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A B S T R A C T

The main effort in the recent literature on equilibrium business cycle theory has been directed towards building models capable of mimicking given sets of time series data. Little attention has been focused on understanding the inner dynamics of these models, despite the fact that they introduce some very special assumptions about the features of preferences and technology over time. This paper establishes global conditions and proofs that in general introducing a "time-to-build" technology for production of capital goods will engender a cyclical accumulation path in a one-sector growth model.

Time Structure of Technology and the Real Business Cycle

At the forefront of the recent literature on business cycle theory has been the development of equilibrium real business cycle models. Their main effort has been directed towards mimicking given sets of time series data. On the other hand, little attention has been focused on understanding the inner dynamics of these models, despite the fact that they introduce some very special assumptions about the features of preferences and technology over time.[1] One would like to assess what role each of these assumptions plays in determining the dynamic properties of the model. However, this task is overwhelming because assumptions on preferences and technology are inextricably tied. Nevertheless, a viable avenue which may be followed is to single out each assumption and measure its impact on the motion of an otherwise standard model. This is the approach taken here.

Moreover, since little is known on the features of preferences across time, whereas empirical evidence may be collected on the time structure of technology, we limit the analysis to the latter. If the impact of the special technology on the dynamics of the model is proved to be significant, unequivocal data supporting it should also be supplied.

This paper establishes global conditions under which introducing a "time-to-build" technology for production of capital goods will have a substantial impact on the dynamic properties of a one-sector growth model. Unless the economy is already at a steady state both consumption and the capital stock will in general exhibit a cyclical behavior.

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[1]We refer here mainly to Kydland and Prescott (1982), and to Long and Plosser (1983).

1. Statement of the Problem and General Conditions

1.1 Introduction

We consider a neoclassical one-sector growth model with a representative agent maximizing utility over an infinite horizon.
The agent derives his utility from consumption of the only good in the economy and also from leisure. He has perfect foresight. The single commodity is produced by a concave production function $f(k_{t-2},k_{t-1},\ell_t)$ which is twice differentiable and increasing in its arguments; where ℓ_t is labor supplied at t, k_{t-2} and k_{t-1} are the capital goods operating at t whose production was originally initiated at t-2 and t-1 respectively.
At each point in time the individual has to decide how much to consume out of current output. Saving is identically equal to investment. Unlike the standard one-sector growth model, we assume that a multiperiod technology is used in producing capital goods. s time periods are required to transform the original inputs into new finished capital. Without loss of generality, we set the time-to-build s=2, in order to preserve analytical manageability. Investment projects initiated at t deliver finished capital only at t+2.

The last paragraph can be condensed into the following equation

$$(1.1) \qquad k_t = f(k_{t-2},k_{t-1},\ell_t) - c_t + (1-g)k_{t-1}$$

where $k_t-(1-g)k_{t-1}$ is gross investment todayand g is the rate of depreciation. The same vintage of capital enters production in heterogeneous forms for two periods in a row. First it enters as young (unfinished) capital one time period after its own production started. At this stage it carries out production jointly with the old capital. It contributes to production to a lesser extent than the old capital, but its contribution is nevertheless non-negligible. Next period -i.e. two periods after they started building it, it will be a fully productive (old) capital, and only then will it pay a depreciation at a rate g.[2] The time subscripts attached to each capital stock refer to the time period when its production is undertaken.

The representative agent faces the following intertemporal optimization problem

$$(1.2) \quad \underset{\{k_t, \ell_t\}}{\text{Max}} \quad \sum_{t=1}^{\infty} \left\{ U(c_t) + R(\ell^* - \ell_t) \right\} \beta^{t-1}$$

subject to: (1.1);

$$k_t \geqq (1-g)k_{t-1};$$

$$0 \leqq \ell_t \leqq \ell^*;$$

$$k_0,\ k_1 \text{ given.}$$

R(.) is strictly concave, whereas we take U(.) to be a linear function for the sake of clarity. ℓ^* is the maximum amount of per period work. The discount factor β is constrained in the range $0 < \beta < 1$.

The above may be stated as a dynamic programming problem

$$(1.3) \quad W(k_1,k_2) = \underset{\{k_3,\ell_3\}}{\text{Max}} \left\{ [f(k_1,k_2,\ell_3)+(1-g)k_2-k_3 + R(\ell^*-\ell_3)] + \beta W(k_2,k_3) \right\}$$

The value function $W(k_1,k_2)$ is strictly concave since R(.) is strictly concave, and U(.) is concave. Hence, the values k_3 and ℓ_3 in the optimal

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[2]The "time-to-build" technology assumed in Kydland and Prescott (1982) would not allow for unfinished capital to be productive at all, which is a limiting case of the model we analyze here.

solution are unique. They only depend on the set of initial conditions (k_1,k_2) and it is possible to express them as continuous functions

$$k_3(k_1,k_2) \quad \text{and} \quad \ell_3(k_1,k_2).$$

Accordingly, since at optimum $\ell_3 = \ell_3(k_1,k_2,k_3)$, (1.3) may be represented as

$$(1.4) \qquad W(k_1,k_2) = \max_{\{k_3\}} \left[V(k_1,k_2,k_3) + \beta\, W(k_2,k_3) \right]$$

or, alternatively[3],

$$(1.4') \qquad W(k_1,k_2) = V(k_1,k_2,k_3) + \beta V(k_2,k_3,k_4) + \beta^2 W(k_3,k_4).$$

1.2 A General Dynamic Property of the Model

Since the capital good is the only stock variable in this economy, the dynamics of the model is wholly characterized by the pattern of accumulation. By using the optimality requirement for the accumulation path it is possible to derive sufficient conditions for the system to obey an oscillatory (monotonic) law of motion through time. These conditions hinge on the properties of the return function $V(k_t,k_{t+1},k_{t+2})$. More specifically, they depend on a particular configuration for the signs of V_{12}, V_{13}, and V_{23}

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[3] Henceforth we drop the $\max_{\{k_3\}}$ symbol as it is understood to always hold.

-where V_{ij} represents a second cross-partial derivative of $V(k_t, k_{t+1}, k_{t+2})$:[4]

Proposition: Consider problem (1.2):

(i) If V_{12}, V_{13}, $V_{23} < 0$, for all $(k_t, k_{t+1}, k_{t+2}) > 0$ then an interior optimal trajectory $\{k_t\}$ cannot be strictly monotonic;

(ii) If $[V_{23} + \beta V_{12}] < 0$ (>0) and $V_{13} = 0$ for all $(k_t, k_{t+1}, k_{t+2}) > 0$ then an interior optimal trajectory $\{k_t\}$ is strictly oscillatory (monotonic): $[(k_{t+2} - k_{t+1})(k_{t+1} - k_t)] < 0$ (>0) for $k_{t+1} \neq k_t$.

Proof: see appendix.

II. Establishing the Impact of the Time-to-Build Technology in the Symple Model with Exogenous Labor.

Here we are going to assume that the representative agent supplies inelastically a fixed amount of labor for each and every period. This is done in order to isolate the impact of the "time-to-build" technology, while ruling out any possible effect from substitution in leisure.

If we keep labor exogenous, problem (1.2) becomes

$$(2.1) \qquad \underset{k_t}{\text{Max}} \sum_{t=0}^{\infty} c_t \beta^t$$

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[4]The type of global analysis used here is related to the one in Benhabib and Nishimura (fothcoming).

subject to:

(2.2) $$k_t = h(k_{t-2},k_{t-1}) - c_t + (1-g)k_{t-1};$$

$$k_t \geq (1-g)k_{t-1};$$

k_0, k_1 given.[5]

In (2.1) and (2.2) variables are per-capita variables. ℓ_t is an exogenously given constant which we set equal to 1 for each and every t. Accordingly, the originary production function $f(k_t,k_{t+1},\ell_{t+2})$ collapses into $h(k_t,k_{t+1})$.

(2.1) may be recast as the dynamic programming problem in (1.4)

(2.3) $$W(k_1,k_2) = \max \left\{V(k_1,k_2,k_3) + \beta W(k_2,k_3)\right\}$$

where $V(k_t,k_{t+1},k_{t+2}) = h(k_t,k_{t+1}) + (1-g)k_{t+1} - k_{t+2}$.

The associated cross-partial derivatives V_{12}, V_{13}, and V_{23} are easily computed:

$$V_{13} = V_{23} = 0; \qquad V_{12} = h_{12}(k_t,k_{t+1}).$$[6]

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[5]When we drop leisure choices from the agent's optimal problem one might think that interiority of solutions is lost. On the contrary, we must point out that although the agent's utility of consumption is linear, consumption itself is strictly concave with respect to past investment decisions due to strict concavity of the production function. Hence, concavity of the problem is always guaranteed as long as we restrict our analysis to strictly concave branches of the production function.

It turns out that part (ii) of the proposition above applies to this problem, as long as $h_{12}(k_t,k_{t+1}) \neq 0$.

Let us consider a Cobb-Douglas production function

$$(2.4) \qquad f(K_t,K_{t+1},\ell_{t+2}) = K_t^{\alpha_1} K_{t+1}^{\alpha_2} \ell_{t+2}^{\alpha_3} \qquad 0 < (\alpha_1+\alpha_2+\alpha_3) \lesseqgtr 1$$

and in our case, assuming constant returns to scale, let it collapse to[7]

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[6] If we were to let the agent determine labor endogenously as in (1.2) the resulting cross-partials would be:

$$V_{31} = V_{32} = 0$$

$$V_{12} = \left\{f_{12}(k_t,k_{t+1},\ell_{t+2})+[f_{33}(.)+R''(\ell*-\ell_{t+2})]\partial\ell_{t+2}/\partial k_{t+1}\right\} \partial\ell_{t+2}/\partial k_t$$

where

$$\partial\ell_{t+2}/\partial k_t = -f_{31}/[f_{33}+R''(.)]; \quad \partial\ell_{t+2}/\partial k_{t+1} = -f_{32}/[f_{33}+R''(.)].$$

Accordingly, the trajectory for capital will be oscillatory if any of the two following conditions is satisfied:

(I) $f_{13}(k_t,k_{t+1},\ell_{t+2}) < 0$ and $f_{12}(.) > f_{23}(.) > 0$.

(ii) $f_{13}(.) > 0$ and $f_{12}(.) < f_{23}(.) < 0$.

Both conditions (I) and (II) are compatible with constant returns to scale. But we limit the discussion to the exogenous labor model because the complementarity and substitutability relations among the three factors of production would require a deeper analysis in order for us to draw precise conclusions from (I) and (II).

[7] Indeed, suppose that if we assume constant returns to scale the production function that applies is

$$f(K_t,K_{t+1},\ell_{t+2}) = h(K_t,K_{t+1})\ell_{t+2} = [K_t + bK_{t+1}]^{(\alpha_1+\alpha_2)} \ell_{t+2}^{-(\alpha_1+\alpha_2)}$$

and finally,

$$h(k_t,k_{t+1}) = [k_t + bk_{t+1}]^{(\alpha_1+\alpha_2)}$$

this kind of production function takes account of the fact that the contribution to production by the young capital is not the same as the one by the old capital, as $b \neq 1$.

$$(2.5) \qquad h(k_t, k_{t+1}) = [\ k_t + bk_{t+1}\ ]^{(\alpha_1+\alpha_2)} \qquad\qquad b<1$$

Here we are assuming $b < 1$ in order to be consistent with the time-to-build technology in giving account of the fact that young capitals are unfinished, and do contribute to production to a limited extent. An alternative way of looking at our model would be to interpret it as a vintage growth model. Capital goods would enter production for two consecutive time periods but would pay a depreciation only in the second period.
Young capitals would be more productive than the old ones as the former would supposedly embody a technological progress. The higher productivity of the young capital goods would be reflected by $b > 1$, in (2.5). However, we must point out that our results are robust to this change, and would carry over to the case of $b > 1$.

In general h_{12} is negative, but this result depends on the returns to scale that obtain. If there are constant or decreasing returns to scale, $h_{12} < 0$. On the other hand, although necessary, increasing returns to scale is not a sufficient condition for non-negativity of the cross-partial of h.

Here the cross-partial of h is clearly negative since $(\alpha_1 + \alpha_2) < 1$:

$$h_{12} = b(\alpha_1+\alpha_2)\ (\alpha_1+\alpha_2-1)\ [\ k_t + bk_{t+1}\ ]^{(\alpha_1+\alpha_2-2)}$$

The rationale for this result lays with the fact that when we raise the amount of young capital (k_{t+1}) this will have an impact on the marginal product of the old capital via the change in the quantity of the fixed labor factor which joins it for production. The increment in per-capita old capital engenders a tendency for its marginal contribution to production to decrease.

Why is the oscillatory accumulation path optimal for the agent? This may be easily shown via a heuristic proof by looking at the first order conditions of the problem. The agent's choice for any k_t must satisfy the condition

$$(2.6) \qquad V_3(k_{t-2},k_{t-1},k_t) + \beta V_2(k_{t-1},k_t,k_{t+1}) + \beta^2 V_1(k_t,k_{t+1},k_{t+2}) = 0.$$

Say he were to raise k_{t-1}. Since $V_{32} = 0$, there would be no impact on $V_3(k_{t-2},k_{t-1},k_t)$ and the only change, as far as k_t is concerned, would be in the marginal benefit of k_t when it enters production as a young capital, $\beta V_2(k_{t-1},k_t,k_{t+1})$, which would decline, because $V_{21} < 0$. But then condition (2.6) could only hold if the agent were to lower k_t, whereby increasing its marginal benefit in each one of the three periods. Now, updating (2.6) one period ahead requires

$$(2.7) \qquad V_3(k_{t-1},k_t,k_{t+1}) + \beta V_2(k_t,k_{t+1},k_{t+2}) + \beta^2 V_1(k_{t+1},k_{t+2},k_{t+3}) = 0.$$

The increase in k_{t-1} would have no impact here since $V_{31} = 0$, but the decline in k_t would raise $\beta V_2(k_t,k_{t+1},k_{t+2})$ as $V_{21} < 0$. Accordingly, k_{t+1} would have to increase in order to preserve the equality in (2.7).

In other words, considering that

$$V_3(k_{t-2},k_{t-1},k_t) = V_3(k_{t-1},k_t,k_{t+1}) = -1;$$
$$V_2(k_{t-1},k_t,k_{t+1}) = h_2(k_{t-1},k_t) + (1-g);$$

$$V_2(k_t,k_{t+1},k_{t+2}) = h_2(k_t,k_{t+1}) + (1-g);$$

$$V_1(k_t,k_{t+1},k_{t+2}) = h_1(k_t,k_{t+1});$$

$$V_1(k_{t+1},k_{t+2},k_{t+3}) = h_1(k_{t+1},k_{t+2}),$$

(2.6) becomes

(2.6′) $-1 + \beta[h_2(k_{t-1},k_t) + (1-g)] + \beta^2[h_1(k_t,k_{t+1})] = 0.$

Raising k_{t-1} lowers the marginal benefit of k_t as a young capital since the marginal product of k_t at t+1 is decreased as $h_{12} < 0$. Then the only way to satisfy (2.6′) is via reducing k_t which will engender higher marginal returns for it both at t+1 and at t+2. By the same token, having decreased k_t, the agent has an incentive to consume less and invest more at t+1 since optimality requires

(2.7′) $-1 + \beta[h_2(k_t,k_{t+1}) + (1-g)] + \beta^2[h_1(k_{t=1},k_{t+2})] = 0.$

We may conclude that, unless the agent is already at a steady state, both consumption and the capital stock will follow a two-period cycling path. We must then remark that the nature of this cycling is solely technological. If it were possible to obtain fully productive capital goods available tomorrow out of investment projects initiated today the optimal accumulation path for the agent would be monotonic. It is the presence of the "time-to-build" technology which, by raising the cost of hedging across time, makes the cycling path for capital optimal to the agent. Therefore, it is evident that introducing such an assumption in the process for building capital is likely to have an impact on the mechanics of the model in which it is embodied.

APPENDIX

Consider the problem,

$$W(k_{-1},k_0) = \underset{\{k_t\}\geqq 0}{\text{Max}} \sum_0^{\infty} \beta^t\, V(k_{t-1},k_t,k_{t+1})$$

$$= \underset{\{k_1\}\geqq 0}{\text{Max}} \left\{ V(k_{-1},k_0,k_1) + \beta\, W(k_0,k_1) \right\}$$

$$= \underset{\{k_1,k_2\}\geqq 0}{\text{Max}} \left\{ V(k_{-1},k_0,k_1) + \beta\, V(k_0,k_1,k_2) + \beta^2\, W(k_1,k_2) \right\}$$

We introduce two different sets of initial conditions (k_{-1},k_0) and (k'_{-1},k'_0). Let k_1 and k_2 be the optimal choices from (k_{-1},k_0) and k'_1 and k'_2 be the optimal choices from (k'_{-1},k'_0).

If $k_{-1} \neq k'_{-1}$ or $k_0 \neq k'_0$ the following two inequalities must hold:

$$\text{(A1)} \quad \left\{ V(k_{-1},k_0,k_1) + \beta V(k_0,k_1,k_2) + \beta^2 W(k_1,k_2) \right\} \geqq$$
$$\geqq \left\{ V(k_{-1},k_0,k'_1) + \beta V(k_0,k'_1,k'_2) + \beta^2 W(k'_1,k'_2) \right\};$$

$$\text{(A2)} \quad \left\{ V(k'_{-1},k'_0,k'_1) + \beta V(k'_0,k'_1,k'_2) + \beta^2 W(k'_1,k'_2) \right\} \geqq$$
$$\geqq \left\{ V(k'_{-1},k'_0,k_1) + \beta V(k'_0,k_1,k_2) + \beta^2 W(k_1,k_2) \right\}.$$

Summing up (A1) and (A2) yields, after simplification and regrouping,

$$\text{(A3)}\quad [V(k_{-1},k_0,k_1) - V(k_{-1},k_0,k'_1)] + [V(k'_{-1},k'_0,k'_1) - V(k'_{-1},k'_0,k_1)] +$$
$$+ \beta \left\{[V(k'_{-1},k'_0,k'_1) - V(k'_0,k_1,k_2)] + [V(k_0,k_1,k_2) - V(k_0,k'_1,k'_2)]\right\} \geqq 0.$$

Condition (A3) may be restated in integral notation as

$$\text{(A4)}\quad \int_{k'_1}^{k_1}[V_3(k_{-1},k_0,s_1)-V_3(k'_1,k'_0,s_1)]ds_1 + \beta\int_{k'_0}^{k_0}[V_1(s_0,k_1,k_2)-V_1(s_0,k'_1,k'_2)]ds_0 \geqq 0$$

By adding and subtracting:

(i) $V_3(k_{-1},k'_0,s_1)$ in the first integral;

(ii) $V_1(s_0,k'_1,k_2)$ in the second integral,

we obtain

$$\text{(A5)}\quad \Big\{ \int_{k'_1}^{k_1}\{[V_3(k_{-1},k_0,s_1)-V_3(k_{-1},k'_0,s_1)]+[V_3(k_{-1},k'_0,s_1)-V_3(k'_{-1},k'_0,s_1)]\}ds_1 +$$
$$+ \int_{k'_0}^{k_0}\{[V_1(s_0,k_1,k_2)-V_1(s_0,k'_1,k_2)]+[V_1(s_0,k'_1,k_2)-V_1(s_0,k'_1,k'_2)]\}ds_0 \Big\} \geqq 0.$$

Finally, condition (A5) may be restated in a more convenient form as

$$\text{(A6)}\quad \int_{k'_1}^{k_1}\int_{k'_0}^{k_0} V_{32}(k_{-1},s_0,s_1)\, ds_0\, ds_1 + \int_{k'_1}^{k_1}\int_{k'_{-1}}^{k_{-1}} V_{31}(s_{-1},k'_0,s_1)\, ds_{-1}\, ds_1 +$$
$$\beta \left(\int_{k'_0}^{k_0}\int_{k'_1}^{k_1} V_{21}(s_0,s_1,k_2)\, ds_0\, ds_1 + \int_{k'_0}^{k_0}\int_{k'_2}^{k_2} V_{31}(s_0,k'_1,s_2)\, ds_0\, ds2\right) \Big\} \geqq 0.$$

(A6) must be satisfied if the path $\{k_t\}$ is to be optimal.

By using condition (A3) we can infer some important restrictions on the shape of the trajectory for capital accumulation. These restrictions depend on the particular configuration of the cross-partials V_{21}, V_{31}, and V_{32} of the return function.

Part (i): If, as in part (i) of the proposition, we have

(A7) $\quad V_{21} < 0, \qquad V_{31} < 0, \qquad V_{32} < 0$

we proceed as follows.

Consider the two different sets of initial conditions

(k_{-1},k_0) and (k'_{-1},k'_0) such that $k_{-1}<k_0$ and $k'_{-1}<k'_0$.

Then, without loss of generality, we may set $k_{i+1} = k'_i$. Hence, the two paths, if monotonicity holds, will yield

(A8) $\quad k_{-1} < k'_{-1};\ k_0 < k'_0;\ k_1 < k'_1;\ k_2 < k'_2 \quad (\text{or } k_{-1}<k_0<k_1<k_2<k_3).$

If (A6) holds, not all inequalities in (A8) can be satisfied at the same time, otherwise condition (A6) would be violated. Therefore, the optimal trajectory cannot be strictly monotonic, as in (A8).

Part (ii): If $V_{31} \equiv 0$ and $V_{21} \leqq 0$ and $V_{32} \leqq 0$ (not being both zero), then (A6) becomes

(A9) $$\int_{k'_1}^{k_1} \int_{k'_0}^{k_0} \left\{ V_{32}(s_0,s_1) + \beta V_{21}(s_0,s_1) \right\} ds_0\, ds_1 \quad \geqq 0.$$

Thus, if $[V_{32} + \beta V_{21}] < 0$, $k'_0 > k_0$ implies $k'_1 \leqq k_1$. (An argument -just identical to the heuristic proof of the two-period cycling in the text-involving the first order conditions shows that the equality cannot hold if $[V_{32} + \beta V_{21}] \neq 0)$. Now if we set $k'_0 = k_1$, it follows that $k_2 = k'_1$ and $k_1 > k_0$ implies $k_2 < k_1$. The proof for the monotonic case with $[V_{32} + \beta V_{21}] > 0$ is identical.

REFERENCES

Benhabib, J. and K. Nishimura, "Competitive Equilibrium Cycles", *Journal of Economic Theory*, forthcoming.

Kydland, F.E., and E.C. Prescott, "Time to Build and Aggregate Fluctuations", *Econometrica*, 50 (1982), 1345-1370.

Long, J.B. and C.I. Plosser, "Real Business Cycles", *Journal of Political Economy*, 91 (1983), 39-69.