching number (Cor. 4.2, Cor. 4.3).

Some effort is made to clarify the structure of equilibrium prices, in particular the discount factors and probabilities implicit in them, and the dependence of these quantities on the normalization (i.e., the choice of numeraire).

The special cases of "short-term" and "long-term" futures trading are also discussed.

Our paper generalizes the work of TOWNSEND '78 to the multiperiod case, and is also closely related both to the literature on "spanning" conditions for incomplete markets (ARROW '64, GUESNERIE & JAFFRAY '74, KREPS '82) and the "arbitrage" approach to the pricing of financial securities (BLACK & SCHOLES '73, HARRISON & KREPS '79).

Our paper complements the existing literature in at least two ways, both by placing the analysis in a full-fledged general equilibrium model, and by concentrating on an actually existing--and increasingly popular--type of contract, viz. futures contracts.

2. Structure of the Model and Arrow-Debreu Equilibrium

We consider a standard pure-exchange economy described by the following data (cf. DEBREU '59, RADNER '72): there is a finite number of goods, indexed $h = 1,2,\ldots,\ell$; a finite set of agents, indexed $i \in I$; a finite set of dates, $t = 0,1,\ldots,T$; and a finite set of states of the world, $\omega \in \Omega$.

The information structure is described by a sequence of partitions of Ω, $F = \{F_0, F_1, \ldots, F_T\}$, such that F_{t+1} is finer than F_t $(0 \leq t \leq T-1)$, $F_0 = \{\Omega\}$, and $F_T = \{\{\omega\} | \omega \in \Omega\}$. F_t represents the agents' (common) information at date t, i.e., at date t it is known to which element A of F_t the true state belongs.

The set of all date-event pairs or "markets" is denoted by

$$M = \{m = (t,A) | t = 0,\ldots,T, \ A \in F_t\}.$$

The date of market $m = (t,A)$ is denoted by

$$\tau(m) = \tau(t,A) = t,$$

and the set of all markets at date t is

$$M_t = \{m \in M | \tau(m) = t\},$$

with typical element $m_t \in M_t$. There exists exactly one <u>initial market</u>, viz. $m_0 = (0,\Omega)$. The set of all <u>terminal markets</u>, M_T, can be identified with the set of states, Ω.

M is partially ordered by the relation

$$m \leq m' \iff t \leq t' \quad \text{and} \quad A \supseteq A'$$

(m' follows m). We write $m < m'$ iff

$$m \leq m' \quad \text{and} \quad m \neq m'.$$

For every market $m \in M$, the set of its (immediate) <u>successors</u> is defined by

$$S(m) = \{m' \in M | m' \geq m, \tau(m') = \tau(m) + 1\}.$$

Every market m except the initial one has a unique <u>predecessor</u>, denoted by $-m$, and characterized by the condition $m \in S(-m)$.

M can be viewed as the set of nodes in the <u>event tree</u> defined by the information structure F. The number of branches leaving a node m is

$$b(m) = \text{card } S(m),$$

and the <u>branching number</u> of the information structure F is defined by

$$b = \max_{m \in M} b(m).$$

A typical agent receives an endowment

$$w = (w(m))_{m \in M}, \quad \text{where} \quad w(m) \in \mathbf{R}^{\ell}_{+} ,$$

and has preferences $\succsim$ over his consumption set $X = R^{\ell \cdot \text{card}\, M}_{+}$. A consumption plan $x \in X$ is of the form

$$x = (x(m))_{m \in M}, \quad x(m) \in \mathbf{R}^{\ell}_{+} .$$

In order to be able to restrict our attention to strictly positive price systems, we assume that preferences are strictly increasing, i.e.,

$$x \underset{\neq}{>} x' \quad \Rightarrow \quad x \succ x'.$$

We shall consider several market structures or regimes in this paper. The first is the Arrow-Debreu complete market structure, in which contracts for delivery at any date, contingent on any event, can be made at the initial market. An AD-price system is of the form

$$p^{*} = (p^{*}(m))_{m \in M},$$

where $p^{*}(m) \in R^{\ell}_{++}$ is the price vector at date zero for delivery on market m.

An agent with endowment w can choose any consumption plan x in the budget set

$$B^{*}(p^{*}, w) = \{x \in X \mid p^{*}(x - w) = 0\}.$$

Let endowments and preferences $(w^{i}, \succsim^{i})$ be given for all agents $i \in I$. An allocation $\underset{\sim}{x} = (x^{i})_{i \in I}$ is an array of consumption plans $x^{i} \in X$, for all agents.

Definition 2.1. $[(x^i)_{i\in I}, p^*]$ is an Arrow-Debreu equilibrium (AD-equilibrium) if

(i) x^i is $\succsim^i$-maximal in $B^*(p^*,w^i)$ $\forall\, i \in I$

(ii) $\sum_{i\in I} (x^i - w^i) = 0.$

Under standard assumptions, such an equilibrium exists, and the resulting allocation is Pareto-efficient.

It will be convenient to normalize prices as follows. Choose a semipositive numeraire commodity bundle $c \in \mathbf{R}_+$, $c \neq 0$, and define normalized AD-prices by the condition

$$p^*(m_0) \cdot c = 1 \tag{2.1}$$

(i.e., c serves as numeraire on the initial spot market). Implicit in the normalized price system are discount factors and probabilities as follows. The one-period discount factors ρ^* are given by

$$\rho^*(m) = \frac{1}{p^*(m)\cdot c} \cdot \sum_{m'\in S(m)} p^*(m') \cdot c, \quad m \in M \setminus M_T, \tag{2.2}$$

and the cumulative discount factors are

$$\begin{aligned} \rho_0^*(m_0) &= 1 \\ \rho_0^*(m) &= \prod_{m' < m} \rho^*(m) \qquad m > m_0 \end{aligned} \tag{2.3}$$

We define the conditional probability of m' given m by

$$\pi^*(m'|m) = \frac{1}{\rho^*(m)} \cdot \frac{p^*(m')\cdot c}{p^*(m) \cdot c}, \qquad m' \in S(m) \tag{2.4}$$

and the unconditional probabilities recursively by

$$\begin{aligned} \pi^*(m_0) &= 1 \\ \pi^*(m) &= \pi^*(m \mid -m) \cdot \pi^*(-m) \qquad m > m_0. \end{aligned} \tag{2.5}$$

For future reference we note:

$$p^*(m)c = \pi^*(m) \cdot \rho_0^*(m) \quad \forall\, m \in M \tag{2.6}$$

i.e., the price of the numeraire, contingent on date-event $m = (t,A)$, is equal to the probability of that event, discounted to date zero.

To see that (2.6) is indeed true, observe that it is obviously true for $m = m_0$, and assume inductively that it is true for $m_t \in M_t$. Then we have for $m_{t+1} \in S(m_t)$:

$$\begin{aligned} p^*(m_{t+1}) \cdot c &= \pi^*(m_{t+1} | m_t) \cdot \rho^*(m_t) \cdot p^*(m_t) \cdot c \\ &= \pi^*(m_{t+1}) \cdot \rho_0^*(m_{t+1}), \end{aligned}$$

by (2.4), (2.3), (2.5), and the induction hypothesis.

Finally, we point out the well-known fact that in an AD-equilibrium all contracts are made at the initial date, i.e., only the market m_0 is "open." Of course this is not a genuine restriction since the possibility of state-contingent trading enables the agents in the AD-world optimally to take into account all possible future developments already at the initial date, so that at later dates nobody would want to "recontract." In other words, even if some or all markets $m > m_0$ were open, they would not be active. This should be contrasted with the incomplete market structure of the next section, in which "sequential" trading is essential.

3. Futures Equilibrium

Now we introduce an incomplete sequential market structure (in the sense of RADNER '72), distinguished by the following feature: state-contingent contracts à la Arrow-Debreu cannot be made, but futures trading for uncontingent delivery at later dates is possible. More precisely, on any market $m \in M$, contracts for delivery at the dates $u = \tau(m), \tau(m)+1, \ldots, T$ can be made (of course $u = \tau(m)$ corresponds simply to spot trading). A futures price system (F-price system) is of the form

$$p = (p(m))_{m \in M},$$

where $p(m) = [p(m,\tau(m)), p(m,\tau(m)+1), \ldots, p(m,T)]$, and $p(m,u) \in \mathbf{R}^{\ell}_{++}$ is the price vector on market m for delivery at date u, $\tau(m) \leq u \leq T$. The "futures component" of p(m) is denoted by $p^f(m) = [p(m,\tau(m)+1), \ldots, p(m,T)]$.

Similarly, a (net) trade plan for an agent is of the form

$$z = (z(m))_{m \in M},$$

where $z(m) = [z(m,\tau(m)), z(m,\tau(m)+1), \ldots, z(m,T)]$, and $z(m,u) \in \mathbf{R}^{\ell}$ is the trade on market m for delivery at date u, $\tau(m) \leq u \leq T$. We adopt the convention that positive components of z(m,u) correspond to purchases, and negative components to sales. $z^f(m)$ is defined in the same way as $p^f(m)$.

A trade plan z generates the consumption stream x from endowment w if

$$x(m) = w(m) + \sum_{m' \leq m} z(m',\tau(m)) \quad \forall\, m, \tag{3.1}$$

and z is feasible if it satisfies the budget constraints

$$p(m) \cdot z(m) = 0 \quad \forall\, m. \tag{3.2}$$

Note that we allow arbitrary long or short sales.

An agent with endowment w can then choose any consumption plan x in the budget set

$$B(p,w) = \{x \in X \mid x \text{ is generated from } w \text{ by some feasible } z\}.$$

Given endowments and preferences $(w^i, \succsim^i)$ for all agents, we define:

Definition 3.1. $[(x^i)_{i\in I}, p]$ is a Futures equilibrium (F-equilibrium) if

(i) x^i is $\succsim^i$-maximal in $B(p,w^i)$ $\forall\, i \in I$

(ii) $\sum_{i\in I} (x^i - w^i) = 0.$

Remark: Since $x^i \in B(p,w^i)$ by (i), x^i is generated by some feasible trade plan z^i, for $i \in I$. The market clearing condition (ii) then implies by (3.1):

$$0 = \sum_i \sum_{m' \leq m} z^i(m',\tau(m)) \quad \forall\, m \in M,$$

i.e., on each market m, the agents' trade plans are compatible.

Now let p be any F-price system (not necessarily equilibrium). We say that p admits arbitrage possibilities if there exists a feasible trading strategy z which generates a semipositive net consumption stream (a "free lunch"), i.e., which satisfies (cf. (3.1)):

$$\sum_{m' \leq m} z(m',\tau(m)) \begin{cases} \geq 0 & \forall\, m \in M \\ \\ \neq 0 & \text{some } m \in M \end{cases} \tag{3.3}$$

Clearly, an arbitrage possibility is incompatible with equilibrium: by adding a free lunch to his consumption, an agent could always move to a strictly preferred consumption stream (remember that preferences are strictly increasing), without violating his budget constraint, so that B(p,w) cannot contain a most preferred element.

We call an F-price system viable if it does not admit arbitrage possibilities, and shall henceforth restrict attention to viable price systems. They are characterized by the following theorem:

Theorem 3.1. For any F-price system p, the following two conditions are equivalent:

(i) p does not admit arbitrage possibilities.

(ii) For every non-terminal market $m \in M$, there exist numbers $\lambda(m,m') > 0$, $m' \in S(m)$, such that

$$p^f(m) = \sum_{m' \in S(m)} \lambda(m,m')p(m'). \tag{3.4}$$

Recall that $p^f(m)$ is the "futures" component of the price system at m. (3.4) says that, up to normalization, $p^f(m)$ is a convex combination of the possible next period's prices. If (3.4) is not satisfied for some market m, then there exists a predictable change of relative prices between m and its successors $m' \in S(m)$, giving rise to an arbitrage possibility. Indeed, using the separating hyperplane theorem, one can then find a net trade vector whose value never decreases, and increases in at least one instance, in the transition from m to any of its successors $m' \in S(m)$. For details see the proof of Th. 3.1.

Given a numeraire commodity bundle $c \in \mathbf{R}^{\ell}_{+}$, $c \neq 0$, we define normalized F-prices by the conditions:

$$p(m,\tau(m)) \cdot c = 1 \quad \forall\, m \in M \tag{3.5}$$

(i.e., c serves as numeraire on all spot markets).

One-period discount factors ρ are then defined by

$$\rho(m) = p(m,\tau(m)+1)\cdot c \qquad m \in M \setminus M_T, \tag{3.6}$$

and cumulative discount factors ρ_0 by

$$\begin{aligned} \rho_0(m_0) &= 1 \\ \rho_0(m) &= \prod_{m' < m} \rho(m) \qquad m > m_0 \end{aligned} \tag{3.7}$$

By (3.4), (3.5)

$$\rho(m) = \sum_{m' \in S(m)} \lambda(m,m')\underbrace{p(m',\tau(m'))c}_{=1} = \sum_{m' \in S(m)} \lambda(m,m') \tag{3.8}$$

and we can define the conditional probability of m', given m:

$$\pi(m'|m) = \frac{\lambda(m,m')}{\rho(m)} \tag{3.9}$$

and unconditional probabilities:

$$\begin{aligned} \pi(m_0) &= 1 \\ \pi(m) &= \pi(m|-m)\cdot\pi(-m) \qquad m > m_0 \end{aligned} \tag{3.10}$$

For a viable normalized F-price system p we have then the following martingale property:

<u>Lemma 3.1</u>. For any date u, $0 \leq u \leq T$, the discounted prices for delivery at date u form a martingale with respect to the probabilities π:

$$\begin{aligned} \rho_0(m_t)\cdot p(m_t,u) &= \sum_{\substack{m_s \in M_s \\ m_s \geq m_t}} \pi(m_s|m_t)\rho_0(m_s)p(m_s,u) \\ &= \underset{m_s \in M_s}{E}\,[\rho_0(m_s)p(m_s,u)\,|\,m_t] \end{aligned} \tag{3.11}$$

for $m_t \in M_t$, $0 \leq t \leq s \leq u \leq T$.

By Th. 3.1 and L.3.1, an F-price system p is viable iff it has the martingale property (3.11) w.r.t. some strictly positive probability measure π (cf. HARRISON & KREPS '79, Th. 2).

4. Equivalence of Futures Equilibria and Arrow-Debreu Equilibria

In this section we show that every AD-equilibrium is an F-equilibrium and conversely, provided there is "sufficient price variability."

As a preliminary step, we establish a correspondence between AD-price systems and viable F-price systems as follows. Given a normalized viable F-price system p, we define the associated AD-prices $p^* = AD[p]$ by

$$p^*(m) := \pi(m) \cdot \rho_0(m) \cdot p(m,\tau(m)). \tag{4.1}$$

Conversely, given a normalized AD-price system p^*, we define the associated F-prices $p = F[p^*]$ by

$$p(m,u) := \frac{1}{\pi^*(m) \cdot \rho_0^*(m)} \sum_{\substack{m_u \in M_u \\ m_u \geq m}} p^*(m_u), \quad u \geq \tau(m) \tag{4.2}$$

Lemma 4.1.

(i) Let p be a normalized viable F-price system. Then $p^* = AD[p]$ is normalized, $\rho^* = \rho$, $\pi^* = \pi$, and $F[p^*] = p$.

(ii) Let p^* be a normalized AD-price system. Then $p = F[p^*]$ is viable and normalized, $\rho = \rho^*$, $\pi = \pi^*$, and $AD[p] = p^*$.

The Lemma establishes a one-to-one correspondence between normalized viable F-price systems p and normalized AD-price systems p^*. We say that p and p^* are equivalent if $p = F[p^*]$ (or, equivalently, $p^* = AD[p]$). More generally, we say that two arbitrary price systems p, p^* are equivalent if their normalized versions are equivalent; and denote by $F[p^*]$ resp. $AD[p]$ the normalized equivalent Futures resp. AD-price system.

Equivalent price systems imply the same discount factors and probabilities.

The interpretation of the latter is obvious: if income (or wealth) is expressed in terms of the given numeraire c, then $\rho(m)$ is the price on market m of unconditional income tomorrow relative to income today; and the vector of conditional probabilities $\{\pi(m'|m)|m' \in S(m)\}$ gives the relative prices, on market m, of conditional income for the $b(m)$ possible markets tomorrow, $m' \in S(m)$.

It is important to understand that these discount factors and probabilities depend on the numeraire c. Of course they need not be equal to any agent's subjective probability assignments or rates of time preference.

Definition 4.1. An F-price system p has sufficient price variability if for every non-terminal market $m \in M$ the $b(m)$ vectors $\{p(m')|m' \in S(m)\}$ are linearly independent. An AD-price system p^* has sufficient price variability if the associated F-price system $p = F[p^*]$ has sufficient price variability.

Let $\underset{\sim}{x} = (x^i)_{i \in I}$ denote an allocation.

Theorem 4.1.

(i) If $[\underset{\sim}{x}, p^*]$ is an AD-equilibrium with sufficient price variability, then $[\underset{\sim}{x}, F[p^*]]$ is an F-equilibrium.

(ii) If $[\underset{\sim}{x},p]$ is an F-equilibrium with sufficient price variability, then $[\underset{\sim}{x},AD[p]]$ is an AD-equilibrium.

Corollary 4.1. If $[\underset{\sim}{x},p]$ is an F-equilibrium with sufficient price variability, then the allocation $\underset{\sim}{x}$ is Pareto-efficient.

If the definition of "sufficient price variability" is strengthened slightly, then short-term futures contracts, i.e., contracts extending at most one period into the future, are enough to achieve efficiency.

Definition 4.2. An F-price system p (resp. its associated AD-price system p^*) has sufficient spot price variability if for every non-terminal market $m \in M$ the $b(m)$ vectors $\{p(m',\tau(m')) \mid m' \in S(m)\}$ are linearly independent.

We say that an F-equilibrium involves only short-term futures trading if the equilibrium allocation can be generated by trading strategies z^i satisfying, for all $i \in I$:

$$z^i(m,u) = 0 \quad \text{whenever } u > \tau(m) + 1 \tag{4.3}$$

Corollary 4.2. Let p and p^* be equivalent and have sufficient spot price variability. Then $[\underset{\sim}{x},p^*]$ is an AD-equilibrium iff $[\underset{\sim}{x},p]$ is an F-equilibrium involving only short-term trading.

In particular, if p has sufficient spot price variability, then the number of traded commodities, ℓ, must be at least as large as the branching number b of the information structure.

Corollary 4.3. If p has sufficient spot price variability, then $\ell \geq b$.

5. Futures Contracts as "Long-Lived" Securities

A case frequently considered in the "spanning" literature is the case where consumption only takes place at the last date (or only at the initial and the last date), and a few "long-lived" securities are traded repeatedly before this final date (HARRISON & KREPS '79, DUFFIE & HUANG '83). The main results of this literature are twofold: first, from arbitrage considerations, it is deduced that security prices must have some martingale property; second, the condition for "completeness" of the market (or "spanning," i.e., equivalence to an AD-market) is that the number of independent financial instruments be at least equal to the branching number of the information structure.

In our context, a futures trade for delivery at the last date T can be viewed as such a long-lived security, whose value changes over time as more information becomes available.

With this interpretation, Th. 3.1 implies that, for viability, futures prices must be a martingale; and the condition of sufficient price variability in Th. 4.1 ensures that futures trading in the ℓ different commodities provides at least b(m) independent financial instruments at every market m.

For completeness' sake, we indicate briefly how the model of the preceding sections can be adapted to the special case where agents consume only at the last date. Noting that the set of terminal markets can be identified with the set of states, $M_T = \Omega$, write

$$p^*(\omega),\ x(\omega),\ w(\omega) \quad \text{for } m_T = \omega;$$

and set

$$p^*(m) = x(m) = w(m) = 0 \quad \text{for} \quad m \in M \setminus M_T \tag{5.1}$$

(consumption and endowments occur only on the last date). Moveover, in the F-market, put

$$p(m,u) = z(m,u) = 0 \quad \text{for} \quad u < T \tag{5.2}$$

(all trades are for delivery at the final date).

With these added restrictions, AD-equilibria and F-equilibria are defined as before. In the present context, given a numeraire $c \in R_+^\ell$, $c \neq 0$, it is natural to normalize all prices "to the last date," i.e.,

$$\sum_{\omega \in \Omega} p^*(\omega)c = 1 \tag{5.3}$$

and

$$p(m,T)c = 1 \quad \forall\ m \in M. \tag{5.4}$$

Then all discount factors are equal to one, and probabilities are given as follows:

$$\begin{aligned} \pi^*(\omega) &= p^*(\omega) \cdot c \\ \pi^*(m) &= \sum_{\omega \geq m} \pi^*(\omega). \end{aligned} \tag{5.5}$$

Moreover, viability of p implies by (3.4), (5.4):

$$\sum_{m' \in S(m)} \lambda(m,m') = 1, \qquad m \in M \setminus M_T$$

and we can define

$$\begin{aligned} &\pi(m'|m) = \lambda(m,m') \\ &\pi(m) \quad \text{recursively} \quad \text{as in } (3.10). \end{aligned} \tag{5.6}$$

The martingale property (3.11) implies

$$\pi(m)p(m,T) = \sum_{\omega \geq m} \pi(\omega)p(\omega,T). \tag{5.7}$$

Then we can define $p^* = AD[p]$ by

$$p^*(\omega) = \pi(\omega) \cdot p(\omega,T) \tag{5.8}$$

and $p = F[p^*]$ by

$$p(m,T) = \frac{1}{\pi^*(m)} \cdot \sum_{\omega \geq m} p^*(\omega). \tag{5.9}$$

Then Lemma 4.1 and Th. 4.1 remain true, where sufficient price variability of p means that for all $m \in M \setminus M_T$ the b(m) vectors $\{p(m',T) \mid m' \in S(m)\}$ are linearly independent. In particular, sufficient price variability implies $\ell \geq b$.

A. Proofs

We shall use the following version of the separating hyperplane theorem:

Lemma A.1. For any matrix A, exactly one of the following two alternatives holds: either $xA = 0$ has a strictly positive solution x, or $Ay \underset{\neq}{\geq} 0$ has a solution y.

Proof: See GALE '60, Cor. 1 on p. 49.

Proof of Th. 3.1

We prove first the implication (i) => (ii). Assume indirectly that (ii) is not satisfied, i.e., there exists a nonterminal market $\bar{m}$ s.t. $p^f(\bar{m})$ is not a strictly positive linear combination of the vectors $p(m')$, $m' \in S(\bar{m})$. This means that $xA = 0$ does not have a strictly positive solution, where A is the matrix whose first $b(\bar{m})$ rows are $p(m')$, $m' \in S(\bar{m})$, and whose last row is $-p^f(\bar{m})$. By Lemma A.1, the equation $Ay \underset{\neq}{\geq} 0$ has then a solution, i.e., there exists y s.t.

$$\left.\begin{array}{l} p(m')y \geq 0 \quad \text{for all } m' \in S(\bar{m}), \text{ and} \\ p^f(\bar{m})y \leq 0, \text{ with strict inequality in at least one instance} \end{array}\right\} \qquad (A.1)$$

Define a trading strategy z by $z^f(\bar{m}) = y$, $z(m') = -y$ for $m' \in S(\bar{m})$, and all other components zero. By definition,

$$\sum_{m' \leq m} z(m', \tau(m)) = 0 \quad \forall\, m \in M.$$

Moreover, by (A.1), $p(m)z(m) \leq 0$ for all $m \in M$, and $p(m)z(m) < 0$ for some $m \in S(\bar{m}) \cup \{\bar{m}\}$. But this means that we can increase at least one component of z by a positive amount without violating the budget constraints, thus generating a free lunch. This proves the implication (i) => (ii).

Next we prove the reverse implication (ii) => (i). W.l.o.g. we can normalize p so that $\lambda(m,m') = 1$, i.e.,

$$p^f(m) = \sum_{m' \in S(m)} p(m') \quad \forall\, m \in M \setminus M_T.$$

Through repeated application of this formula we obtain

$$p(m,u) = \sum_{\substack{m_u \in M_u \\ m_u \geq m}} p(m_u,u) \quad \text{for} \quad u \geq \tau(m) \tag{A.2}$$

Assume indirectly that there exists a "free lunch" z, satisfying (3.3). From the budget constraints (3.2), and rearranging terms, we have, by (A.2):

$$0 = \sum_{m \in M} p(m)z(m) = \sum_{m \in M} \sum_{u=\tau(m)}^{T} p(m,u)z(m,u)$$

$$= \sum_{u=0}^{T} \sum_{\substack{m \in M \\ \tau(m) \leq u}} \Big(\sum_{\substack{m_u \in M_u \\ m_u \geq m}} p(m_u,u)\Big) \cdot z(m,u)$$

$$= \sum_{u=0}^{T} \sum_{m_u \in M_u} p(m_u,u) \cdot \sum_{\substack{m \in M \\ m \leq m_u}} z(m,u)$$

By (3.3), each term in this sum is nonnegative, and at least one is strictly positive, a contradiction. This proves the implication (ii) => (i), and the Theorem. Q.E.D.

Proof of Lemma 3.1

We prove it by induction on u - t. For u - t = 0, there is nothing to prove. For u - t $\geq$ 1, we have, from (3.4), (3.9):

$$p(m_t,u) = \sum_{m_{t+1}\in S(m_t)} \rho(m_t)\pi(m_{t+1}|m_t)p(m_{t+1},u).$$

Multiplying this by $\rho_0(m_t)$, and applying the induction hypothesis to $p(m_{t+1},u)$ gives, by (3.7):

$$\rho_0(m_t)\cdot p(m_t,u) = \sum_{m_{t+1}\in S(m_t)} \pi(m_{t+1}|m_t)\cdot \sum_{m_s\in M_s} \pi(m_s|m_{t+1})\rho_0(m_s)p(m_s,u),$$

i.e., (3.11). Q.E.D.

Proof of Lemma 4.1

(i) Let p be normalized and viable, and $p^* = AD[p]$. By (4.1), (3.5)

$$p^*(m) \cdot c = \pi(m) \cdot \rho_0(m) \quad \forall\ m. \tag{A.3}$$

Hence, by (3.10), (3.7), $p^*(m_0)c = 1$, i.e., p^* is normalized. By (2.2), (A.3), (3.8), (3.9)

$$\rho^*(m) = \frac{1}{\pi(m)\rho_0(m)} \cdot \sum_{m'\in S(m)} \pi(m')\rho_0(m') = \rho(m).$$

By (2.4), (A.3)

$$\pi^*(m'|m) = \frac{1}{\rho(m)} \quad \frac{\pi(m')\rho_0(m')}{\pi(m)\rho_0(m)} = \pi(m'|m).$$

By (4.2), (4.1), $p' = F[p^*]$ is given by

$$p'(m,u) = \frac{1}{\pi(m)\rho_0(m)} \cdot \sum_{\substack{m_u \in M_u \\ m_u \geq m}} \pi(m_u)\rho_0(m_u)p(m_u,u) = p(m,u)$$

by Lemma 3.1. This proves (i).

(ii) Let p^* be normalized, and $p = F[p^*]$. By (4.2), (2.6)

$$p(m,\tau(m)) \cdot c = \frac{1}{\pi^*(m) \cdot \rho_0^*(m)} \cdot p^*(m)c = 1,$$

i.e., p is normalized. For $u \geq \tau(m) + 1$, (4.2) can be written

$$p(m,u) = \frac{1}{\pi^*(m)\rho_0^*(m)} \cdot \sum_{m' \in S(m)} \sum_{\substack{m_u \in M_u \\ m_u \geq m'}} p^*(m_u)$$

$$= \sum_{m' \in S(m)} \pi^*(m'|m)\rho^*(m) \cdot \frac{1}{\pi^*(m') \cdot \rho_0^*(m')} \cdot \sum_{m_u \geq m'} p^*(m_u),$$

i.e., again by (4.2),

$$p(m,u) = \sum_{m' \in S(m)} \lambda(m,m')p(m',u),$$

where

$$\lambda(m,m') = \pi^*(m'|m)\rho^*(m) > 0,$$

i.e., p is viable. This implies, by (3.6)

$$\rho(m) = \sum_{m' \in S(m)} \pi^*(m'|m)\rho^*(m)\underbrace{p(m',\tau(m'))c}_{=1} = \rho^*(m)$$

and by (3.9)

$$\pi(m'|m) = \frac{\lambda(m,m')}{\rho(m)} = \pi^*(m'|m).$$

Finally, by (4.1), (4.2), $p^{**} = AD[p]$ is given by

$$p^{**}(m) = \pi(m)\rho_0(m)p(m,\tau(m)) = \pi(m)\rho_0(m)\,\frac{1}{\pi^*(m)\rho_0^*(m)}\,p^*(m)$$
$$= p^*(m).$$

This proves (ii) and the Lemma. Q.E.D.

Proof of Th. 4.1

Let p and p^* be equivalent, and have sufficient price variability. We show that then the agents' opportunity sets $B(p,w)$ and $B^*(p^*,w)$ coincide, for arbitrary endowments w. Clearly this suffices to establish the theorem, since the agents will choose the same allocation in both market regimes.

First we show that $B(p,w) \subseteq B^*(p^*,w)$. Let $x \in B(p,w)$ be generated by the feasible trade plan z. By (3.1), (3.2), (4.2):

$$p^*(x - w) = \sum_{m\in M} p^*(m)[x(m) - w(m)] = \sum_{m\in M} p^*(m) \sum_{m' \leq m} z(m',\tau(m))$$

$$= \sum_{m'\in M}\ \sum_{m \geq m'} p^*(m)z(m',\tau(m))$$

$$= \sum_{m'\in M}\ \sum_{u=\tau(m')}^{T} \Big(\sum_{m_u \geq m'} p^*(m_u)\Big) \cdot z(m',u)$$

$$= \sum_{m'\in M} \pi^*(m')\rho_0^*(m')p(m')z(m') = 0,$$

i.e., $x \in B^*(p^*,w)$.

Next we show that $B^*(p^*,w) \subseteq B(p,w)$. Let $x \in B^*(p^*,w)$. We have to find a feasible trade plan z which generates x. To simplify the notation in this proof, we renormalize p w.l.o.g. so that

$$p(m,\tau(m)) = p^*(m) \quad \forall\, m \in M. \tag{A.4}$$

This implies

$$p^f(m) = \sum_{m' \in S(m)} p(m') \quad \forall\, m \in M \setminus M_T. \tag{A.5}$$

Now define

$$\begin{aligned} r(m) &= p^*(m)[x(m) - w(m)] \\ R(m) &= \sum_{m' \geq m} r(m') \end{aligned} \tag{A.6}$$

Then clearly

$$R(m) = r(m) + \sum_{m' \in S(m)} R(m') \tag{A.7}$$

and the AD-budget constraint $p^*(x - w) = 0$ takes the form

$$R(m_0) = 0 \tag{A.8}$$

$R(m)$ is the agent's total net wealth when he arrives on market m, and $r(m)$ is what he spends for consumption on market m (net).

Any trading strategy z can be represented equivalently in terms of the associated cumulative trading strategy y, defined by

$$y(m,u) = \sum_{m' \leq m} z(m,u) \qquad \tau(m) \leq u \leq T,\ m \in M. \tag{A.9}$$

$y(m,u)$ is the total amount of commitments for delivery at date u which the agent has accepted on or before market m. We use the same notation for y as for z, in particular $y(m) = [y(m,\tau(m)), y^f(m)]$, and note that

$$z(m) = y(m) - y^f(-m) \quad \forall\, m \in M \tag{A.10}$$

(where $y^f(-m_0) := 0$) is the unique non-cumulative strategy associated with a given cumulative trading plan y.

Now choose y such that it satisfies, for all $m \in M$:

$$y(m,\tau(m)) = x(m) - w(m) \tag{A.11.a}$$

$$p(m')y^f(m) = R(m') \quad \forall\, m' \in S(m) \tag{A.11.b}$$

and define z by (A.10). For every $m \in M$, (A.11.a) defines a unique cumulative spot trade, and (A.11.b) is a system of b(m) equations whose unknowns are the components of $y^f(m)$. By the assumption of sufficient price variability, the coefficient matrix of (A.11.b) has rank b(m), i.e., a solution $y^f(m)$ exists (not necessarily unique).

It remains to verify that the resulting non-cumulative trade plan z satisfies (3.1), (3.2), i.e., generates x and is feasible.

The first is trivial since by construction ((A.9), (A.11.a)):

$$\sum_{m' \leq m} z(m',\tau(m)) = y(m,\tau(m)) = x(m) - w(m),$$

i.e., (3.1). To see that z satisfies the budget constraints, note that by (A.11), (A.4), (A.6), (A.5), (A.7)

$$\begin{aligned} p(m)y(m) &= p^*(m)[x(m) - w(m)] + p^f(m)y^f(m) \\ &= r(m) + \sum_{m' \in S(m)} p(m')y^f(m) = r(m) + \sum_{m' \in S(m)} R(m') = R(m) \end{aligned} \tag{A.12}$$

Hence $p(m_0)z(m_0) = p(m_0)y(m_0) = R(m_0) = 0$ by (A.8). For $m > m_0$, we have, by (A.10), (A.12), (A.11.b)

$$p(m)z(m) = p(m)y(m) - p(m)y^f(-m) = R(m) - R(m) = 0,$$

i.e., z satisfies (3.2). This proves Th. 4.1. Q.E.D.

Remark: The trading strategy y constructed in the proof of Th. 4.1 can be understood as follows. Since by (A.4) the spot prices are the same in both market regimes, all we have to do is find a futures trade which gives the agent the income distribution $r(m)$, $m \in M$. By (A.7), this can be done in one-period steps, viz.: for each market m, given $R(m)$, find a trade which allocates $r(m)$ to current (net) consumption, and transfers $R(m')$ to m', for each of the $b(m)$ possible successor states $m' \in S(m)$. By the assumption of sufficient price variability, such trades exist, and the whole problem can be solved recursively, starting at m_0 with $R(m_0) = 0$.

Proof of Cor. 4.2

The system of equations (A.11.b) in the proof of Th. 4.1 can be written

$$p(m',\tau(m'))y(m,\tau(m)+1) + \sum_{u=\tau(m)+2}^{T} p(m',u)y(m,u) = R(m') \qquad \text{(A.13)}$$

$$\forall\ m' \in S(m).$$

If there is sufficient spot price variability, then (A.13) has a solution with $y(m,u) = 0$ for $u > \tau(m) + 1$, i.e., the cumulative strategy y is of the form

$$y(m) = [y(m,\tau(m)),y(m,\tau(m)+1),0,\dots,0] \quad \forall\ m.$$

The associated non-cumulative strategy z is, by (A.10):

$$z(m) = [y(m,\tau(m)) - y(-m,\tau(m)),y(m,\tau(m)+1),0,\dots,0],$$

i.e., z involves only short-term trading. Q.E.D.

Proof of Cor. 4.3

Let $\bar{m}$ be the market with the maximum branching number, i.e., $b(\bar{m}) = b$. The vectors $p(m',\tau(m'))$, $m' \in S(\bar{m})$, are b vectors of dimension ℓ; if they are linearly independent, we must have $\ell \geq b$. Q.E.D.

References

K. J. ARROW '64. The role of securities in the optimal allocation of risk-bearing. Rev. Econ. Stud., 31.

F. BLACK & M. SCHOLES '73. The pricing of options and corporate liabilities. J. Pol. Econ., 81.

G. DEBREU '59. Theory of Value.

D. DUFFIE & C-F. HUANG '83. Implementing Arrow-Debreu Equilibria by Continuous Trading of Few Long-lived Securities. Stanford Univ., manuscript.

D. GALE '60. The Theory of Linear Economic Models.

R. GUESNERIE & J-Y. JAFFRAY '74. Optimality of Equilibrium of Plans, Prices, and Price Expectations. In: J. DREZE (ed.), Allocation under Uncertainty.

J. M. HARRISON & D. KREPS '79. Martingales and arbitrage in multiperiod securities markets. J. Econ. Th., 20.

D. KREPS '82. Multiperiod Securities and the Efficient Allocation of Risk: A Comment on the Black-Scholes Option Pricing Model. In: J. McCALL (ed.), The Economics of Uncertainty and Information.

R. RADNER '72. Existence of equilibrium of plans, prices and price expectations in a sequence of markets. Econometrica, 40.

R. M. TOWNSEND '78. On the optimality of forward markets. Amer. Econ. Rev., 68.