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1. Introduction

The bargaining problem is one of the oldest in economic theory and yet remains unresolved. The first rigorous formulation was due to Edgeworth [2], whose solution contained a multiplicity of equilibria, as can easily be checked in a textbook exposition of the Edgeworth box. Economic theorists have taken two routes to overcome this problem. One, started by Nash [5], imposes some axiomatic restrictions on agents' behavior, on the assumption that the agents make demands in the utility space. If the demands of the various agents are feasible they get at least what they demanded, while if they are infeasible the agents get the utility level corresponding to the threat strategy. The most important result was that if we impose the conditions of Pareto-optimality, symmetry, independence or irrelevant alternatives and of linear transformations on the solution, then there is one and only one equilibrium which is the maximum of the function $\prod_{i=1}^{n} (x_i - d_i)$ over the feasible set, where n is the number of players, x_i is the utility of player i, and d_i its utility at the threat strategy level.[1]

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[1] For the definitions of all properties quoted in the text, the proof of the above statement, and further results in the axiomatic approach we refer to Roth [7].

Even if these theories allowed for some advances in the treatment of the bargaining problem, they have several shortcomings, the most crucial being that all the properties of the game cannot be justified only on the basis of the individual rationality assumption, nor on appealing intuitive reasons. Dissatisfaction with this approach was first implicitly demonstrated by Nash [6] himself, when he tried to justify his solution concept as being the limit of a series of equilibria of a corresponding series of what he called "smoothed games." Notwithstanding this serious problem there was no treatment of the bargaining problem which could give rise to desired properties (such as Pareto-optimality) and at the same time was rigorous in game theoretic terms, at least until Rubinstein's [8] work. His game will be described in detail later on, but the problem is to divide a unit cake between two players, who have to make proposals and counter-proposals until they reach an agreement. The two agents have linear utility functions and bear some "costs" in delaying the agreement. It turns out that this game has perfect Nash equilibrium and in some particular cases that equilibrium is unique.

There were various attempts to generalize this model, e.g., Binmore [1]. This paper will show that Rubinstein's game can be, in a sense, generalized and that a wide class of games has a unique equilibrium. The results are not completely new, but the proofs certainly are. The proof of the central theorem makes use of the Fudenberg-Levine [3] theory of infinite horizon games. This new proof is simpler than those available in the literature, and it is constructive. This means that we are able to provide an algorithm for computing perfect equilibrium strategies with any degree of approximation. This property is obviously very useful for any theory if it ever has to be tested. Furthermore, it will be easy to check whether the perfect equilibrium so constructed does have the properties assumed in Nash's axiomatic bargaining theory. The results on this matter are that the perfect equilibrium here found satisfies conditions which can be thought of as the most natural generalizations of some of Nash's properties. The need for such generalizations arises from the fact that our game is dynamic, while the axiomatic bargaining theory is inherently static.

The next two short sections will be devoted respectively to describing the theory of games with infinite horizon mainly due to Fudenberg-Levine [3], and to describing the game. In the fourth section we shall derive the main properties of the equilibrium, and in the fifth we will give some examples that are of particular interest in economic theory.

2. Perfect Equilibria in Infinite Horizon Games[2]

We must introduce some notation. For the sake of generality we give definitions for an n-person game, even if later on we shall deal only with a two-person game.

Let us assume that each player has a set of strategies open to him in each period, say X_t^i. Hence the space of all strategies for all players at time t is the Cartesian product: $\chi_t = \prod_{i=1}^{N} X_t^i$. We are now able to introduce the following definitions: A history of the game is

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[2] The reader already acquainted with Fudenberg-Levine [3] can skip this section without loss of continuity.

the sequence of moves of all agents up to a certain point of time. So a history has the form: $\underline{x}_T = ((x_1^1, x_1^2, \ldots, x_1^N), (x_2^1, x_2^2, \ldots, x_2^N), \ldots, (x_T^1, x_T^2, \ldots, x_T^N))$. In each point of time the history must be feasible; that is, $\underline{x}_T \in \chi_T$, where $\chi_T = \prod_{t=1}^{T} X_t$. Let $x_T^i = \prod_{i=1}^{T} X_t^i$ denote the set of feasible histories for player i.

At each point in time all agents must choose a strategy. This choice can be represented by a mapping defined as follows:

$$g_{T+1}^i : \chi_T^i \to X_{T+1}^i.$$

The set of all strategies at time T is called the strategy space of agent i and is denoted: $S^i(T)$.

Let g denote the strategies followed by all agents. Hence we can define the outcome function $\underline{x}_T(g)$ to be the outcome of the game when the initial history is $\underline{x}_T$ and afterwards each player plays g^i.

Finally, let g^{-i} denote the strategies followed by all agents other than agent i and (g'^i, g^{-i}) be the strategy where agent i chooses g'^i instead of g^i. We are now able to introduce the following very important definition:

Definition 2.1. $g^* \in S(T) = \prod_{i=1}^{N} S^i(T)$ is an ε-perfect equilibrium for the game truncated at T if for each $t \geq 0$, history $\underline{x}$, strategy $g \in S(T)$:

$$(1) \qquad u_i(\underline{x}_t(g^i, g^{*-i})) - u_i(x_t(g^*)) \leq \varepsilon;$$

i.e., the utility of the equilibrium outcomes of any player cannot be improved in any way, for any past history, by more than ε, given the strategies of other players. Of course if we set $\varepsilon = 0$ we have the definition of a perfect equilibrium.

Now we can state, without proof,[3] the following theorem:

Fudenberg-Levine Theorem (Part B). If u_i are uniformly continuous for some metric, a necessary and sufficient condition for g^* to be perfect in $S(\infty)$ is that there be sequences ε_n, $T(n)$, and g_n such that g_n is ε-perfect in $S(T(n))$ and as $n \to \infty$, $\varepsilon_n \to 0$, $T(n) \to \infty$, and $g_n \to g^*$.

3. Description of the Bargaining Model

There are two players, 1 and 2; each has a known utility function (not necessarily the same) which satisfies the following conditions:

(2i) $v_i(\underline{x}_1,\underline{x}_2,t) > v_i(\underline{x}_1,\underline{x}_2,\tau)$, for $t < \tau$, $\underline{x}_i \in X_i$, $i = 1,2$,

(2ii) $\lim\limits_{t\to\infty} v_i(\underline{x}_1,\underline{x}_2,t) = b_i, |b_i| < \infty, \forall\ \underline{x}_i$, $i = 1,2$,

(2iii) $v_i(\cdot)$ is bounded for all $\underline{x}_i \in X_i$ and t, $i = 1,2$,

(2iv) $v_i(\cdot)$ is continuous in t, $\forall\ \underline{x}_i \in X_i\ \ \forall\ t$, $i = 1,2$,

(2v) for any Δ we can find an η such that $v_i(\cdot,t+\Delta) \leq \eta v_i(\cdot,t)$, $0 < \eta < 1$,

where $\underline{x}_i$ is the level of a set of variables under the control of agent i, while X_i is a compact subset of a vector space. The above conditions are not particularly restrictive and do not need any particular explanation with the exception of (2v), which implies that can be found an upper bound to the rate of decrease of $v_i(\cdot)$. It should be noted that we are not assuming utility functions representing the same preference ordering over $X = X_1 \times X_2$ at different points in time. Hence the utility functions

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[3] The proof is contained in Fudenberg-Levine [3].

here considered are not of stationary type. Besides, we are not considering utility functions which are restricted to be linear. In this sense this model can be thought of as a generalization of Rubinstein's [8] model.

At $t = 0$ player 1 makes a demand in expected utility terms. Player 2 can accept it or not. If he accepts, the game is over; if he does not, he will make a demand in expected utility terms after a period of time, Δ. Now player 1 can accept it or not; if he does, the game is over; otherwise, after a period of time equal to Δ the game goes on as at $t = 0$. Unless otherwise stated Δ is set equal to one.

Two properties of this game should be underlined. First, the strategy space of this game equals the expected utility space of each period. Second, in this game we allow not only for pure and mixed strategies, but we consider also the possibility that the players agree on a joint distribution function over the strategy space X. We shall call these agreements joint mixed strategies.

4. Properties of the Equilibrium

Before stating the main theorem about the existence and uniqueness of the equilibrium for the game described above, we need to derive some lemmas. These will enable us to compare some of the properties of the equilibrium, if it exists, with those of the Nash-cooperative equilibrium. First of all a definition:

Definition 4.1. At any time t, the strong Pareto-optimal frontier $u_2 = \phi(u_1,t)$, $\underline{u}_1(t) \leq u_1 \leq \bar{u}_1(t)$ is the locus of points such that there is no probability measure F: $X_1 \times X_2 \rightarrow [0,1]$, F nondecreasing in its arguments, such that:

(i) $$u'_1 = \underset{F}{E}(v_1(t)) \geq u_1,$$

(ii) $$u'_2 = \underset{F}{E}(v_2(t)) \geq u_2 = \phi(u_1),$$

with $u'_i > u_i$ for some i, and for any $u_1 \in [\underline{u}_1(t), \bar{u}_1(t)]$.

We can now state the following lemma. The proof is very easy, therefore we shall skip it.

Lemma 4.1. $\phi(u_1,t)$ is an invertible function and if $\underline{u}_1(t) < \bar{u}_1(t)$, then $\phi(\cdot)$ is decreasing and concave in u_1.

We now can define the weak Pareto-optimal frontier in the usual way: For definition 4.1 and lemma 4.1 we can give this other definition.

Definition 4.2. The weak Pareto-optimal frontier at time t is the locus of points $u_2 = \phi_w(u_1,t)$, $\underline{u}_{1,w}(t) \leq u_1 \leq \bar{u}_{1,w}(t)$ such that there is no probability measure F: $X_1 \times X_2 \rightarrow [0,1]$ such that (i) and (ii) of definition 4.1 hold with strict inequality.

Remark. The space of all vectors $\underline{u}(t) = (u_1(t),u_2(t))$ which are not outside the weak Pareto-optimal frontier at time t is the set of all feasible strategies for the two players at time t. Since we proved that this space is compact and convex and the strategy set is in expected utility terms, we have also proved that the utility function is uniformly continuous.

It is very easy to prove the following results which we shall only state here. The weak and the strong Pareto-optimal frontiers are equal, otherwise the weak Pareto-optimal frontier presents some flat regions in the tails. In the latter case the weak Pareto-optimal frontier need not be a function, nor its inverse.

So far we have explored the relation between u_1 and $\phi(\cdot)$; now we would like to know also some of the properties of $\phi(\cdot)$ with respect to the other variable in its domain, namely, t. Since $v(\cdot)$ is decreasing in t so will the weak Pareto-optimal frontier; further, the continuity of $v(\cdot)$ with respect to t ensures also the continuity of the weak Pareto-optimal frontier with respect to the same variable. We can summarize what has been said as follows:

Lemma 4.2. If $t < \tau$, and $u_1 \leq \bar{u}_1(\tau)$ then: $\phi_w(u_1,t) > \phi_w(u_1,\tau)$. Further, we have that:

$$\lim_{\varepsilon\to 0} \phi_w(u_1,t+\varepsilon) = \phi_w(u_1,t).$$

We said in the introduction that one of the properties imposed on the solution concept in the axiomatic approach is that it be Pareto-optimal. In the dynamic set-up of this article the above property has to be translated into two different ones. The first is that the equilibrium should satisfy: $u_2^* = \phi(u_1^*,t)$, but for lemma 4.2 the agreement should also be reached at $t = 0$. The next two lemmas will explore these properties.

Lemma 4.3. If the players reach an agreement at time t, the outcome of the game must satisfy $u_2^* = \phi(u_1^*,t)$; hence the equilibrium is strong Pareto-optimal.

Proof. Assume the contrary and denote the equilibrium as $u_1 = x$, $u_2 = y$. Hence we can define

$$u_1' = \max[\phi_w^{-1}(y,t), \bar{u}_1(t)],$$

$$u_2' = \max[\phi_w(x,t), \phi(\underline{u}_1,t)].$$

By definition $u_1' \geq x$ and $u_2' \geq y$ and one of the two is satisfied with strict inequality. Hence one of the two players can improve his position even if the other does not change his own strategy. Thus (x,y) cannot be a Nash perfect equilibrium. Q.E.D.

Lemma 4.4. The agreement is reached at time 0.

Proof. Assume the contrary and that for instance the agreement is reached at time 1. For the preceding lemma the equilibrium allocation must satisfy:

$$u_2^* = \phi(u_1^*, 1).$$

By lemma 4.2 we certainly have that:

$$u_1^* < \phi_w^{-1}(u_2^*, 0).$$

Hence player 1 can improve his position without player 2 changing his strategies. Q.E.D.

There is another important property of the perfect equilibrium of this game which resembles one of the axioms of Nash bargaining theory; the following lemma will be devoted to show it.

Lemma 4.5. Independence of Linear Transformations of the Utility Functions: Let Γ denote our bargaining game and $\underline{u}^*$ the expected utility vector corresponding to the equilibrium outcome. Let Γ' be another game identical to the first one, but the utility functions defined by: $v_i' = a_i + c_i v_i$, $c_i > 0$. Then the expected utility vector corresponding to the Nash perfect equilibrium is given by: $\underline{u}_i'^* = a_i + c_i \underline{u}_i^*$.

Proof. Also the proof of this lemma is quite straightforward. From the definition of perfect equilibrium we have:

$$u_i(g^i, g^{*-i}) - u_i(g^*) \leq 0,$$

where $u_i(g^*) = u_i^*$. Hence we have:

$$u_i'(g^i, g^{*-i}) - u_i'(g^*) = a_i + c_i u_i(g^i, g^{*-i}) - a_i - c_i u_i(g^*) =$$

$$= c_i[u_i(g^i, g^{*-i}) - u_i(g^*)] \leq 0,$$

since $c_i > 0$. Q.E.D.

There is another important similarity between the equilibrium of the game presented here and the Nash solution concept for the bargaining problem, namely, the independence of irrelevant alternatives. This is a direct consequence of the definition of perfect equilibrium and of the fact that the preference ordering on a set is the same if the set is considered in itself or as a subset of a larger set. We can state without proof the following:

Lemma 4.6. Independence of Irrelevant Alternatives: Let Γ denote our game and g^* its equilibrium. Let Γ' be another game identical to Γ

except for the fact that $S'(T) \subsetneq S(T)$. If $g^* \in S'(\infty)$ then g^* is the equilibrium also for Γ'.

Remark. It should be noted that the definition of relevant alternatives implicit in Lemma 4.6 is different from that of the axiomatic approach. In fact, it is not sufficient that the point $u_1^* = \phi(u^*,0)$ is not excluded from $S'(\infty)$, but also all the points in the sequence of proposals and counterproposals conjectured by the players in equilibrium must belong to $S'(\infty)$. This is a consequence of the fact that in our dynamic setup $S(T) \subsetneq S(\infty)$ for all T.

So far we have described some of the properties that the equilibrium should have if it existed. We did not yet prove the most important of the properties of the equilibrium points, i.e., their existence and uniqueness. Before doing that, we need to introduce some further observations. First of all we wish to emphasize that the game can be treated as if its feasible strategy set were only the strong Pareto-optimal frontier. This is a consequence of lemma 4.3. In that case it seems natural to treat separately the games where $\underline{u}_1(t) = b_1$ and $\phi(\bar{u}_1(t)) = b_2$ for all t. Finally, for lemma 4.5, we can set $b_i = 0$ without loss of generality. This convention will be held for all the remaining parts of the paper. For notational convenience we shall adopt also the following convention:

(3i) $\qquad \phi_1(u_2) = \phi^{-1}(u_2)$,

(3ii) $\qquad \phi_2(u_1) = \phi(u_1)$.

In Figure 1 we represent the equilibria of the game truncated at period 4 and at period 3. In the vertical axis we have also written the bargaining period, while in the horizontal axis we show which player has to move. The equilibrium of the game truncated at T = 4 is the lower, while that of the game truncated at T = 3 is the upper line delimiting the shaded area. Their meaning is the following: No player will accept in this period a lower utility level than that he can assure himself in the next; the player who has to make a demand will choose the highest utility level among those which satisfy the preceding constraint.

It is also possible to prove that these equilibria are unique. Take for instance T = 4 and assume that the past history did not bring to any agreement at t = 3. We can see in Figure 2 that the only perfect equilibrium outcome is (b,0), since points like c are infeasible and player 2 has no positive incentive to refuse allocations between 0 and b. Since the outcome in case of disagreement at t = 3 is (b,0), in case of disagreement at t = 2, the equilibrium outcome can be only (b,f). It cannot be (a,d) because if 1 refuses he can get b > a in the next period. It cannot be (e,c), because suppose that the initial history led to no agreement until 2 and that at 3 player 2 demands a utility level between f and e; then the strategy of one to refuse would not be optimal. This argument can be extended to the equilibrium strategy at the preceding points in time. Of course we can generalize this procedure to any other T and we have that the perfect equilibrium outcome for player one is given by

$$(4) \qquad u_1 = \phi_2(\phi_2(\ldots(\phi_1(\phi_2(u_1,T-1),T-2),\ldots),1),0) = F(u_1,T)$$

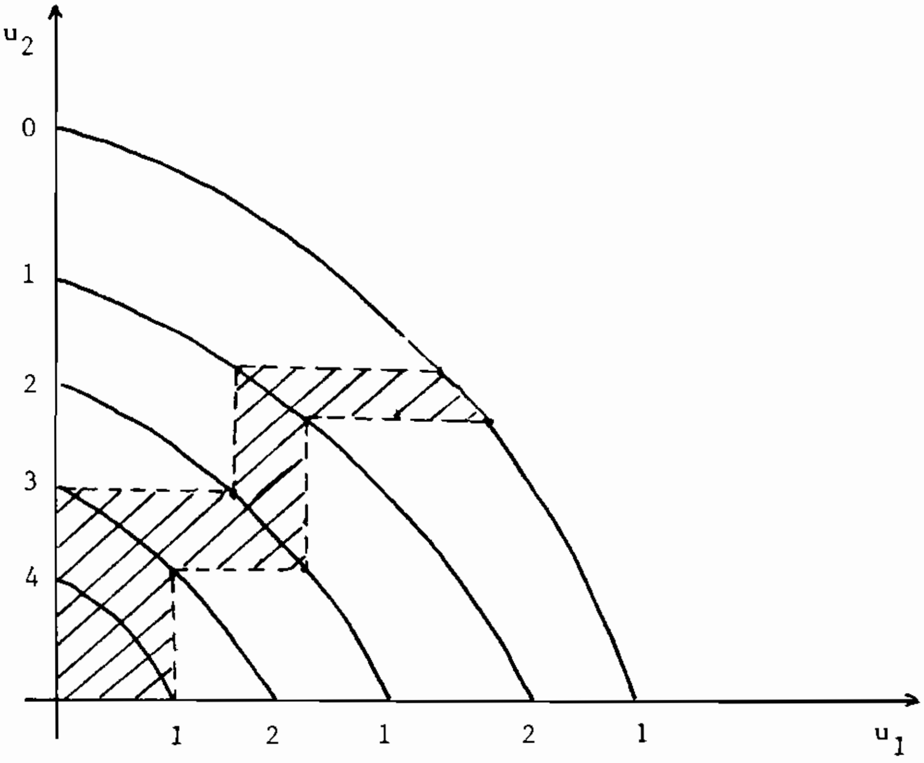

Figure 1

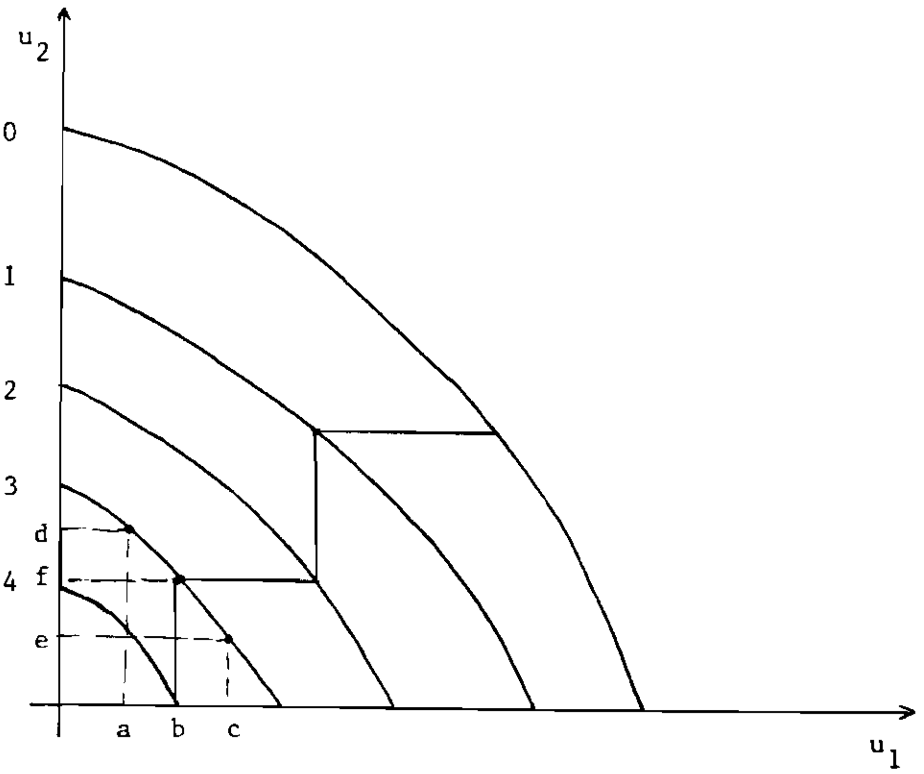

Figure 2

where:

$$(5) \qquad u_1 = 0 \qquad \text{if the game is truncated at } T - 1$$

$$(6) \qquad u_1 = \bar{u}_1(T) \quad \text{if the game is truncated at } T,$$

where $T = 2n$, $n = 1,2,3,\ldots$.

We can summarize the preceding arguments in the following lemma:

Lemma 4.7. Any truncated game has a unique perfect equilibrium whose outcome for player 1 is given by (4) - (6).

With similar arguments to those used for proving lemma 4.7 it is also very easy to prove that the perfect equilibrium strategies in $S(\infty)$ cannot be outside the shaded area in Figure 1, and analogously for any other T. If this is true then any sequence ε-perfect equilibrium has to converge in the region inside two perfect equilibria of the game truncated at two subsequent periods. Then if we prove that the perfect equilibria for the games truncated at T and T - 1 converge to each other as $T \to \infty$ we have proved both existence and uniqueness, since there are two series of perfect equilibria (and hence also ε-perfect equilibria) which are converging to the same equilibrium and no sequence of ε-perfect equilibria can converge to some other strategy. This is a consequence of the fact that the conditions of the Fudenberg-Levine theorem are both necessary and sufficient.

Lemma 4.8. The game here described has one, and only one, equilibrium if the following is true:

$$(7) \qquad \underline{u}_1(t) = \phi(\bar{u}_1(t),t) = 0 \quad \forall\, t$$

Proof. First of all we should prove that $\phi(*)$ is uniformly continuous in u_1. To show that we shall refer to Figure 3. The functional forms of the three lines $\overline{ab}$, $\overline{ac}$, and $\overline{ad}$ are, respectively:

$$u_2 = a - c_1 u_1 \tag{8}$$

$$u_2 = a - c_2 u_1 \tag{9}$$

$$u_2 = a - c_3 u_1 \tag{10}$$

where

$$a > 0 \qquad c_3 \geq c_2 \geq c_1 > 0 \tag{11}$$

Hence we derive

$$\phi(u_2') - \phi(u_1'') = a - c_1 u_1' - a + c_2 u_1'' \leq c_1(u_1'' - u_2') \leq c_3(u_1'' - u_1') \tag{12}$$

If we set:

$$c_t = (\bar{u}_1(t)/\phi(0,t))^i, \tag{13}$$

where $i = 1 - 4(t/2 - \text{INT}(T/2))$, we can also easily prove that:

$$F(\bar{u}_1(T),T) - F(0,T) \leq \prod_{t=0}^{T-1} c_t \phi_1(0,T) \tag{14}$$

Let us take $T = 1$. We can derive:

$$c_0\phi(0,1) = c_0 c_1 \bar{u}_1(1) \leq c_0 c_1 \bar{u}_1(2) \tag{15}$$

From (2v) we have:

$$c_0 c_1 \bar{u}_1(2) \leq \eta c_0 \phi(0,1). \tag{16}$$

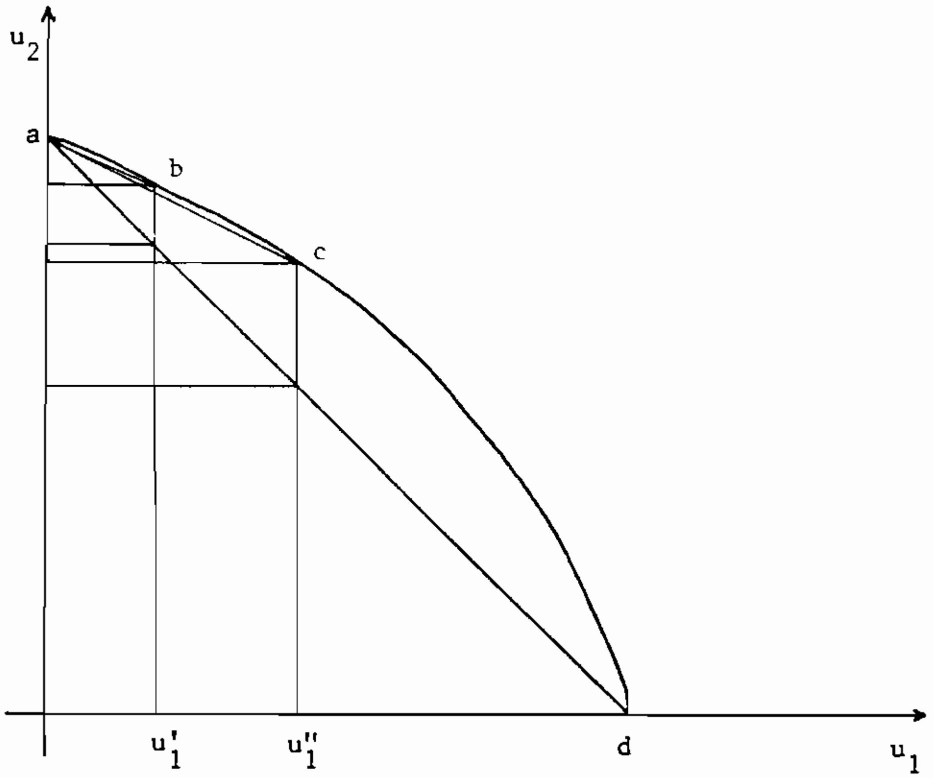

Figure 3

(16) holds since $\bar{u}_1(i)$ is the solution of the programming problem:

$$(17) \qquad \max_F E_F(v_1(\cdot,t))$$

That maximum is certainly reached by some pure strategy, not necessarily unique. If the sets of pure strategies which solve (17) are the same in period (1) and (2), then (16) follows directly from (2v). If the two sets are different this means that for one point $(\underline{x}_1,\underline{x}_2)$ which maximizes u_1 at $t = 1$ the following should be true:

$$(18) \qquad \bar{u}_1(2) > v_1(\underline{x}_1,\underline{x}_2,2) \geq \eta\bar{u}_1(1).$$

Besides (16) and (17) can be generalized to any T, then we can write by induction:

$$(19) \qquad \prod_{t=0}^{T-1} c_t \cdot \phi_1(0,T) \leq \eta^{T-1} c_0 \bar{u}_1(1)$$

This last inequality implies that the limit for $T \to \infty$ of the r.h.s. of (14) equals 0. Since the argument above can be extended to the perfect equilibrium strategy at any other point in time we proved the lemma. Q.E.D.

Remark: It is very easy to check that $F(*)$ is an increasing function in u_1, then:

$$(20) \qquad F(0,T) < F(\phi_1(0,T)T) \text{ for all } T.$$

It is also very easy to prove that:

$$(21) \qquad F(0,T) < F(0,T+2) \qquad \text{for all } T$$

$$(22) \qquad F(\phi_1(0,T),T) > F(\phi_1(0,T+2),T+2) \quad \text{for all } T$$

(20) - (22) imply that the player who plays last has the opportunity to bend the outcome of the game in his or her favor, but this opportunity is decreasing with T, the time at which the game is truncated.

At this point we are ready to prove our:

Theorem 4.1. The game here described has one, and only one, equilibrium.

Proof. If (7) holds the statement has already been proved. We now wish to generalize that result. Before treating the general case, it is useful to deal with two other particular cases.

a. $\underline{u}_1(t) < 0, \ \phi(\bar{u}(t),t) < 0, \ \forall \ t$

In any point outside the positive orthant one player is certainly worse off than in the point of no agreement. Then it is sure that the perfect equilibrium strategy will never lie in any such point, hence the areas outside the positive orthant are irrelevant. Now the proof runs as in the preceding lemma.

b. $\underline{u}_1(t) > b_1 = 0, \ \phi(\bar{u}_1(t)) > b_2 = 0, \ \forall \ t$

As usual we supply a figure which can help the understanding of the mathematical treatment. This class of games can in fact be represented as in Figure 4. The easiest way to deal with these games is to think of them as games of the kind discussed in the preceding lemma, where, for some reason, some strategies are not available to the players. The interpretation of the game does not influence its equilibrium. Anyway, if we interpret the game in the above sense we can always extend the strong Pareto-optimal frontier until they reach the two axes, as illustrated by the dotted parts of the curves in Figure 4. From now on

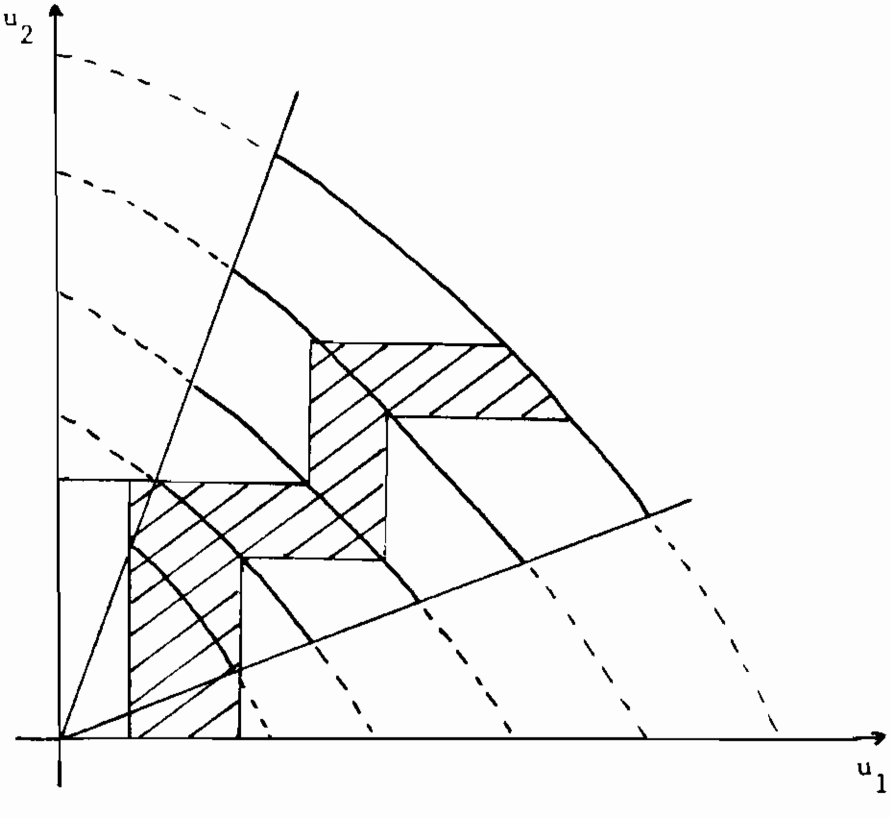

Figure 4

we have to change a bit our notation conventions for what we have just said. $\phi(\cdot)$ will now denote the strong Pareto-optimal frontier of the enlarged game. Of course, if for a certain t, there is no need to enlarge the strong Pareto-optimal frontier, that of the enlarged game will be identical to it. On the contrary $\underline{u}_1(t)$ and $\bar{u}_1(t)$ will denote the minimal and maximal values of u_1 at time t of the true game (the restricted one). Accordingly $\phi(\bar{u}(t),t)$ and $\phi(\underline{u}(t),t)$ will be those of u_2, and will be denoted respectively $\underline{u}_2(t)$ and $\bar{u}_2(t)$. Now we can write the analog of (4) - (6) in the following way:

(23) $$G(u_1,T) = \psi_1(\psi_2(\ldots(\psi_1(\psi_2(u_1,T-1),T-2),\ldots),1),0),$$

where:

(24) $$\psi_i(u_j,t) = \min[\phi_i(z,t),u_i(t)],$$

(25) $$u_1 = \underline{u}_1(T) \text{ if the game is truncated at } T-1,$$

(26) $$u_1 = \bar{u}_1(T) \text{ if the game is truncated at } T,$$

where as usual $T = 2n$, $n = 1,2,3,\ldots$. Since the following certainly holds:

(27) $$G(\underline{u}_1(T),T) - G(\bar{u}_1(T),T) \leq F(0,T) - F(\bar{u}_1(T),T),$$

the proof can follow the same steps as in the preceding lemma.

c. General Case

The argument used in part a. of this proof can be easily generalized to any situation in which the line joining $\underline{u}_i(t)$ and $\underline{u}_i(t-1)$ is negatively sloped, while the argument used in part b. can be easily generalized to the case where that line is positively sloped. Q.E.D.

Remark. It should be noted that the system (23) - (26) can be extended in order to provide an algorithm for computing the Nash-perfect equilibrium strategies for the two agents, with any desired degree of approximation. In fact, for any given ε, our system can actually compute the minimal T and the associated perfect equilibrium such that $\varepsilon(T) < \varepsilon$, where $\varepsilon(T)$ is the greatest distance between the perfect equilibrium strategies at time t for the game truncated at T and those at the same time for the game truncated at T - 1.

5. Some Examples

So far we have proved that the perfect equilibrium shares many properties with the Nash bargaining solution concept. Namely, it exists, it is unique, it is (strongly) Pareto-optimal, it is independent of linear transformations and of irrelevant alternatives. We still don't know, though, whether the game is also symmetric. In this section we shall deal only with games where the utility is discounted. Hence we can define the property of symmetry as follows: If $\phi_1 = \phi_2$ and the players have the same discount rate then the perfect equilibrium outcome assigns the same utility to both players.

We shall prove with the following example that this property need not hold. Further, the example suggests that the player who starts the bargaining has an advantage.

a. Rubinstein Bargaining Model with Discounting

As we have already said, Rubinstein's model can be thought of as a simplified version of the present model. The bargaining model is the same as in section 2, with the exception of the assumptions on the utility functions, which are as follows:

$$u_1 = x\delta_1^t, \ \forall \ t, \tag{23}$$

$$u_2 = y\delta_2^t, \ \forall \ t, \tag{24}$$

$$x = 1 - y, \ 0 \leq y \leq 1, \ \forall \ t \tag{25}$$

We now apply the algorithm (4) - (6) to this particular game. At time T and for x = 1 we have that the utility level for player 1 is:

$$u_1^T = \delta_1^T. \tag{26}$$

If player 2 wants to guarantee to player 1 at time T - 1 the same level of utility he must give him:

$$x^{T-1} = \delta_1. \tag{27}$$

Hence the corresponding level of utility for player 2 will be:

$$u_2^{T-1} = (1 - \delta_1)\delta_2^{T-1}. \tag{28}$$

Solving backward the system we find the following:

$$u_i^{T-k} = (1 - (1 - (1 - \ldots - (1 - \delta_1)\delta_2)\delta_1 \ \ldots)\delta_i^{T-k}, \ k = 1,\ldots,T, \tag{29}$$

where

$$i = (\frac{k+1}{2} - \mathrm{INT}\,(\frac{k+1}{2}))\cdot 2 .$$

For $k = T$, after some easy algebraic manipulations and remembering that $T = 2n$, we derive the equation:

$$(30) \qquad x^0 = u_1^0 = (1 - \delta_2) \cdot \sum_{i=0}^{n-1} (\delta_1\delta_2)^i + (\delta_1\delta_2)^n.$$

If we take the limit for n approaching infinity we obtain the unique perfect equilibrium outcome:[4]

$$(31) \qquad x = \frac{1 - \delta_2}{1 - \delta_1\delta_2} \qquad y = 1 - x = \frac{\delta_2(1 - \delta_1)}{1 - \delta_1\delta_2} .$$

It is quite obvious that if player 2 had started the bargaining, the equilibrium outcome would have been:

$$(32) \qquad x = \frac{\delta_1(1 - \delta_1)}{1 - \delta_1\delta_2} < \frac{1 - \delta_2}{1 - \delta_1\delta_2} ,$$

this inequality holding even if $\delta_1 = \delta_2$. Hence we have that the player starting the bargaining has some sort of advantage over the opponent, which implies that the game is not symmetric. It is worthwhile to note that if we solve the following system:

$$(33) \qquad y = \delta_1 x,$$

$$(34) \qquad x = 1 - \delta_2 + \delta_2 y,$$

for x we obtain (31), while if we solve it for y, we obtain (32). Why this should be the case will be explained in the next example.

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[4] Actually the result holds only if $0 < \delta_i < 1$, $i = 1,2$. For details see Rubinstein [8].

b. Generalized Rubinstein Model with Discounting

The simplest way to generalize the preceding model is to allow for non-linearity in the utility functions. Hence the model becomes:

(35) $u_1 = x\delta_1^t, \forall t,$

(36) $u_2 = y\delta_2^t, \forall t$

(37) $y = \phi(x), \forall t.$

For this kind of game it is possible to prove the following lemma:

Lemma 5.1. If the utility functions satisfy (35) - (37) and if x^*, $\phi(x^*)$ is a perfect equilibrium outcome for the game, assuming that the players start bargaining at time t, it will also be a perfect equilibrium outcome if the players start bargaining at time t + 2.

Proof. The lemma is a straightforward consequence of lemma 4.5, in fact it is sufficient to multiply the utility functions of the game starting at t + 2 respectively for δ_1^2 and δ_2^2 and we will obtain the game starting at t. Q.E.D.

Further, we can observe that if at time 0 the player 1 can choose his demand in expected utility level and at time 1 the perfect equilibrium guarantees him y^1, then the perfect equilibrium outcome for player 2 at time 0 will be:

(38) $\phi(x^0) = \phi(y^1)\delta_2.$

Analogously, if at time 2 the perfect equilibrium guarantees player 1 x^2, the perfect equilibrium at time 1 will guarantee him y^1:

$$y^1 = x^2\delta_1. \tag{39}$$

For lemma 5.1 we have that $x^1 = x^2 = x$, hence:

$$\phi(x) = \phi(x\delta_1)\delta_2 \tag{40}$$

defines the perfect equilibrium outcome when player 1 starts the bargaining.

It is also possible to carry out a similar proof for games where player 2 starts the bargaining and we find that the two perfect equilibrium outcomes are respectively the solution for x and the solution for y of the system:

$$y = x\delta_1, \tag{41}$$

$$\phi(x) = \phi(y)\delta_2, \tag{42}$$

which is clearly the generalized form of system (33) - (34).

So far we have proved that in general the game does not have the property of symmetry. It would be interesting to find a general class of games which have this property. Intuition suggests a first candidate in precisely those games whose bargaining interval, i.e., the period of time between one stage of the bargaining process and the other, tends to 0.

c. Generalized Rubinstein Model with Discounting and Indefinitely Small Bargaining Interval

Let Δ denote the time interval between one stage of the bargaining and the following. We can transform our original variable indicating time according to the following:

(43) $$t = \Delta s.$$

The model is that presented in (35) - (37) where we set, for mathematical convenience, $\delta_1 = e^{-\alpha\Delta}$ and $\delta_2 = e^{-\beta\Delta}$. Further, we assume that $\phi(\cdot)$ is differentiable in a neighborhood of the equilibrium. Now we can write (40) in the following way:

(44) $$\phi(x) = \phi(xe^{-\alpha\Delta})e^{-\beta\Delta}.$$

Then we can differentiate (46) with respect to Δ and obtain

(45) $$\phi'(x)x' = -\beta\phi(xe^{-\alpha\Delta})e^{-\beta\Delta} + \phi'(xe^{-\alpha\Delta})(x'e^{-\alpha\Delta} - \alpha xe^{-\alpha\Delta})e^{-\beta\Delta}.$$

Taking the limit for $\Delta \to 0$, after some simplifications, we obtain:

(46) $$\frac{\beta y}{\alpha x} = -\phi'(x)$$

(47) $$y = \phi(x)$$

In this case the system (46) - (47) fully characterizes our perfect equilibrium outcome, and obviously satisfies our definition of symmetry.

Remark. In the previous section we have seen that this game for T finite assigns some "power" to the player who plays last. This property is very well known in zero-sum Stackelberg games and it is maintained in the present which can be thought of as a repeated Stackelberg game. We have seen that the "asymmetry" of the last move disappears as T → ∞. In games with discounting we have seen that there is also another form of asymmetry, the first move asymmetry. The first player has an advantage since he has to discount less. This asymmetry disappears as $\Delta \to 0$, but it is important to notice that if the two players make their demands and decide whether to accept the other's demand simultaneously, the game has infinite equilibria. Hence some degree of asymmetry is needed to obtain the results of the preceding section, even if we can make this asymmetry as little as we please.

Lastly, it can easily be proved that system (46) - (47) is the solution of the following programming problem:

$$\text{(48)} \qquad \max x^{1/\alpha} y^{1/\beta} \quad \text{s.t. } y = \phi(x)$$

It should be clear that the solution of the programming problem (48) is not symmetric in the static sense, since the utilities of the two players are weighted by the inverse of the respective instantaneous discount rate. The system (48) is known, in the axiomatic approach, as the asymmetric solution[5] and the two weights are usually interpreted as an index of the power, or of the bargaining capacility, of the two agents. Here the bargaining capability of one agent is given by his own capability of waiting, synthesized by the inverse of the instantaneous discount rate. The rationale for this result is the following: The more valuable is the future for one agent, the less effective is the threat of the other of not reaching an agreement in the present.

6. Conclusions

In this paper we analyzed a sort of generalized version of the Rubinstein bargaining model and we succeeded in proving the existence and uniqueness of the equilibrium. The proof made use of the Fudenberg

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[5] See Roth [7], pp. 15-19.

and Tirole treatment of the perfect equilibria in infinite horizon games seen as the limit of a series of ε-perfect equilibria. The proof provides us also with a technique for calculating the equilibrium strategies with any desired degree of approximation.

The main aim of this paper was to compare the properties of Rubinstein's strategic approach to the axiomatic one. The results are that the equilibrium of our games shares many of the properties of the solution concepts proposed in the axiomatic approach. Beside being unique it is also (strong) Pareto-optimal, independent of linear transformations and of irrelevant alternatives. Some of these properties had to be restated in order to cope with the new dynamic setup. In the definition of Pareto-optimality we had to add explicitly the property that the agreement should be reached in the first period.

With a few examples we showed that our game is not symmetric; in general the player who starts the bargaining has an advantage. This was done through the computation of the equilibrium in the simple case of discounting, both in Rubinstein's model and in a slightly generalized version. We have proved also that in case the interval between one step of the game and the following goes to 0 the perfect equilibrium of those games converges to a symmetric solution. Finally, in the case of discounting and of interval of bargaining infinitely small, we could derive endogenously a sort of welfare function.

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