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by

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1. Introductory Remarks

Efforts to axiomatize the measurement of technical efficiency were stimulated by the Färe/Lovell [1978] suggestion that (input) efficiency measures should satisfy three conditions:

(I) indication of efficient input vectors (the measure is equal to one if and only if the input bundle is technically efficient in the sense of Koopmans [1951]),

(M) monotonicity (increasing one input quantity while holding all others constant lowers the measure), and

(H) homogeneity of degree minus one (e.g., doubling all input quantities cuts the measure in half).

These criteria have been used to assess several proposed efficiency measures -- most notably, the Debreu [1951]/Farrell [1957] measure, E_{DF} (essentially the inverse of the Malmquist [1951]/Shephard [1970] distance function, which measures the maximum amount that an input vector can be shrunk along a ray while holding output levels constant), the Färe/Lovell [1978] measure, E_{FL} (the maximum sum of proportionate reductions in individual inputs in coordinate directions), and the Zieschang [1984] measure, E_Z (essentially an

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amalgam of the Debreu/Farrell and Färe/Lovell measures, shrinking input vectors along a ray to the isoquant and then shrinking in coordinate directions along the isoquant until an efficient point is reached).

Bol [1985], however, has recently shown that there does not exist a measure satisfying the three Färe/Lovell conditions for the broad class of technologies considered in this literature. Bol's result leaves wide open the issue of what combinations of desirable properties of a measure of technical efficiency are feasible. As Bol points out, there are two approaches to identifying properties that can be satisfied by an efficiency measure: (1) relaxation of the desired properties and (2) narrowing the class of technologies to which the efficency measure is to be applied. In this paper, I pursue the first of these avenues of inquiry.

Three results along this line have held up:

R1: (H) is satisfied by E_{DF} for all technologies (Färe and Lovell [1978]),

R2: (I)∩(H) is satisfied by E_Z for all technologies (Zieschang [1984]), and

R3: (I)∩(M)∩(H) cannot be satisfied for all technologies (Bol [1984]).

Questions that have not been answered are the following:

Q1: Does there exist a measure satisfying (M) for all technologies?

Q2: Does there exist a measure satisfying (M)∩(H) for all technologies?

Q3: Does there exist a measure satisfying (I)∩(M) for all technologies?

Another approach is to weaken (rather than eliminate) conditions. Russell [1985] suggested the following weakening of monotonicity:

(WM) weak monotonicity (increasing one input quantity while holding others constant cannot increase the efficiency measure).

The one positive result regarding (WM) is

R4: (I)∩(WM) is satisfied by E_{FL} for all technologies (Russell [1985]).

Since the Färe/Lovell measure fails to satisfy the homogeneity condition, the following question remains unanswered:

Q4: Does there exist a measure satisfying (WM)∩(H)?

I attempt to answer these questions in this paper. I also consider a weakening of the indication condition (I) to a requirement that the measure is equal to 1 if and only if the input vector is weakly efficient (it is not possible to lower all input quantities while holding output levels constant). As might be expected, this condition, which is the indication counterpart of (WM), interacts significantly with other conditions in producing both possibility and impossibility results.

More important, I also consider a condition that is perhaps more fundamental than any of those identified above: invariance with respect to changes in the units of measurement. This condition also interacts profoundly with the other axioms. Most important -- and perhaps surprising -- is the result that this invariance condition and monotonicity, (M), are mutually inconsistent.

After laying down the groundwork in Section 2, I state and prove the results in Section 3 and conclude in Section 4.

2. Production Technologies and Efficiency Measures

Let $\mathbf{R}_+^n$ be the nonnegative Euclidean n-orthant and adopt the following notations for $\langle x,\hat{x}\rangle \in \mathbf{R}_+^{2n}$:

$x \geqq \hat{x}$ if $x_i \geqq \hat{x}_i$ $\forall$ i,

$x \underline{>} \hat{x}$ if $x_i \geqq x_i$ $\forall$i and $x \neq \hat{x}$,

$x > \hat{x}$ if $x_i > \hat{x}_i$ $\forall$ i,

and

$x \not> \hat{x}$ if $\hat{x}_i \geqq x_i$ for some i.

Also, $\underline{0}$ is the n-dimensional zero vector and $\hat{\mathbf{R}}_+^n = \mathbf{R}_+^n \setminus \{\underline{0}\}$.

The production technology is characterized by the upper-level-set mapping, $L : \mathbf{R}_+^m \to 2^{\mathbf{R}_+^n}$. L(u) is the set of input vectors that can produce the output vector $u \in \mathbf{R}_+^m$. Let

$\Gamma = \{u \in \mathbf{R}_+^n | L(u) \neq \emptyset \cap \underline{0} \notin L(u)\}$

and define $\underline{L}$ as the collection of upper-level-set mappings that satisfy

(L1) $x \in L(u)$ implies $\lambda x \in L(u)$ $\forall \lambda \in [1,+\infty)$, $\forall$ $u \in \Gamma$,

and

(L2) L(u) is closed $\forall$ $u \in \Gamma$.

An input vector $x \in L(u)$ is efficient (in the sense of Koopmans [1951]) if $x \underline{>} \hat{x}$ implies $\hat{x} \notin L(u)$. A vector $x \in L(u)$ is weakly efficient if $x > \hat{x}$ implies $\hat{x} \notin L(u)$.

An (input) efficiency measure is a mapping $E : \Gamma \times \hat{R}^n_+ \times L \rightarrow (0,1] \cup \{+\infty\}$ with the property that $E(u,x,L) \in (0,1]$ if and only if $x \in L(u)$. The properties of E proposed by Färe and Lovell [1978] are

(I) $E(u,x,L) = 1$ iff x is efficient,

(M) $E(u,\hat{x},L) < E(u,x,L)$ if $\hat{x} \geq x$, $\hat{x} \in L(u)$, and $x \in L(u)$,

and

(H) $E(u,\lambda x,L) = \lambda^{-1}E(u,x,L)$ $\forall$ $\lambda \in [D(u,x,L)^{-1},+\infty)$, $\forall$ $x \in L(u)$,

where D is the distance function, defined by

$$D(u,x,L) = \max_{\lambda}\{\lambda \in \mathbf{R}_+ | x/\lambda \in L(u)\}.$$

The weaker version of (M) suggested by Russell [1975] is

(WM) $E(u,\hat{x},L) \leq E(u,x,L)$ if $\hat{x} \geq x$, $\hat{x} \in L(u)$, and $x \in L(u)$.

The efficiency measures proposed by Debreu [1951] and Farrell [1957], Färe and Lovell [1978], and Zieschang [1984] are defined, for $x \in L(u)$, by

$$E_{DF}(u,x,L) = D(u,x)^{-1},$$

$$E_{FL}(u,x,L) = \min_{k} \{ \sum_{i=1}^{n} k_i / \sum_{i=1}^{n} \delta(x_i) \mid Kx \in L(u) \cap k_i \in [0,1] \forall i \}$$

and

$$E_Z(u,x,L) = E_{FL}(u,x/D(u,x,L^+)/D(u,x,L^+),$$

where K is the positive diagonal matrix with $[k_1,\ldots,k_n]$ on the diagonal,

$$\delta(x_i) = \begin{cases} 1 \text{ if } x_i > 0 \\ 0 \text{ if } x_i = 0, \end{cases}$$

and $L^+(u) = L(u) + R^n_+$ (the free-disposal hull of L(u)).

3. Results

The first theorem shows that, by dispensing with the homogeneity condition (H), we can salvage the indication and monotonicity conditions, (I) and (M). To this end, define the efficiency measure, E^{ϵ}_{FL}, by

$$E^{\epsilon}_{FL}(u,x,L) = \begin{cases} \min_k \{ \sum_{i=1}^{n} k_i \mid K(x+\underline{\epsilon}) \in L(u)+\{\underline{\epsilon}\} \cap k_i \in [0,1]\ \forall i \} & \text{if } x \in L(u) \\ +\infty & \text{if } x \notin L(u), \end{cases}$$

where K is the n x n diagonal matrix with k on the diagonal and $\underline{\epsilon} \in R^n_+$ is the vector with each element equal to $\epsilon \in (0,+\infty)$. Note that, although the definition only restricts k_i to the closed interval [0,1], the minimizing vector k^* satisfies $k^* > 0$, since $L(u)+\{\underline{\epsilon}\}$ is contained in the interior of R^n_+.

Thus, E^{ϵ}_{FL} is a modification of the Färe/Lovell measure, obtained by displacing the level set L(u) and the vector x by the factor $\underline{\epsilon}$. This displacement, which can be arbitrarily small, eliminates those cases where E_{FL} fails the monotonicity criterion, as shown by Theorem 1.

Theorem 1. E^{ϵ}_{FL} satisfies (I) and (M) for all $L \in \underline{L}$.

Proof: To prove (I), suppose the contrary: there exists an efficient $x^* \in L(u)$ such that $E^{\epsilon}_{FL}(u,x^*L) < 1$ or, alternatively, there exists an inefficient $x^* \in L(u)$ with $E^{\epsilon}_{FL}(u,x^*,L) = 1$. In the first case, for some i, say j, $k_j^* < 1$ where

(a) $k^* = \arg\min_k \{ \sum_i k_i \mid K(x^*+\underline{\epsilon}) \in L(u)+\{\underline{\epsilon}\} \cap k_i \in [0,1] \ \forall i \}$.

Since $K^*(x^* + \underline{\epsilon}) \in L(u) + \{\underline{\epsilon}\}$, we have

(b) $K^*x^* \in L(u) + \{\underline{\epsilon} - K^* \underline{\epsilon}\}$.

Together with $\underline{\epsilon} - K^* \underline{\epsilon} > \underline{0}$, (b) implies that $K^*x^* \in L(u)$. Since $K_j^*x_j^* < x_j^*$, this contradicts the assumption that x^* is efficient.

In the second case, there exists an $\hat{x} \in L(u)$ such that $\hat{x}_j < x_j^*$ for some j and $\hat{x}_i = x_i^*$ for all $i \neq j$. Thus, $\hat{x} + \underline{\epsilon} \in L(u) + \{\underline{\epsilon}\}$. Consequently, for $\hat{k}_j \leq (\hat{x}_j + \epsilon)/(x_j^* + \epsilon) < 1$, $\hat{K}(\hat{x}+\underline{\epsilon}) \in L(u) + \{\underline{\epsilon}\}$, contradicting $E_{FL}^{\epsilon}(u,x^*,L) = 1$.

To prove (M), consider $x^* \in L(u)$ and $\hat{x} \in L(u)$ satisfying $x^* \leq \hat{x}$. As k^*, defined in (a), is positve, $K^*(\hat{x} + \underline{\epsilon}) \geq K^*(x^* + \underline{\epsilon})$ and there exists a $\hat{k}$ satisfying $\hat{k} \leq k^*$ such that $\hat{K}(\hat{x} + \epsilon) \in L(u) + \{\underline{\epsilon}\}$. Hence, $E_{FL}^{\epsilon}(u,\hat{x},L) < E_{FL}^{\epsilon}(u,x^*,L)$. ||

While the construction E_{FL}^{ϵ} provides a way of salvaging two of the three Färe/Lovell conditions, it unfortunately has a perverse property that undermines its attractiveness: it is not invariant with respect to changes in the units of measurement. Essentially, this is because the shrinkage factor, k, is proportional with respect to $\underline{\epsilon}$ rather than the origin, $\underline{0}$. I believe that insensitivity to the choice of units of measurement is a more fundamental property of efficiency measures than either (I) or (M). Let us, therefore, explore the implications of adding the following dimensionality axiom:

(D) If $\hat{u} = \Omega u$ and $\hat{x} = \Lambda x$, where Ω and Λ are, respectively, m x m and n × n positive diagonal matrices and $\hat{L}(\hat{u}) = \{\hat{x} \in \mathbb{R}^n_+ | x \in L(u)\}$,

then $E(\hat{u},\hat{x},\hat{L}) = E(u,x,L)$.

The next result shows that not only does E^{ε}_{FL} fail to satisfy (D) but, in fact, <u>no</u> measure satisfying (M) can meet the criterion (D) for all technologies.

<u>Theorem 2.</u> There does not exist an efficiency measure E satisfying (D) and (M) for all $L \in \underline{L}$.

<u>Proof:</u> Consider the technology with the level set,

$$L(\bar{u}) = \{ x \in \mathbb{R}^n_+ \mid x_j \geqq 0 \cap x_i \geqq \tilde{x}_i > 0 \ \forall \ i \neq j \}$$

for some $\bar{u} \in \Gamma$. It is easy to show that this level set is consistent with (L1) and (L2).

For $\bar{x} \in L(\bar{u})$, define $\hat{x}$ by $\hat{x}_j = \lambda \bar{x}_j$, where $\lambda \in (1,+\infty)$, and $\hat{x}_i = \bar{x}_i \neq 0$ for all $i \neq j$. From the definition of $L(\bar{u})$, $\bar{x} \in L(\bar{u})$ implies that $\hat{x} \in L(\bar{u})$. By (D),

$$E(\bar{u},\bar{x},L) = E(\bar{u},\hat{x},\hat{L}),$$

where $\hat{L}$ is defined by

$$\hat{L}(\bar{u}) = \{\hat{x} \in \mathbb{R}^n_+ | \bar{x} \in L(\bar{u})\}$$

$$= \{\hat{x} \in \mathbb{R}^n_+ | \hat{x} \in L(\bar{u})\}$$

$$= L(\bar{u}).$$

Consequently,

$$E(\bar{u},\bar{x},L) = E(\bar{u},\hat{x},L),$$

contradicting (M).‖

If one takes the condition (D) to be fundamental, it would appear that the monotonicity condition (M) should be jettisoned from the list of desirable criteria for efficiency measures. The question this raises is whether the weaker monotonicity condition (WM) is compatible with invariance with respect to changes in units of measurement. The next theorem answers this question affirmatively.

Theorem 3. E_{FL} satisfies (D), (I), and (WM) for all $L \in \underline{L}$.

Proof: The properties (I) and (WM) were established by Färe and Lovell [1978] and Russell [1985], respectively. To prove that E_{FL} satisfies (D), let

$$k^* = \arg\min_k \{ \sum_i k_i / \sum_i \delta(x_i) \mid Kx \in L(u) \cap k_i \in [0,1] \; \forall i \},$$

and consider

$$E(\Omega u, \Lambda x, \hat{L}) = \min_k \{ \sum_i k_i / \sum_i \delta(\lambda_i x_i) \mid K\Lambda x \in \hat{L}(\Omega u) \cap k_i \in [0,1] \; \forall i \},$$

where, as in the definition of (D), Ω and Λ are positive diagonal matrices, and

$$\hat{L}(\Omega u) = \{\Lambda x \in R^n_+ \mid x \in L(u)\}.$$

This last identity implies that $x \in L(u)$ if and only if $\Lambda x \in \hat{L}(\Omega u)$. Hence $Kx \in L(u)$ if and only if $K\Lambda x \in \hat{L}(\Omega u)$. Moreover, since Λ is positive, $\delta(\lambda_i x_i) = \delta(x_i)$ $\forall i$. Consequently,

$$E(\Omega u, \Lambda x, \hat{L}) = E(u, x, L). \; \|$$

The next theorem shows that homogeneity (H), along with the indication condition (I), is also compatible with (D).

Theorem 4. E_Z satisfies (D), (I), and (H) for all $L \in \underline{L}$.

Proof: Zieschang [1984] showed that E_Z satisfies (I) and (H). To prove that E_Z satisfies (D), in the light of Theorem 3, it suffices to show that

$$D(\Omega u, \Lambda x, \hat{L}^+) = D(u, x, L^+).$$

To see that this is so, note that

$$x \in L^{+}(u) \iff \exists \hat{x} \in R^{n}_{+} \mid x - \hat{x} \in L(u)$$

$$\iff \exists \hat{x} \in R^{n}_{+} \mid \Lambda x - \Lambda \hat{x} \in \hat{L}(\Omega u)$$

$$\iff \Lambda x \in \hat{L}(\Omega u) + R^{n}_{+} = \hat{L}^{+}(\Omega u),$$

where the last equivalence exploits the positivity of Λ.

Consequently,

$$D(\Omega u, \Lambda x, \hat{L}^{+}) = \max_{\alpha} \{ \alpha \in R_{+} \mid \Lambda x/\alpha \in \hat{L}^{+}(\Omega u) \}$$

$$= \max_{\alpha} \{ \alpha \in R_{+} \mid x/\alpha \in L^{+}(u) \}$$

$$= D(u, x, L^{+}). \; \|$$

Theorems 3 and 5 indicated that (WM) and (H) are separately compatible with (D). This raises the question of whether (WM) and (H) are jointly consistent with (D). In fact, Bol [1985] has shown that, given (I), (WM) and (H) are incompatible whether or not (D) is satisfied.

Theorem 5. There does not exist an E satisfying (I), (WM), and (H) for all $L \in \underline{L}$.

Proof of this theorem can be found in Bol [1985], although he states the result in the weaker form with (M) substituted for (WM).

Thus, in order to salvage (WM) and (H), we would have to modify (I). To this end, consider the condition

(WI) $E(u,x,L) = 1$ iff x is weakly efficient.

As it turns out, this indication condition is incompatible with homogeneity:

Theorem 6. There does not exist an E satisfying (WI) and (H) for all $L \in \underline{L}$.

Proof: Consider any level set with $\hat{x} \in L(u)$ where $\hat{x} \geq 0$. By (L1), $\lambda\hat{x} \in L(u)$ for $\lambda > 1$. Since $L(u) \subseteq R^n_+$, $\hat{x}$ and $\lambda\hat{x}$ are necessarily weakly efficient, so that (WI) implies $E(u,\hat{x},L) = E(u,\lambda\hat{x},L) = 1$. But this violates (H). ‖

It is, however, possible to salvage (WM) and (H) if we weaken (WI) to

($\overset{\circ}{W}I$) For all $x > 0$, $E(u,x,L) = 1$ iff x is weakly efficient.

This condition has some appeal (as compared to (WI)) because input vectors on the boundary of R^n_+ are trivially weakly efficient and, consequently, a measure satisfying (WI) provides no (indication) information about vectors on the boundary. Define the efficency measure E^+_{DF} by

$$E^+_{DF}(u,x,L) = \begin{cases} D(u,x,L^+)^{-1} & \text{if } x \in L(u) \\ +\infty & \text{if } x \notin L(u) \end{cases}$$

where, it will be recalled, $L^+(u) = L(u) + R^n_+$. This modification of the Debreu/Farrell measure is equivalent to the "weak input measure of technical efficiency" formulated by Färe and Grosskopf [1983] (see, also, Färe, Lovell, and Grosskopf [1985, pp. 57-64]).

Theorem 7. E^+_{DF} satisfies (D), ($\overset{\circ}{W}I$), (WM), and (H) $\forall L \in \underline{L}$.

Proof: The invariance of $D(u,x,L^+)$ with respect to changes in units of measurement was shown in the proof of Theorem 4. To prove that E^+_{DF} satisfies ($\overset{\circ}{W}I$), suppose that $x^* > 0$, x^* is weakly efficient, and $E(u,x^*,L) < 1$. Then $x^* \in L(u)$ and $\lambda x^* \in L^+(u)$ for some $\lambda < 1$. Thus, there exists an $\hat{x} \in L(u)$ such that $\hat{x} \leqq \lambda x^* < x^*$, contradicting the supposition that x^* is weakly efficient. Hence, $E(u,x^*,L) = 1$.

Now suppose that x^* is not weakly efficient. Then there exists an $\hat{x} \in L(u)$ such that $\hat{x} < x^*$. If $\epsilon = \min \{x_i^* - \hat{x}_i | i = 1,\ldots,n\}$, and $\lambda = (\|x^*\| - \epsilon)/\|x^*\|$, then $\lambda x^* \geq \hat{x}$. Hence $\lambda x^* \in L^+(u)$. However, as $\lambda < 1$,

$$E(u,x^*,L) = D(u,x^*,L^+)^{-1} < 1,$$

completing the proof of $(\mathring{WI})$.

To prove that E_{DF}^+ satisfies (WM), consider $\hat{x} \in L(u)$ and $x \in L(u)$ satisfying $\hat{x} \geq x$. By the definition of D, $x/D(u,x,L^+) \in L^+(u)$. Moreover, as $\hat{x}/D(u,x,L^+) \geq x/D(u,x,L^+)$, we have $\hat{x}/D(u,x,L^+) \in L^+(u)$. Consequently, $D(u,\hat{x},L^+) \geq D(u,x,L^+)$.

Finally, the property (H) follows immediately from the first-degree homogeneity of the distance function D. ‖

4. Concluding Remarks

The summary and upshot of the foregoing results are as follows:

(1) If we insist on the fundamental dimensionality property (insensitivity to changes in units of measurement), (D), we can't have monotonicity (M).

(2) If we weaken monotonicity to (WM), we can have indication of efficient vectors (I) or homogeneity (H), but not both.

(3) We can have (H) as well as (WM) if we modify (I) to the weak indication condition $(\mathring{WI})$.

(4) Consequently, in the the context of the foregoing axiom system, we must choose between

(a) (D) ∩ (I) ∩ (WM), satisfied by E_{FL},

and

(b) (D) ∩ ($\mathring{WI}$) ∩ (WM) ∩ (H), satisfied by E^{+}_{DF}.

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