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ON THE OPTIMAL PRICING POLICY OF A MONOPOLIST

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Abstract

The paper presents a simple explanation of price dispersion by a monopolist assuming only that consumers arrive in a random order and are served on a first come, first serve basis. Under these conditions, it is shown that a firm can sometimes increase its profits by charging different prices for different units of a homogeneous product. However, it is never necessary to charge more than two prices, and a single price is sufficient as long as either the marginal revenue curve is everywhere downward sloping or the marginal cost of production is constant.

1. <u>Introduction</u>

When will a monopolist charge the same price for every unit that it sells? It is well known that a monopolist may have an incentive to charge different prices to different consumers if it can identify the demand curve of the different consumers and prohibit arbitrage. Even if individual consumers cannot be identified, it is sometimes possible to price discriminate by offering quantity discounts. If the cost of search is explicitly modelled, Salop (1977) has shown that a monopolist can sometimes benefit from randomizing over its prices across outlets (or presumably across time) in order to exploit differences in the search costs of agents with different reservation values. In this note, I examine the extent to which any of these explanations are <u>necessary</u> to explain nonuniform pricing by a monopolist.

Consider a "static" model of a monopolist facing a demand curve generated by a large number of consumers and suppose the firm is permitted to choose not only the quantity it will sell but also a separate price for each unit it offers for sale. The firm is unable to identify the demand curve of the individual consumers who arrive in random order and purchase at the lowest available price up to the point where the price of the next unit exceeds their reservation value. In such a model will the firm ever have an incentive to charge different prices for different units of the same good? The answer is yes as the following example illustrates.

Suppose that the firm has 300 units to sell. 100 consumers are willing to purchase a single unit at a price of 2 and 400 consumers are willing to purchase at a price of 1. Clearly, if the firm must charge the same price for all of the units, it must charge a price of 1, resulting in a total revenue of 300. However, if we permit the firm to charge a different price for different units, it could increase it revenue by adopting the following policy. Sell the first 250 units at price 1 and sell the last 50 units at price 2. Since there are a total of 500 consumers with a reservation value in excess of 1, on average, only one half of the consumers with a reservation value of 2 will be among the first 250 consumers. Consequently, after the first 250 units are sold, there will still be an average of 50 consumers who are willing to pay the price of 2 for the last 50 units. The expected revenue of the firm is consequently increased from 300 to approximately 350.

The next two sections contain a characterization of the optimal pricing policy for any left continuous, downward sloping demand curve generated by a large number of consumers. For a fixed level of output, I show that the optimal pricing policy can be found as the solution to a linear programming problem in the space of nonnegative measures on the space of prices. From this, a number of properties of the solution can be established. First of all, the firm need never charge more than two prices to maximize its revenue. Second, in contrast the case where the monopolist is constrained to using a single price, the _marginal revenue_ will always be a nonincreasing function of the quantity. In fact, it is precisely in those cases where the single price marginal revenue curve is strictly increasing over some range that the firm will have an incentive to charge more than two prices. Finally, at any level of output at which the firm charges more than two prices, the marginal revenue is constant. This implies immediately that if the firm can produce output at a constant marginal cost, then its revenue can be maximized by choosing a level of output for which a single price is optimal.

To what extent the model presented here is an adequate explanation of price dispersion is difficult to tell. Not all markets satisfy the conditions required for price dispersion. In those markets where there is limited capacity and the distribution of reservation prices has more than one peak, however, it may be optimal to for a monopolist to use more than one price. In these cases, it may provide and simpler and more plausible alternative to more sophisticated explanations, such as Salop's, which presume that the firm is exploiting some correlation between the demand curves of individual consumers and the cost of acquiring information. It may, for instance, be a sufficient explanation for the rationing of super saver fares by airlines or even by the price randomization practiced by some retail distributors such as grocery stores.

In the concluding section, I consider briefly how the results might be affected in the presence of competition from other firms.

2. The Optimal Pricing Policy of a Firm Selling a Fixed Quantity

Suppose that the demand curve $D(\cdot)$ facing a monopolist is generated by a continuum of individual consumers. Suppose also that the firm may charge a different price for each unit that it sells. Consumers arrive at the store at random. When it is his turn to purchase, a consumer purchases from the available units up to the point where the price of the next unit exceeds the price the consumer is willing to pay for it. The first question to be addressed is what pricing policy will maximize the revenue of firm from selling q units of the good.

Without loss of generality, we may order the goods as an increasing function of price. Then assuming that goods are perfectly divisible, the pricing policy for q units of the good can be represented as a nondecreasing function $p: [0,q] \to R_+$. Consider first the characterization of the set of price functions which are consistent with the sale of q units of the good. I present here a heuristic derivation for the case where D(p) is strictly decreasing and continuous on $[0,\infty)$. A similar argument establishes the same restrictions for any demand curve which is nonincreasing and continous from below.[1]

We begin by considering pricing policies which are step functions. Partition the quantity space into intervals of Δ and suppose that prices must be constant on each Δ interval. Let p_i be the price of goods in the i^{th} interval and let $D_i = D(p_i)$. Finally, let E_i be the measure of consumers who are willing to purchase at price p_i after the first $i\Delta$ units have been sold. Then

$$E_1 = D_1 - \Delta = D_1[1-(\Delta/D_1)].$$

If $E_1 \geqq 0$, then the demand for goods priced at p_2 will be equal to the excess demand at price p_1 times the fraction of those consumers who stay in the market when the price rises from p_1 to p_2. The excess demand at price p_1 is just E_1. The fraction of consumers who stay in the market is the ratio of the measure of consumers with reservation value greater than or equal to p_2 to the number of consumers with reservation value greater than or equal to p_1, $[D(p_2)/D(p_1)]$. Consequently, the excess demand at price p_2 is equal to

$$E_2 = E_1[D_2/D_1] - \Delta = D_2[1-\Delta[(1/D_1)+(1/D_2)]].$$

Using mathematical induction, we may conclude that as long as $E_{i-1} \geqq 0$, then

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[1]We may even allow $D(0) = \infty$.

$$\varepsilon_i = \varepsilon_{i-1}[D_i/D_{i-1}]-\Delta = D_{i-1}[1-\Delta\sum_{j=1}^{i-1}(1/D_j)][D_i/D_{i-1}]-\Delta$$
$$= D_i[1-\Delta\sum_{j=1}^{i}(1/D_j)].$$

The requirement that the pricing policy be chosen so that the first q units are sold is then equivalent to imposing the constraint that $\varepsilon_{q/\Delta} \geqq 0$.

Letting Δ go to zero we may conclude that a price function $p(\cdot)$ is consistent with selling q units if and only if it satisfies:

(1) $$\int_0^q [1/D(p(t))]dt \leqq 1.$$

We turn next to the determination of the price function which maximizes the revenue of a firm which must sell q units. The problem of the firm is to choose a nondecreasing function $p(\cdot)$ to maximize

$$\int_0^q p(t)dt$$

subject to inequality (1). Note, however, that since both the objective function and the constraint are integrals of functions of $p(t)$, requiring $p(\cdot)$ to be nondecreasing will not affect the value of the problem. It simply provides a way to order the prices. This allows us to reformulate the problem as a <u>linear</u> maximization problem.

Let $P^* = \{p \geqq 0: D(p) > 0\}$ be the set of prices for which demand is strictly positive and let Σ be the set of nonnegative measures over P^*. For any $\sigma \in \Sigma$, and any measurable subset of prices $S \subset P^*$, we may interpret $\sigma(S)$ as the number of units which are assigned a price in S. Then the problem may be restated as follows:

(I) Choose $\sigma \in \Sigma$ to maximize

$$\int p\sigma(dp)$$

subject to

$$\int \sigma(dp) = q, \text{ and}$$

$$\int [1/D(p)]\sigma(dp) \leq 1.$$

The problem is thus reduced to the maximization of a linear function subject to two linear constraints. It follows immediately, therefore, that if a solution exists, there is a solution with all mass concentrated on at most two prices.[2] In order to guarantee the existence of a solution, however, some additional restrictions must be added. One sufficient condition is to require that $\lim_{p\to\infty} pD(p) = 0$.

To characterize the solution, we appeal to the Kuhn-Tucker conditions. If $0 < q < D(0)$, a measure σ is a solution if and only if there is a λ and μ[3] such that

(2) $p - \lambda - \mu/D(p) \leq 0$ for all $p \in P^*$, $\sigma(\cdot) \geq 0$, and

$\int [(p-\lambda)-\mu/D(p)]\sigma(dp) = 0,$

(3) $\int \sigma(dp) = q$, and

(4) $\int [1/D(p)]\sigma(dp) \leq 1$, $\mu \geq 0$, $\mu[\int [1/D(p)]\sigma(dp) - 1] = 0.$

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[2]See, for example, Chernoff and Reiter, (1954).

[3]The necessity and sufficiency of these conditions follows from the fact there is an $\epsilon > 0$ such that for any $\delta \in (-\epsilon,\epsilon)$, the set of σ satisfying $\int [1/D(p)]\sigma(dp) < q-\epsilon$ and $\int \sigma(dp) \leq 1-\epsilon$ is non-empty.

Note first that constraint (2) is equivalent to the requirement that the firm charge only those prices which maximize $(p-\lambda)D(p)$. Let $P(\lambda) = \{p \in \mathrm{argmax}_p (p-\lambda)D(p)\}$. Second, note that $\mu = 0$ implies that $\lambda = p^* \equiv \sup P^*$ and hence that $\sigma(p^*) = q$; otherwise relation (2) will be violated. We may conclude, therefore, that either $q \leqq D(p^*)$, in which case $\sigma(p^*) = q$, or $q > D(p^*)$, in which case the necessary and sufficient conditions for σ to be a solution may be expressed as the requirement that there be a $\lambda \geqq 0$ such that

(5) $\sigma(P(\lambda)) = q$,

(3) $\int \sigma(dp) = q$, and

(6) $\int [1/D(p)]\sigma(dp) = 1$.

Since $D(\cdot)$ is nonincreasing and continuous from below, it follows that $P(\lambda)$ is closed. Furthermore, $P(\cdot)$ is upper hemi-continuous and nondecreasing in λ in the sense that if $p' \in P(\lambda')$ and $p'' \in P(\lambda'')$, then $\lambda'' > \lambda'$ implies $p' \geqq p''$. Let $\underline{p}(\lambda) = \inf P(\lambda)$ and $\bar{p}(\lambda) = \sup P(\lambda)$. Then constraints (3), (5) and (6) requires that λ satisfy $D(\bar{p}(\lambda)) \leqq q \leqq D(\underline{p}(\lambda))$. A solution to (I) is then determined by a value of λ and values of $\sigma(\underline{p}(\lambda))$ and $\sigma(\bar{p}(\lambda))$ which satisfy

(7) $\sigma(\underline{p}(\lambda)) + \sigma(\bar{p}(\lambda)) = q$.

(8) $\sigma(\underline{p}(\lambda))/D(\underline{p}(\lambda)) + \sigma(\bar{p}(\lambda))/D(\bar{p}(\lambda)) = 1$,

Note that if $\underline{p} = \bar{p}$, then equations (7) and (8) reduce to the requirement that $D(\underline{p}) = q$, so that only a single price is used.

We may summarize our conclusions as

Proposition 1: For a fixed level of sales, a monopolist can always acheive its maximum revenue with at most two prices.

3. The Optimal Choice of Quantity and the Conditions for a Single Price

Consider next how the revenue changes with an increase in the number of units to be sold. For any level of $q \in (0, D(0))$, let $R(q)$ denote the maximum revenue than can be obtained from selling q units and let $\Lambda(q)$ be the set of values of λ consistent with equations (2) to (4). Then since (I) is a concave programing problem, it follows $R(\cdot)$ is a concave function and that $\Lambda(q)$ is an upper hemi-continuous compact, convex valued correspondence with the property that if $\lambda \in \Lambda(q)$ and $\lambda' \in \Lambda(q')$, then $q > q'$ implies $\lambda \leqq \lambda'$.

Let $\underline{\lambda}(q) = \inf \Lambda(q)$. Then $\underline{\lambda}(q)$ measures the right hand derivative of $R(q)$ which I will call the marginal revenue of q.[4] We have already established that $\underline{\lambda}(\cdot)$ is nonincreasing. We can also show that whenever more than one price is used, $\partial\underline{\lambda}(q)/\partial q = 0$. To see this, suppose that $\lambda \in \Lambda(q)$ and that $\sigma(\underline{p}(\lambda)) < 1$. For any q' sufficiently close to q, relations (2) to (4) imply that either $\underline{\lambda}(q) = p^* = \underline{\lambda}(q')$ or equations (7) and (8) are satisfied, in which case $\sigma(p) > 0$ for some $p > \underline{p}(\lambda)$. For any λ' such that $p \in P(\lambda')$, relation (5) then implies that $\lambda' \leqq \lambda$. But if $q' > q$, we know

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[4]Note that if $D(0) = \infty$, relation (2) implies that marginal revenue must be nonnegative.

that $\lambda' \in \Lambda(q')$ implies that $\lambda' \leqq \underline{\lambda}(q)$. We may conclude, therefore, that $\underline{\lambda}(q') = \underline{\lambda}(q)$.

These conclusions are summarized as

Proposition 2: (i) Marginal revenue is a nonincreasing function of the quantity q. (ii) The marginal revenue is constant at quantities at which the firm charges more than one price.

In contrast, when the firm is constrained to charge a single price, marginal revenue will generally not be downward sloping throughout the domain of output levels. In fact, it is precisely when the single price marginal revenue curve as an increasing section that firm will have an incentive to charge more than one price at some output levels. To see this, let $\bar{R}(q)$ be the revenue to the firm when it is constrained to sell q units at the single price $D^{-1}(q)$[5]. Let $H = \{(r,q) \in RXP^*: r \leqq R(q)\}$ and $\bar{H} = \{(r,q) \in RXP^*: r \leqq \bar{R}(q)\}$. Since the firm always has the option of using a single price, it follows immediately that $R(q) \geqq \bar{R}(q)$ for all $q \geqq 0$. The concavity of $R(\cdot)$ then implies that $co\bar{H}$[6] $\subset coH = H$. On the other hand, we have already shown that $R(q) = p^*q = \bar{R}(q)$ for $q \leqq D(p^*)$, and for any $q > D(p^*)$, the solution is defined by a pair of prices $\underline{p}$ and $\bar{p}$ combined with weights $\sigma(\underline{p})$ and $\sigma(\bar{p})$ satisfying equations (7) and (8). Letting $\alpha = \sigma(\underline{p})/D(\underline{p})$ in this case, we may express the value of the objective function for (I) as

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[5]$D^{-1}(q)$ is the highest price such that $D(p) = q$.

[6]$co\bar{H}$ denotes the convex hull of $\bar{H}$.

$$R(q) = \alpha \underline{p} D(\underline{p}) + (1-\alpha)\bar{p}D(\bar{p}) = \alpha \bar{R}(\underline{p}) + (1-\alpha)\bar{R}(\bar{p}).$$

Since $\alpha \in [0,1]$, this implies that $H \subset \text{co}\bar{H}$, and hence that $H = \text{co}\bar{H}$. Since $\bar{H} = \text{co}\bar{H}$ if and only if $\bar{R}(\cdot)$ is a concave function, it follows immediately that $R(q) = \bar{R}(q)$ for all q if and only if $\bar{R}(\cdot)$ is a concave function. We may therefore conclude that revenue can be maximized with a single price at all quantities if and only if $\bar{R}(\cdot)$ is a concave function.

The quantity actually produced will depend upon both the marginal revenue and the marginal cost functions. In general, we cannot rule out cases where the marginal revenue curve cuts the marginal cost curve at a quantity where it is optimal for the firm to charge two distinct prices. Suppose, however, that marginal cost is constant. Then since Proposition 2 guarantees that the marginal revenue is also constant at any quantity in which the firm charges more than one price, it follows immediately that revenue can always be maximized at a quantity where the firm charges a single price.

These conclusions are stated as

Proposition 3: (i) The firm can maximize its revenue at all output levels using a single price if and only if the (single price) marginal revenue function is nonincreasing. (ii) If marginal cost is constant, then the firm can always maximize revenue by producing a quantity which equates the (single price) marginal revenue to marginal cost and then charging a single price.

4. An Example With Two Prices

For any demand function with points of discontinuity, there will be a range of output levels at which it is optimal to charge more than one price. I consider here such a demand curve and contrast the optimal pricing policy and the corresponding revenue function with the pricing policy and revenue function which result when the firm is constrained to charge a single price.

Consider the following demand function:

$D(p) = 0$ for $p > 2$;

$D(p) = 1$ for $1 < p \leqq 2$;

$D(p) = 5$ for $0 < p \leqq 1$;

If the firm is constrained to charge a single price, then it will earn a revenue of 2q for $q \leqq 1$ and a revenue of q for $1 < q \leqq 4$. The marginal revenue will be undefined at $q = 1$. Now suppose we permit the firm to charge more than one price. Then for $q \leqq 1$, its optimal pricing strategy is to charge a price of 2 for all units. For $1 < q \leqq 4$, its optimal policy is to charge price 1 for the first $5(q-1)/4$ units and price 2 the remaining $(5-q)/4$ units. The marginal revenue is 2 for the first unit and 3/4 for the next four units.

The example is illustrated in Figure 1. The demand curve is presented in the top half of the figure. In the bottom half of the figure are the two revenue functions. Both the optimal revenue function and the single price revenue function are identical for the first unit. At $q = 1$, the single price revenue function $\bar{R}$ has a downward discontinuity whereupon it increases with slope 1 up to $q = 5$. In contrast, the optimal revenue function is found by connecting the single price revenue at $q = 1$ with the single price revenue at $q = 5$ with a straight line. Over this range of outputs, the firm will charge a price of 1 for some units and a price of 2 for the other units.

5. Concluding Remarks

Although I have concentrated on the analysis of a static model, a similar result could have been established in a dynamic model where the firm faces a flow of consumers with random demand curves. In such a case, a firm with an upward sloping marginal cost curve might find it optimal to randomize from day to day (or week to week) over two different prices in order to achieve a flow demand equal (on average) to its flow supply.

I should emphasize, however, that our argument depended crucially on our implicit informal appeal to the law of large numbers so that we could identify the realization of the effective demand curve with its average. If we explicitly take into account the discreteness of the product space and the consumers, we are confronted with a complicated integer programing problem. For this problem, Krishna and Perry (1985) have been able to show that the firm will charge a single price when marginal cost is zero; otherwise, very little seems to be known about the optimal pricing policy.

Finally, it would be interesting to examine the extent to which competition would remove or enhance the possibility of an individual firm using price dispersion. I have not yet examined this question in much detail. However, some preliminary analysis indicates that the answer may not be simple. Suppose consumers have zero search costs and firms face capacity constraints. Then the Nash equilibrium sometimes implies price randomization even when each firm is constrained to charge the same price for each unit. Typically, however, there will be a range of capacities which imply a deterministic equilibrium price.[7] In this case, it can be shown that if a monopolist would use two different prices to sell an output equal to the total capacity level, then a duopolist also has an incentive to use two prices as

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[7]This occurs whenever neither firm can benefit from raising its price when both firms charge the price where aggregate demand equals aggregate quantity.

long as the other firm charges the single market clearing price. In such a case, the equilibrium of the market requires not only that each firm charge different prices for different units but that the pricing policy itself be random.

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Figure 1. Comparison of Revenue Under Optimal Pricing with the Single Price Revenue.