ECONOMIC RESEARCH REPORTS

LEARNING TO BELIEVE IN SUNSPOTS

by

Michael Woodford

R.R. #86-16

May 1986

C. V. STARR CENTER
FOR APPLIED ECONOMICS

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

LEARNING TO BELIEVE IN SUNSPOTS *

Michael Woodford

Columbia University
and
New York University

June 1986

Preliminary draft; Comments welcome

Abstract

An adaptive learning rule is exhibited for the Azariadis (1981) model under which the economy may converge to a stationary sunspot equilibrium, even if agents do not initially believe that outcomes are significantly different in different "sunspot" states.

A number of authors have shown that competitive economies may possess "sunspot equilibria", that is, rational expectations equilibria in which purely extrinsic uncertainity affects equilibrium prices and allocations.[1] Such results demonstrate that it does not require a lack of faith in the rationality of market participants to believe that competitive markets may be subject to purely speculative fluctuations, driven solely by expectations.[2]

The mere existence of sunspot equilibria as solutions to a system of market-clearing conditions, however, might not be judged sufficient to indicate that competitive markets with rational participants could ever be subject to speculative instability. The sunspot equilibria represent states of affairs in which agents act differently in the case of different realizations of the "sunspot" variable, and it is rational for each agent to do so. But it might be thought unlikely that the beliefs of all the participants in the market could ever come to be coordinated so as to bring about an equilibrium of that kind. It is rational to believe that sunspots convey information about future states of affairs once the economy is in a sunspot equilibrium, but why would rational agents ever begin to believe in such a thing, so as to create the conditions under which the belief is rational?

In order to address such a question, one must go beyond the mere statement of the conditions for equilibrium and discussion of what states of affairs satisfy them; one must specify an explicit dynamic process according to which the beliefs of agents adjust when out of equilibrium. Any exercise of this kind is necessarily unsatisfactory, as there is no univocal meaning for the postulate of "rational" behavior outside of an equilibrium.[3] It may well be the case that different "learning" processes -- all equally plausible or implausible, in that all satisfy some weak criteria for rational decision-making and all involve quite arbitary choices -- yield different conclusions as to the stability of a given equilibrium. Yet there seems no other way to address doubts about the economic significance of sunspot equilibria. And the exercise is not without value, even when it must be inconclusive. An example of instability of the non-sunspot rational expectations equilibrium, even for a particular learning rule, indicates that a coherent story can be told in which speculative instability arises in a competitive economy. And contrariwise, an example of stability of the non-sunspot equilibrium even when sunspot equilibria exist would indicate that competitive economies may be less subject to speculative fluctuations than a mere consideration of the set of equilibria would suggest.

In fact, an exercise of this kind is, in our view, more interesting in a case where there are multiple rational expectations equilibria to which the learning process might converge, than in the case more often considered in the learning literature, where there is only one. For in the usual case, if the analysis indicates that the learning process fails to converge to the rational expectations equilibrium, one may well conclude that agents will not in fact follow the postulated rule of inference forever, as they should eventually realize that their forecasts are not becoming more accurate despite the accumulation of data.[4] In the case of multiple equilibria, by contrast, it is possible for one equilibrium to be found to be unstable under a learning process that nonetheless converges to another equilibrium, so that agents need not ever modify their rule of inference (and hence possibly somehow get back to the equilibrium that was unstable under the original rule).

In section I, we review the Azariadis (1981) example of a simple infinite horizon general equilibrium model for which stationary sunspot equilibria may exist. In section II, we introduce a plausible rule by which agents in this economy might seek to learn whether the sunspot variable is of any use in forecasting future variables of interest to them, and in section III we discuss the convergence of the dynamics generated by this learning process to rational expectations equilibrium. We exhibit conditions under which the non-sunspot stationary equilibrium is unstable under the learning process, and under which certain sunspot equilibria are, on the contrary, stable. Section 4 concludes.

I. STATIONARY SUNSPOT EQUILIBRIA IN THE AZARIADIS MODEL

The example of an infinite horizon economy with stationary sunspot equilibria considered here was first presented by Azariadis (1981). The analysis of this example has subsequently been extended by Azariadis and Guesnerie (1982, 1984), Spear (1983), and Grandmont (1985b). We choose to consider this example not only because it is so well-known, but because models with stationary sunspot equilibria of this kind are of interest for modeling repetitive "business cycle" fluctuations. We re-derive here a sufficient condition for the existence of stationary sunspot equilibria in this model, because certain topological concepts used in this proof also play an important role in the stability analysis of section III. (Our analysis here is a variant of that given in Azariadis and Guesnerie (1982).

Consider a stationary overlapping generations exchange economy in which all agents live for two consecutive periods, there is a single perishable consumption good each period, and fiat money is the only asset. Let us suppose further that all agents have identical preferences, represented by a utility function $u(c)-v(n)$, where n is the amount of labor supplied during the first period of life, and c is the amount of the single good consumed during the second period of life.[5] The good is produced at constant returns to scale using the labor of the young, and units are chosen so that one unit of labor produces one unit of the good.

The utility function is assumed to satisfy

(A.1) u,v are C^3; $u(c)$ is defined for all $c > 0$, $v(n)$ for all $0 \leq n < \bar{n}$;

(A.2) $u' > 0$, $u'' < 0$, $v' > 0$, $v'' > 0$, for all $c > 0$, $0 \leq n < \bar{n} < \infty$;

(A.3) $v'(n) \to \infty$ as $n \to \bar{n}$; and

(A.4) there exists $\epsilon > 0$ such that $u'(n) > v'(n)$ for all $n \leq \epsilon$.

The differentiability assumed in (A.1) is necessary in order for us to be able to use results from the stability analysis of smooth dynamical systems in analyzing convergence in section III. Conditions (A.2) state that preferences are monotone and concave in consumption and leisure. (A.3) is assumed in order to avoid the possibility of a corner solution for optimal labor supply (this simplifies some equations but is inessential). Condition (A.4) is the only assumption that is at all restrictive; this is the condition (defining the "Samuelson case" in Gale's (1973) typology) well known to be necessary in order for an equilibrium to exist with valued fiat money.

A young agent in period t, facing a known price p_t for his period t supply and expecting a (possibly stochastic) price p_{t+1} for the good in period t+1, chooses his labor supply n_t to maximize

$$E_t[u(p_t n_t/n_{t+1}) - v(n_t)] \tag{1.1}$$

where E_t denotes the expectations of agents in period t. (Throughout, we assume that all agents form expectations in the same way.) Since (1.1) is a concave function of n_t, the optimal labor supply is the unique n_t satisfying

$$v'(n_t) = E_t\left[\frac{p_t}{p_{t+1}} u'\left(\frac{p_t n_t}{p_{t+1}}\right)\right]$$

assuming that there is a solution for $n_t \geq 0$. If $E_t(p_t/p_{t+1}) < v'(0)/u'(0)$, there is no solution and $n_t = 0$. In either case, n_t is the unique solution to

$$(1.2) \qquad n_t v'(n_t) = E_t\left[\frac{p_t n_t}{p_{t+1}} u'\left(\frac{p_t n_t}{p_{t+1}}\right)\right]$$

We assume a constant supply of fiat money $M > 0$. This must be held by the old at the beginning of each period; hence the consumption demand each period is $c_t = M/p_t$. Goods market clearing then implies $p_t n_t = M$ each period, which together with (1.2) implies

$$(1.3) \qquad n_t v'(n_t) = E_t[n_{t+1} u'(n_{t+1})].$$

A stationary rational expectations equilibrium (s.r.e.e.) is then a stationary stochastic process for n_t that satisfies (1.3).

One stationary equilibrium is $n_t = n^*$ for all t, where n^* is the unique solution to $u'(n^*) = v'(n^*)$. ((A.1) - (A.4) imply that the solution exists, is unique, and satisfies $0 < n^* < \bar{n}$.) This is the familiar monetary steady state of the overlapping generations model of fiat money. If $\lim_{c \to 0} cu'(c) = 0$, then another stationary solution is $n_t = 0$ for all t. This is the familiar non-monetary (autarchic) steady state of the overlapping generations model. These are the only equilibria in which n_t is constant for all t. Azariadis shows that there may also exist s.r.e.e. in which prices and allocations are stochastic, despite the absence of any random element in preferences, endowments, or technology.

Azariadis considers the case in which agents observe a random variable s_t (the "sunspot" variable), which takes a finite number of values $\{1,\ldots,m\}$ and follows a Markov process with transition probabilities $\pi_{ij} > 0$ for $i,j = 1,\ldots,m$ (π_{ij} = probability of moving to state j from state i). The sunspot variable has no effect upon the economy except through agents' expectations that may be conditioned upon it. He considers the existence of stationary rational expectations equilibria in which $n_t = n_i$ whenever $s_t = i$, for $i = 1,\ldots,m$. In this case, (1.3) becomes a system of m coupled equations for $(n_1,\ldots,m)$:

$$n_j v'(n_j) = \Sigma_k \pi_{jk} n_k u'(n_k) \tag{1.4}$$

for $j = 1,\ldots,m$. A stationary sunspot equilibrium is any s.r.e.e. (i.e., solution to (1.4)) in which $n_i \neq n_j$ for some i,j.

If we define, for any vector $n \in (0,\bar{n})^m$, the vector F by

$$F_j(n) = \frac{\Sigma_k \pi_{jk}(n_k u'(n_k)) - n_j v'(n_j)}{n_j[v''(n_j) - \Sigma_k \pi_{jk}(n_k/n_j)^2 u''(n_k)]}$$

then s.r.e.e. in which money is always valued are just zeroes of F. (The denominator, which is always positive, is included in the above expression in order to simplify certain expressions that arise in connection with the stability analysis of section III.) Let us assume the regularity condition

(R) At each s.r.e.e. with $n \gg 0$, no eigenvalue of DF has zero real part. Consequently, $\Delta = (-1)^m \operatorname{Det} DF \neq 0$.

Condition (R) implies that 0 is a regular value[6] of the map F.

We now observe the following consequences of our assumptions on preferences.

Lemma 1. There exists an $\underline{n} > 0$ such that $0 < n_k \leq \underline{n}$, and $n_k \leq n_j < \bar{n}$ for all $j \neq k$, imply $F_k(n) > 0$.

Lemma 2. There exists an $\hat{n} < \bar{n}$ such that $\hat{n} \leq n_k \leq \underline{n}$, and $\underline{n} \leq n_j \leq n_k$ for $j \neq k$, imply $F_k(n) < 0$.

(The proofs, which are simple, are omitted.) Note that necessarily $\underline{n} < \hat{n}$, since we know that $F(n^*,\ldots,n^*) = 0$. The consequences of these Lemmas are depicted geometrically in Figure 1, for the case m=2. Two consequences are of interest to us. First, one observes that all zeroes of F with $n \gg 0$ must lie in the interior of a cube $B = [\underline{n},\hat{n}]^m$. Second, one observes that the vector field F points into the interior at all points on the boundary of B.

Since no zeroes of F lie on the boundary of B, we can define a map $\phi: \partial B \to S^1$ by $\phi(n) = F(n)/|F(n)|$. Because F points into the interior of B at all points on the boundary, ϕ is a map of of degree one.[7]

Then the Poincare-Hopf index theorem[8] immediately yields the following:

Theorem 1. Let s.r.e.e. be assigned an index of +1 if $\Delta > 0$, and -1 if $\Delta < 0$, where Δ is defined in (R). Then the sum of the indices of all s.r.e.e. other than $n = 0$ is +1.

Theorem 1, in turn, has several immediate consequences for the existence of s.s.e.

Corollary 1.1. The number of s.s.e. is even.

Corollary 1.2. If any s.s.e. exist, at least one has an index +1.

Corollary 1.3. If $u'(c) + cu''(c) > 0$ for all $c > 0$ (i.e., preferences satisfy the condition of gross substitutability[9]), then no s.s.e. exist.

If the condition of gross substitutability holds, $\Delta > 0$ at any s.r.e.e. $n \gg 0$. For let F_{jk} denote the partial derivative of F_j with respect to n_k. Then for any $j \neq k$,

$$F_{jk} = \frac{\pi_{jk}[u'(n_k) + n_k u''(n_k)]}{n_j[v''(n_j) - \Sigma_h (n_h/n_j)^2 u''(n_h)]}$$

Gross substitutability implies that the numerator is positive, while (A.2) implies that the denominator is. Hence all off-diagonal elements of DF are positive. Furthermore, regardless of whether gross substitutability holds, one can show that $[DF]n = -n$ at any s.r.e.e. $n \gg 0$. Hence all diagonal elements of DF are negative in the case of gross substitutability, and DF has a dominant diagonal. By the familiar theorem of McKenzie,[10] it follows that all eigenvalues of DF have negative real part, so that Det DF has the same sign as $(-1)^m$, and $\Delta > 0$. Hence all s.r.e.e. other than $n = 0$ have an index of +1, and there can be only one, the monetary steady state.

Corollary 1.4. If $\Delta < 0$ at the monetary steady state, then there exist at least two s.s.e., and at least two of them have index +1.

Corollary 1.5 If

$$n^* v''(n^*) + n^* u''(n^*) + 2u'(n^*) < 0 \qquad (1.5)$$

(i.e., if perfect foresight equilibrium is indeterminate[11] near the monetary steady state), then there exist sets of transition probabilities $\pi_{ij} > 0$ such that $\Delta < 0$ at the monetary steady state. Thus there exist sunspot variables for which s.s.e. exist.

Note that if (1.5) holds, $\Delta < 0$ for the probabilities $\pi_{12} = 1$, $\pi_{1j} = 0$ for all $j \neq 2$, $\pi_{j1} = 1$ for all $j \neq 1$, and $\pi_{jk} = 0$ for all $j,k \neq 1$. By continuity, Δ continues to be negative even when the zeroes are made small positive quantities.

Here we have drawn attention, not only to conditions for s.s.e. to exist, but also for them to have index +1, because the index is related to the stability of a s.r.e.e. under the learning dynamics discussed below. Finally, it should be noted that condition (1.5) can hold for well-behaved utility functions (even when both leisure and consumption are normal goods); hence s.s.e. of the kind described here can exist.

II. AN ADAPTIVE LEARNING PROCESS

We now wish to consider a learning process appropriate for this economy. Agents seek to determine the relevant parameters of the process generating the rates of return on money holdings $R_{t+1} = p_t/p_{t+1}$. We suppose that agents believe the rates of return to be generated by one of a certain class of stationary processes, and that they use standard statistical procedures to estimate the relevant parameters. In particular, agents use a rule of inference that is consistent if the rates of return really are generated by one of the models in that class. The set of models considered by the agents is too small to include the true process (which is non-stationary) generating the rates of return while learning takes place, but is broad enough to include the s.r.e.e. -- so that <u>if</u> the economy eventually converges to a s.r.e.e., the maintained assumption of the agents comes to be correct. In all of these respects, our learning process is similar to those considered for other types of dynamic economic models by Cyert and DeGroot (1974), Frydman (1982), Bray (1982,1983), Bray and Savin (1984), and Marcet and Sargent (1986).

We wish to consider whether the economy could ever converge to an equilibrium in which sunspots matter. Hence we must assume that agents entertain the possibility that the current state s_t provides information about the distribution of R_{t+1}, and that they determine how much difference s_t makes by looking at past outcomes. The simplest case which allows this issue to be addressed is that in which agents suppose that the distribution of R_{t+1} depends only upon s_t (if upon that). The class of models in which R_{t+1} is drawn independently from a distribution G_j whenever $s_j = j$, for $j = 1,\ldots,m$, is broad enough to include the true model in the case of any s.r.e.e. of the kind discussed above. (In a s.r.e.e., G_j is a discrete distribution that assigns probability π_{jk} to the value n_k/n_j, for each $k = 1,\ldots,m$.) We may suppose, then, that agents seek to estimate the relevant parameters of G_1 using the set of observed values of R_{t+1} in periods following periods in which $s_t = 1$, and independently seek to estimate the corresponding parameters of G_2 using the set of observed values for periods following periods in which $s_t = 2$, and so on. If R_{t+1} is in fact drawn from the same distribution regardless of the value of s_t, the estimates based upon the m sets of data should eventually converge, and agents' actions will cease to be any different in the different states.

The only parameter of the distribution of R values of interest to an agent is the optimal choice of n_t for a given distribution of possible values for R_{t+1}. Let us then suppose that agents seek directly to estimate this parameter, using the set of past observations. Let n_j denote the optimal choice of n_t when R_{t+1} is to be drawn from G_j, for $j = 1,\ldots,m$. A reasonable estimator $\hat{n}_j$ is then

$$(2.1) \qquad \hat{n}_j = \arg\max_{0 \le n \le \bar{n}} \left\{ \frac{1}{M} \Sigma_{m=1}^{M} [u(nR_m) - v(n)] \right\}$$

where $\{R_m\}$ is a sequence of M observations of R believed to have been drawn from distribution G_j. (Note that a maximum exists for some $n < \bar{n}$ because of (A.3).) This estimator is easily shown to be consistent.[12] That is, if a sequence of observations $\{R_m\}$ is in fact drawn from a single distribution G_j, then as $M \to \infty$,

$$\hat{n}_j \to n_j = \arg\max \int [u(nR) - v(n)] \, dG_j(R)$$

The estimator (2.1) is an example of what is known as an <u>off-line identification procedure</u> in the optimal control literature; that is, the entire data sequence is used again at each sucessive step in the procedure. A <u>recursive identification procedure</u>[13] requires only a constant finite number of quantities to be stored at any stage in the procedure. The following recursive procedure represents a "stochastic approximation"[14] to procedure (2.1). Two quantities, $\hat{n}_{jM}$ and $\hat{H}_{jM}$, are stored at the end of stage M; when a new observation R_{M+1} is obtained, these are updated according to the rule[15]

$$(2.2a) \qquad \hat{n}_{jM+1} = \hat{n}_{jM} + (M+1)^{-1}\hat{H}^{-1}_{jM+1}[R_{M+1}u'(R_{M+1}\hat{n}_{jM}) - v'(\hat{n}_{jM})]$$

$$(2.2b) \qquad \hat{H}_{jM+1} = \hat{H}_{jM} + (M+1)^{-1}[v''(\hat{n}_{jM}) - R^2_{M+1}u''(R_{M+1}\hat{n}_{jM}) - \hat{H}_{jM}]$$

Here $\hat{n}_{jM}$ is the estimate of n_j after M drawings from G_j have been observed, and $\hat{H}_{jM}$ is an estimate of minus the second derivative of the criterion function $V_j(n) = \int[u(nR) - v(n)]\, dG_j(R)$. The quantity $\hat{H}_{jM}$ plays the role here of the Hessian of the criterion function in Newton's method for numerical maximization of a function;[16] larger $\hat{H}_{jM}$ results in a smaller adjustment in response to a given size of gradient of the criterion function. The approximation of (2.2) to (2.1) becomes increasingly accurate as the number of data observed increases; hence not only is (2.2) an equally consistent procedure, but (2.2) has the same asymptotic efficiency as (2.1).[17] We therefore suppose that agents update their estimates of each of the n_j using the recursive procedure (2.3); stability analysis is greatly simplified as we need only trace the evolution of a finite number of state variables.

Let us suppose furthermore that in each period t, the young agents observe s_t and choose n_t accordingly, using the estimate of the best response that they form from observations of $\{R_m\}$ for periods $m = 1,\ldots,t-1$. That is, they choose n_t <u>before</u> updating the estimates $\hat{n}_j$ on the basis of their observation of R_t; this choice of n_t then determines p_t and hence R_t. This assumption simplifies the dynamic equations; if agents used their observations of R_t in choosing n_t, we would have to write down equations for the simultaneous determinaton of n_t and R_t. The simplified procedure is slightly less efficient (agents fail to use one available observation), but the simplification does not affect our analysis of asymptotic convergence,[18] since for large t the most recent observation has little effect upon the estimates anyway.

Then the complete state of the economy evolves stochastically as follows:

$$\hat{n}_{jt} = \hat{n}_{jt-1} + t^{-1}(q_{jt}\hat{H}_{jt})^{-1}[R_t u'(R_t\hat{n}_{jt-1}) - v'(\hat{n}_{jt-1})]\ \Psi_j(s_{t-1}) \tag{2.3a}$$

$$\hat{H}_{jt} = \hat{H}_{jt-1} + t^{-1}q_{jt}{}^{-1}[v''(\hat{n}_{jt-1}) - R_t^2 u''(R_t\hat{n}_{jt-1}) - \hat{H}_{jt-1}]\Psi_j(s_{t-1}) \tag{2.3b}$$

(2.3c) $q_{jt-1} = t^{-1}[\Psi_j(s_{t-1}) - q_{jt-1}]$

(2.3d) $R_t = \Sigma_j \hat{n}_{jt-1}\Psi_j(s_t)/\Sigma_j \hat{n}_{jt-2}\Psi_j(s_{t-1})$

for $j = 1,\ldots,m$, and where $\Psi_j(s) = 1$ if $s = j$, and 0 if $s \neq j$. Here $\hat{n}_j$ and $\hat{H}_j$ are the estimates described in (2.2). The state variable q_j indicates what fraction of the total number of observations of R thus far have ocurred in periods following periods in which the sunspot state was j. The functions Ψ_j indicate that $\hat{n}_j$ and $\hat{H}_j$ are updated in period t only if $s_{t-1} = j$. The observed rate of return R_t is determined by their relation $R_t = p_{t-1}/p_t = n_t/n_{t-1}$, and equation (2.3d) then follows from the behavioral rule $n_t = \Sigma_j \hat{n}_{jt-1}\Psi_j(s_t)$.

III. CONVERGENCE TO RATIONAL EXPECTATIONS EQUILIBRIUM

The asymptotic behavior of the system (2.3) can be studied using the method of Ljung (1977).[19] Briefly, it can be shown that as t grows larger, the stochastic trajectories of a system like (2.3) come progressively closer to following the deterministic trajectories of a certain system of differential equations. Asymptotic convergence of (2.3) to constant values for $(\hat{n}_j, \hat{H}_j, q_j)$ then depends upon whether trajectories of the differential equation system converge to a rest point or not.

Ljung treats algorithms of the form

(3.1a) $x_t = x_{t-1} + t^{-1}Q(x_{t-1}, z_t)$

(3.1b) $z_t = A(x_{t-1})z_{t-1} + B(x_{t-1})e_t$

where e_t is a vector of stochastic forcing variables with compact support. System (2.3) can be put in this form by writing

$$x_t = \begin{bmatrix} \hat{n}_{1t} \\ \cdot \\ \cdot \\ \hat{n}_{mt} \\ \hat{H}_{1t} \\ \cdot \\ \cdot \\ \hat{H}_{mt} \\ q_{1t} \\ \cdot \\ \cdot \\ q_{mt} \end{bmatrix}, \qquad z_t = \begin{bmatrix} n_t \\ n_{t-1} \\ s_t \\ s_{t-1} \end{bmatrix}, \qquad e_t = \begin{bmatrix} \Psi_1(s_t) \\ \cdot \\ \cdot \\ \cdot \\ \Psi_m(s_t) \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B(x_t) = \begin{bmatrix} \hat{n}_1 & \hat{n}_2 & \ldots & \hat{n}_m \\ 0 & 0 & \ldots & 0 \\ 1 & 2 & \ldots & m \\ 0 & 0 & \ldots & 0 \end{bmatrix}$$

Then the functions $Q(x,z)$ are given by equations (2.3a-c), with R_t in equations (2.3a-b) replaced by n_t/n_{t-1}.

The regularity conditions required for the Ljung therorem are easily seen to be satisfied; we use the set of conditions (A) of Ljung (1977). In particular, we observe that the functions $Q(x,z)$ are C^1 in x, and that A and B are Lipschitz continuous functions of x, as required. It remains only to verify condition (A.5): that, if for any fixed x, one constructs the sequence z_t^x defined by $z_0^x = 0$, $z_t^x = A(x)z_{t-1}^x + B(x)e_t$, then the sequence $t^{-1}\Sigma_{k=1}^t Q(x,z_k^x)$ converges with probability 1 to some f(x). In our case, z_t^x always takes on one of m^2 values, depending only upon the values of s_t and

s_{t-1}; let z^x_{ij} denote the value when $s_{t-1} = i$, $s_t = j$. Then the law of large numbers implies convergence to $f(x) = \Sigma_{ij} q^*_i \pi_{ij} Q(x, z^x_{ij})$, where q^*_i is the fraction of all periods in which state i occurs over the long run.

When these conditions are satisfied, Ljung shows that the asymptotic behavior of x_t approaches a trajectory of the deterministic differential equation $\dot{x} = f(x)$, where f(x) is defined above. For the system (2.3), the associated differential equation system is

$$\text{(3.3a)} \qquad \dot{n}_j = (q^*_j/q_j H_j)[\Sigma_k \pi_{jk}(n_k/n_j)u'(n_k) - v'(n_j)]$$

$$\text{(3.3b)} \qquad \dot{H}_j = (q^*_j/q_j)[v''(n_j) - \Sigma_k \pi_{jk}(n_k/n_j)^2 u''(n_k) - H_j]$$

$$\text{(3.3c)} \qquad \dot{q}_j = (q^*_j - q_j)$$

where the dots denote derivatives with respect to a rescaled time variable τ, which increases as $\tau_t = \Sigma^t_{s=1} s^{-1}$, and where we have dropped the hats on the state variables $\hat{n}_j$, $\hat{H}_j$.

If the system (3.3) is such that all trajectories converge to a unique fixed point, then with probability 1 the system (2.3) converges to that fixed point as well. This is what happens in the usual applications of the approach in control theory, and in the case of the learning mechanisms treated by Marcet and Sargent (1986). In our case, it is evident that the subsystem consisting of equations (3.3c) for the state variables q_j has a globally stable fixed point q^*; hence for the system (2.3), q_j converges to q^*_j with probability 1 (as also follows from the law of large numbers). But the full system (3.3) often does not have a globally stable fixed point.

Nonetheless, the system (3.3) reveals a good deal about the possible asymptotic behavior of (2.3). If an open subset D of $\mathfrak{R}^{3m}$ is such that almost all points in D lie on trajectories that eventually leave D under the dynamics

(3.3), then it follows that under the dynamics (2.3) as well, the state variables (n_j, H_j, q_j) eventually leave D with probability 1. Hence the state variables cannot converge to any constant values that are not a locally stable fixed point of the system (3.3).[20] On the other hand, if $\bar{x}$ is a locally (but not globally) stable fixed point of (3.3), then (2.3) converges to $\bar{x}$ with a positive probability (but less than 1), assuming that some pattern of stochastic realizations can take x_t to a neighborhood of $\bar{x}$ starting from the given initial conditions. If the system (3.3) has an attracting set that is not a fixed point (e.g., a limit cycle), then (2.3) need never converge. (The same is true if all trajectories of (3.3) do not eventually remain within some compact set, although we show below that this case can be ruled out for our example.) In such a case, presumably agents will eventually modify the learning process described above, to allow for more complicated hypotheses about the process generating $\{R_t\}$.

Inspection of (3.3) indicates that $\dot{n} = \dot{H} = \dot{q} = 0$ if and only if n,H,q satisfy $F(n) = 0$,

$$(3.4) \qquad H_j = v''(n_j) - \Sigma_k \pi_{jk} (n_k/n_j)^2 u''(n_k)$$

and $q_j = q_j^*$. Accordingly, the system (2.3) cannot converge except to a s.r.e.e. Conversely, one observes that to each s.r.e.e. $\tilde{n}$ there corresponds a unique fixed point $(\tilde{n}, \tilde{H}, q^*)$ of the dynamics (3.3); $\tilde{H}$ is simply given by (3.4).

One also observes that, for all $n,H,q \gg 0$, the sign of $\dot{n}_j$ is the same as the sign of $F_j(n)$. It follows from Lemmas 1 and 2 of section I that under the dynamics (3.3), n always moves into the interior of the cube B from any point on the boundary of B. If we furthermore define

$$\overline{H}_j = \max_{n\in B} [v''(n_j) - \Sigma_k \pi_{jk}(n_k/n_j)^2 u''(n_k)]$$

and $\underline{H}_j$ as the corresponding minimum value, and choose $\overline{q}_j, \underline{q}_j$ so that $0 < \underline{q}_j < q^*_j < \overline{q}_j < 1$, then all trajectories of (3.3) that start inside $\Re^{3m}_{++}$ eventually enter and remain forever within the rectangular region $D \subset \Re^{3m}$ defined by $\underline{n} \leq n_j \leq \hat{n}$, $\underline{H}_j \leq H_j \leq \overline{H}_j$, $\underline{q}_j \leq q_j \leq \overline{q}_j$. Hence specification of the rule used by agents when (2.2) implies a negative value for $\hat{n}_j$ or $\hat{H}_j$ is not important; whatever rule is specified will not affect the asymptotic dynamics.

The mere fact that the asymptotic dynamics remain confined to a compact set does not suffice to guarantee convergence to any of the s.r.e.e. contained within the cube B; a result of this kind is possible only if the dynamics (3.3) have a unique, globally stable fixed point,[21] or if (3.3) represents dynamics of the gradient type.[22] Still, it does suffice to insure that under the stochastic dynamics (2.3), the values of n_j are bounded away from zero in the long run with probability 1. Hence the learning dynamics specified will <u>never</u> lead agents to the s.r.e.e. in which money is not valued and $n = 0$. This suggests that the attention given in the overlapping generations literature to the possibility of a self-fulfilling hyperinflation may be misplaced.

Local stability of a fixed point of (3.3) is evaluated by linearizing (3.3) around the fixed point. One obtains

$$(3.5) \qquad \begin{bmatrix} \dot{n} \\ \dot{H} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} B & 0 & 0 \\ C & -I & 0 \\ 0 & 0 & -I \end{bmatrix} \begin{bmatrix} n-\tilde{n} \\ H-\tilde{H} \\ q-q^* \end{bmatrix}$$

where B is exactly the matrix DF of section I, and where the elements of C involve second and third derivatives of u and v evaluated at $\tilde{n}$. The eigenvalues of the matrix in (3.5) are just the eigenvalues of its three diagonal blocks, i.e., -1 (occurring 2m times) and the m eigenvalues of DF. Condition (R) then guarantees that every fixed point is hyperbolic, so that local stability depends only upon the linearization (3.5).[23]

We thus obtain our main result.

Theorem 2. The adaptive learning dynamics described by (2.3) cannot converge except to a s.r.e.e. with $n \gg 0$. They can converge to such a s.r.e.e. if and only if all eigenvalues of DF, evaluated at that s.r.e.e., have negative real parts.

This has the following consequences.

Corollary 2.1. If preferences satisfy the condition of gross substitutability (see Corollary 1.3), then the monetary steady state is stable (and no s.s.e. exist).

Corollary 2.2. Any s.r.e.e. with index -1 is unstable (i.e., the learning dynamics cannot converge to it). Hence if (1.5) holds, then there exist transition probabilities $\pi_{ij} > 0$ such that the monetary steady state is unstable.

Corollary 2.1 obtains because, as noted above, gross substitutability implies that all eigenvalues of DF have negative real part. More generally, Det DF must equal the product of the eigenvalues of DF, so that if $\Delta < 0$ (i.e., the index is -1), there must be an odd number of eigenvalues (and at least one) with positive real part. Hence Corollary 2.2 obtains. When the index of a s.r.e.e. is +1, one can say only that the number of eigenvalues with positive real part is even, while stability requires that that number be zero. However, we observed in section I that $[DF]n = -n$, so that -1 is always an eigenvalue of DF. Hence when the index of a s.r.e.e. is +1, the number of eigenvalues with positive real part must be an even integer no greater than m-1. Hence in the case m=2 (the case considered by Azariadis), we obtain:

Corollary 2.3. If m = 2, the learning dynamics can converge to a s.r.e.e. if and only if the s.r.e.e. has index +1.

Hence in the special case m = 2, we obtain the following results, as a consequence of Theorems 1 and 2.

Corollary 2.4. If m = 2, at least one s.r.e.e. is stable.

Corollary 2.5. If m = 2, and any s.s.e. exist, then at least one of the s.s.e. is stable.

Corollary 2.6. If m = 2, and the monetary steady state is unstable, there exist at least two s.s.e. that are stable.

These results establish that it is indeed possible for an adaptive learning process of the kind described in section II to lead the economy into a stationary sunspot equilibrium. This may occur with positive probability even if the monetary steady state is locally stable, as long as the initial estimates (n_1, n_2, H_1, H_2) are not exactly those needed for the monetary steady state, i.e., $n_1 = n_2 = n^*$ and $H_1 = H_2 = H^* \equiv v''(n^*) - u''(n^*)$. When the monetary steady state is stable, however, convergence to a sunspot equilibrium requires a long chain of low-probability events, that perturb agents' expectations sufficiently far from (n^*, n^*, H^*, H^*) for them to be attracted to the s.s.e. While this may occur with positive probability, it is probably safe to regard it as a relatively unlikely outcome.

On the other hand, when the monetary steady state is unstable, the learning dynamics cannot converge to it, even if expectations are initially quite close to (n^*,n^*,H^*,H^*), as long as the initial expectations are not exactly consistent with the steady state. If agents begin with exactly the estimates (n^*,n^*,H^*,H^*), then they will never be lead to change them. But even a slight deviation in the initial conditions will result in n_1 evolving separately from n_2 (and H_1 from H_2) due to sampling effects; the estimate that the suspot state matters slightly will then lead agents to act differently in the two states, in such a way as to produce outcomes that further reinforce this belief. The estimate of the extent to which the suspot state matters will grow, driving the economy away from the monetary steady state; it may eventually converge to one of the s.s.e. We cannot prove that it must (since the dynamics (3.3) may have limit cycles or a strange attractor), but simulations indicate that this can occur.

It is not clear that results like corollaries 2.4-2.6 are true for the case $m > 2$. Nonetheless, it is possible to construct robust examples with an arbitrarily large number of sunspot states, for which the monetary steady state is unstable and at least some of the s.s.e. are stable.

Suppose that, in addition to the sunspot variable s_t, agents also observe another, independent, sunspot variable r_t (the "moon-spots" of Azariadis and Guesnerie (1982)). Suppose that the variable r_t is also a Markov process, taking values in the set $\{1,\ldots,p\}$, with transition probabilities σ_{ab}. Then the total number of distinct states is mp. Every s.r.e.e. of the m-state model will also be a s.r.e.e. of the mp-state model (these are s.r.e.e. in which n_t does not depend upon r_t), although there will often be other s.r.e.e. of the mp-state model as well. How is the stability of s.r.e.e. of the m-state model affected if agents also observe (and seek to determine the effects of) another variable r_t? Stability depends upon the eigenvalues of DF. The mp-vector whose elements are e_{ja} $(j = 1,\ldots,m,\ \ a = 1,\ldots,p)$ is an eigenvector of DF, with eigenvalue λ, if and only if

$$(3.6) \qquad \textstyle\sum_{kb}\pi_{jk}\sigma_{ab}U_k e_{kb} = (V_j + \lambda n_j H_j)e_{ja}$$

where $U_j = u'(n_j) + n_j u''(n_j)$, $V_j = v'(n_j) + n_j v''(n_j)$, and H_j is given by (3.4). In the case of the m-state model, the eigenvector equation is simply

$$(3.7) \qquad \textstyle\sum_k \pi_{jk}U_k e_k = (V_j + \lambda n_j H_j)e_j.$$

It is evident that, if the m-vector e_j satisfies (3.7) with eigenvalue λ, then the mp-vector such $e_{ja} = e_j$ for all a satisfies (3.6), for the same eigenvalue λ. Hence if the m-state model has an eigenvalue with positive real part, the mp-state model also does, and we obtain:

Corollary 2.7. If a s.r.e.e. of the m-state model is unstable, that s.r.e.e. continues to be unstable if an independent p-state random variable is added to the model.

If the variable r_t is i.i.d., then σ_{ab} does not depend upon a, and all solutions of (3.6) must be such that e_{ja} is the same for all a. But then all solutions of (3.6) must solve (3.7), so that the eigenvalues of DF in the case of the mp-state model are exactly those of the m-state model (each repeated p times). Accordingly we obtain:

Corollary 2.8 If a s.r.e.e. of the m-state model is stable, that s.r.e.e.

continues to be stable if an i.i.d. p-state random variable is added to the model.

On the other hand, a s.r.e.e. of the m-state model may be stable when agents only consider the use of s_t in forecasting, and yet become unstable if agents begin to consider an additional, independent variable r_t, if r_t is not i.i.d. This possibility is illustrated by our results above regarding the possible instability of the monetary steady state. For the monetary steady state is a s.r.e.e. of a 1-state model, and if only the 1-state variable is considered in forecasting (i.e., if agents assume that the rate of return R_{t+1} is drawn from the same distribution G each period), the monetary steady state is necessarily stable.[24] (Recall that one eigenvalue of DF is always -1; when $m = 1$, this is the only eigenvalue.) Corollary (2.2) can then be interpreted as referring to a case in which $m = 1$ but an additional variable with $p > 1$ states is observed; evidently the stability of the monetary steady state is not preserved when another variable is considered. Note that these considerations also imply the following general result:

Corollary 2.9. If agents observe only i.i.d. sunspot variables, the monetary steady state is stable.

Corollary 2.8 implies that we can construct examples with an arbitrarily large number of sunspot states, in which the monetary steady state is unstable and at least some s.s.e. are stable. For Corollary 2.6 guarantees the existence of such cases when $m = 2$. But then by adding an i.i.d. p-state variable, one obtains an example with 2p states. One can then perturb slightly the transition probabilities in the 2p-state example without affecting the stability results; hence existence of an example of that kind does not depend upon factorization of the 2p sunspot states into a 2-state variable and a p-state i.i.d. variable.

Remark 1. Our stability result is related to that to Marcet and Sargent (1986). They consider the general class of models in which the true dynamics of some vector of state variable z_t are given by

$$z_t = T(\beta_t)z_{t-1} + V(\beta_t)u_t$$

where u_t is a vector of stochastic forcing variables and β_t is the matrix of parameters in the dynamic law that agents believe to be true, i.e.,

$$z_t = \beta_t x_{t-1} + \omega_t$$

Our system (2.3) does not have this structure, so we are unable to directly apply their result. Yet we can define an operator somewhat analogous to the Marcet-Sargent operator T that maps the structure agents believe in to the actual structure.

Suppose that agents' estimates remain fixed for some time at $\hat{n}$. These beliefs generate a distribution of values for R given by (2.3d). Whenever $s_t = j$, R_{t+1} takes the value $\hat{n}_k/\hat{n}_j$ with probability π_{jk}, for $k = 1,\ldots,m$. There is then an optimal choice n_j corresponding to this distribution G_j. Let it be denoted $n = T(\hat{n})$. A s.r.e.e. is then a fixed point of the map T. Furthermore, one can show that the derivative matrix DT, evaluated at a s.r.e.e., is equal to I + DF, where DF is defined in section I. A s.r.e.e. is a stable fixed point of T if and only if all eigenvalues of DT have modulus less than one; this is only possible if all eigenvalues of DT have real part less than one. Since for each eigenvalue λ of DF, $1+\lambda$ is an eigenvalue of DT, stability under the map T implies that all eigenvalues of

DF have negative real part, which implies stability under the dynamics (3.3). Hence stability of a fixed point under the map T is a sufficient (though not necessary) condition for it to be stable under the learning dynamics. Hence the insight of Marcet and Sargent, into the connection between the notion of "expectational stability" introduced by DeCanio (1979) and Evans (1983,1985) and stability under adaptive learning dynamics, appears to apply to an even broader class of models than is considered in their paper.

Remark 2. Our stability result is also related to that obtained by Grandmont (1985a) and Grandmont and Laroque (1985) in their analysis of deterministic temporary competitive equilibrium (t.c.e.) dynamics in an overlapping generations model like that considered here. They exhibit conditions upon an adaptive learning rule (designed to pick out repetitive deterministic cycles of certain orders) under which local uniqueness of the monetary steady state within the class of perfect foresight equilibria (p.f.e.) guarantees stability of the steady state under the t.c.e. dynamics. On the other hand, when p.f.e. is indeterminate near the steady state, they show that a period-2 cyclic p.f.e. exists, that for some learning rules the steady state is then unstable under the t.c.e. dynamics, and that the t.c.e. dynamics may then converge to the cyclic equilibrium instead.

A period-2 deterministic cycle is a limiting case of our (and Azariadis') s.s.e. -- the limit of a model with $m = 2$ in which π_{12} and π_{21} are reduced to zero. In that limit, condition (1.5) is equivalent to $\Delta < 0$ at the monetary steady state, so that determinacy of p.f.e. at the monetary steady state is both necessary and sufficient for stability of the steady state under our learning dynamics, and indeterminacy is a sufficient condition for the existence of a period-2 cyclic p.f.e. that is stable under our learning dynamics. Hence the result of Grandmont and Laroque, relating "stability of the backward perfect foresight dynamics" (i.e., determinacy of p.f.e.) and stability under learning dynamics, is obtained as a limiting case of our Theorem 2.

IV. CONCLUSIONS

Our results indicate that stationary sunspot equilibria cannot be dismissed as a possible positive model of economic fluctuations on the ground that rational agents could never come to have such beliefs. We have exhibited an example in which, when agents use a plausible learning rule (one that is not optimal, but that agents will not have reason to reject if it leads to convergence to a rational expectations equilibrium), the economy can converge to a stationary sunspot equilibrium.

Our results also indicate that the mere consideration of stability under learning dynamics does not remove the large multiplicities of rational expectations equilibria found for many economic models. It might have been hoped that one could show that, in the case of some reasonable sort of learning rule, only one of the s.r.e.e. is stable, so that one could then confidently predict that as the long run state of the economy. We have found, however, that while considerations of stability under learning dynamics can rule out <u>some</u> rational expectations equilibria (e.g., the monetary steady state under certain conditions, several equilibria may all be locally stable. (In the case of the model considered here, whenever multiple s.r.e.e. exist with $n \gg 0$, at least two of them must be locally stable.) When several locally stable equilibria exist, which one the economy eventually converges to (if it converges) depends upon random events during the learning process -- not simply upon tastes, endowments, and technology. Not even specification of the initial state of expectations is enough to allow precise prediction in many cases, for the space of possible initial conditions cannot be divided into disjoint basins of attraction of the several fixed points, as in the case of a deterministic system.

The long run state of the economy is even more indeterminate if one supposes that more than one publicly observable random variable exists that may play the role of a "sunspot variable". Even if one supposes that all extrinsic variables follow two-state Markov processes, the set of transition matrices for which s.s.e. exist can be very large. For example, if (1.5) holds, there is an (m^2-m)-dimensional manifold of matrices such that $\Delta < 0$ at the monetary steady state. If agents seek to determine the effects of <u>any</u> variable of this kind, their actions drive them away from the monetary steady state and may result in convergence to one of the two or more s.s.e. But the set of s.s.e. that may be reached is different for each such transition matrix, i.e., it depends which random variable agents first decide to try using in their forecasts. If one takes into account "sunspot" variables that follow more complicated Markov processes as well,[25] or sunspot equilibria in which state variables depend upon lagged as well as current realizations of the sunspot variable,[26] the set of s.s.e. to which the economy can converge is expanded still further. It is impossible to predict the long run state of such an economy independently of the vagaries of the historical process through which agents reach a consensus about what to expect of each other.[27]

FOOTNOTES

* The author gratefully acknowledges the C.V. Starr Center for Applied Economics for its support and technical assistance.

1 The earliest examples of sunspot equilibria in general equilibrium models were given by Shell (1977) and Cass and Shell (1983). Equilibria of this kind in ad hoc macroeconomic models were previously exhibited by Black (1974), Taylor (1977), Shiller (1978), and others. Other general equilibrium examples are cited in section I.

2 For a discussion of the potential relevance of such models for equilibrium business cycle theory, see Woodford (1986b).

3 Townsend (1983a, 1983b) seeks to ground the evolution of beliefs upon Bayesian updating, but his solution to the problem requires an assumption of pre-existing coordination of agents' beliefs at a higher level, e.g., common knowledge of a certain covariance matrix describing the joint distribution of agents' beliefs. On the general need for more than a simple assumption that agents understand the structure of the economy in order to derive determinate learning dynamics, see Frydman (1982).

4 See, e.g., the comment of Marcet and Sargent (1986), foonote 11.

5 This specification follows Azariadis (1981). A model in which agents supply labor and consume in both periods of life results in equilibrium conditions of the same form as those derived here, so that extension of the results to that case is trivial. See, e.g., Grandmont (1985a). In that case, n_t in the equations below is to be interpreted as excess supply by the young (i.e., labor supplied in excess of their own consumption), and c_t as excess

demand by the old.

The model presented here also has a structure quite similar to the cash-in-advance monetary economy of Lucas and Stokey (1984), in the case of no endowment shocks or money growth shocks. (On the existence of stationary sunspot equilibria in the Lucas-Stokey model, see Woodford, 1986b, sec. 2A.) Hence the results of this paper immediately indicate that the monetary steady state may be unstable, and that stationary sunspot equilibria may be stable under adaptive learning dynamics in that model as well.

[6] Mas-Colell (1985), p. 37.

[7] Ibid., p. 46.

[8] Milnor (1965). For applications of this theorem to general equilibrium theory, see Mas-Colell (1985), pp. 188-222.

[9] The condition of gross substitutability is also known to rule out deterministic cycles in this model (Grandmont, 1985a), and to preclude indeterminacy of perfect foresight equilibrium near the monetary steady state (Woodford, 1984). For extension of both of those results to the general stationary overlapping generations exchange economy, see Kehoe et al. (1986).

[10] See Theorem 4.C.2 of Takayama (1974), p. 382.

[11] Woodford (1984).

[12] For example, this can be proved using the Ljung (1977) differential equation approach discussed in section III.

[13] For the contrast between off-line and recursive identification procedures, see Ljung and Söderström (1983), chapter 2, and secs. 3.3-3.4.

[14] Ibid., sec. 2.4.

[15] Equations (2.2) must be modified so as to prevent $\hat{n}_j$ from ever leaving the interval $(0,\bar{n})$, and $\hat{H}_j$ from ever leaving $(0,\infty)$. We do not bother to

write this explicitly, since, as shown in section III, the boundary modifications have no effect upon the asymptotic dynamics.

[16] Ljung and Söderström (1983), secs. 2.4.5-2.4.6 and 3.5.2-3.5.3.

[17] Ibid., sec. 4.2.1.

[18] See the discussion by Marcet and Sargent (1986), sec. 7.

[19] See also the exposition of Ljung's method in Ljung and Söderström (1983), sec. 4.3.3. The technique was first applied to the problem of convergence to rational expectations equilibrium by Marcet and Sargent (1986).

[20] Ljung (1977), Theorem 2.

[21] Ibid., Theorem 1.

[22] Ibid., Corollary 2 to Theorem 2.

[23] Hirsch and Smale (1974), p. 187.

[24] This learning process is closely related to the one for which Lucas (1985) shows the monetary steady state to be stable.

[25] Grandmont (1985b).

[26] Farmer and Woodford (1984), Woodford (1986a).

[27] This does not, however, mean that the model implies no testable predictions about the long run state of the economy, if we assume that it must converge to <u>some</u> s.r.e.e. Certain restrictions upon the data are implied by all s.r.e.e. of a given model; see Woodford (1986b), sec. 4.

REFERENCES

Azariadis, C., "Self-fulfilling prophecies," J. Econ. Theory 25 (1981), 380-396.

Azariadis, C., and R. Guesnerie, Propheties creatrices et persistance des theories, Revue Economique 33 (1982), 787-806.

————, "Sunspots and cycles", CARESS w.p. no. 83-22, revised March 1984.

Black, F., "Uniqueness of the price level in monetary growth models with rational expectations," J. Econ. Theory 7 (1974), 53-65.

Bray, M.M., "Learning, estimation, and the stability of rational expectations," J. Econ. Theory 26 (1982), 318-339.

————, "Convergence to rational expectations equilibrium," in Frydman and Phelps (1983).

Bray, M.M., and N.E. Savin, "Rational expectations equilibria, learning, and model specification," Econ. Thy. Disc. Paper no. 79, Univ. Cambridge, June 1984.

Cass, D., and K. Shell, "Do sunspots matter?" J. Political Econ. 91 (1983), 193-227.

Cyert, R.M., and M.H. DeGroot, "Rational expectations and Bayesian analysis," J. Political Econ. 82 (1974), 521-536.

DeCanio, S.J., "Rational expectations and learning from experience, Q.J. Econ. 93 (1979), 47-57.

Evans, G., "The stability of rational expectations in macroeconomic models," in Frydman and Phelps (1983).

————, "Expectational stability and the multiple equilibrium problem in linear rational expecations models," Q. J. Econ. 100 (1985), 1217-1234.

Farmer, R.E.A., and M. Woodford, "Self fulfilling prophecies and the business cycle," CARESS w.p. no. 84-12, Univ. Pennsylvania, April 1984.

Frydman, R., "Towards an understanding of market processes: Individual expectations, learning, and convergence to rational expectations equilibrium," Amer. Econ. Rev. 72 (1982), 652-668.

Frydman, R., and E.S. Phelps, eds., Individual Forecasting and Aggregate Outcomes, Cambridge U. Press, Cambridge, 1983.

Gale, D., "Pure exchange equilibrium of dynamic models," J. Econ. Theory 6 (1973), 12-36.

Grandmont, J.-M., "On endogenous competitive business cycles," Econometrica 53 (1985a), 995-1045.

———, "Stabilizing competitive business cycles," CEPREMAP w.p. no. 8518, Paris, August 1985b.

Grandmont, J.-M. and G. Laroque, "Stability of cycles and expectations," CEPREMAP w.p. no. 8519, Paris, revised July 1985.

Hirsch, M.W. and S. Smale, Differential Equations, Dynamical Systems, and Linear Algebra, Academic Press, New York, 1974.

Kehoe, T.J., D.K. Levine, A. Mas-Colell and M. Woodford, "Gross substitutes in large square economies," mimeo, U.C.L.A., April 1986.

Ljung, L., "Analysis of recursive stochastic algorithms," IEEE Trans. Auto. Control 22 (1977), 551-575.

Ljung, L. and T. Söderström, Theory and Practice of Recursive Identification, M.I.T. Press, Cambridge, Mass. 1983.

Lucas, R.E., Jr., "Adaptive behavior and economic theory," mimeo, U. of Chicago, October 1985.

Lucas, R.E., Jr., and N.L. Stokey, "Money and interest in a cash-in-advance economy," CMSEMS w.p. no. 628, Northwestern Univ., October 1984.

Marcet, A., and T.J. Sargent, "Convergence of least squares learning mechanisms in self referential linear stochastic models," mimeo, Stanford Univ., April 1986.

Mas-Colell, A., The Theory of General Economic Equilibrium: A Differentiable Approach, Cambridge U. Press., Cambridge, 1985.

Milnor, J., Topology from the Differentiable Viewpoint, Univ. of Va. Press, Charlottesville, Va., 1965.

Shell, K., "Monnaie et allocation intertemporelle," CNRS seminaire de E. Malinvaud, Nov. 21, 1977.

Shiller, R.J., Rational expectations and the dynamic structure of macroeconomic models, J. Monetary Econ. 4 (1978), 1-44.

Spear, S., "Sufficient conditions for the existence of sunspot equilibria," J. Econ. Theory 34 (1984), 360-370.

Taylor, J.B., Conditions for unique solutions to stochastic macroeconomic models with rational expectations, Econometrica 45 (1977), 1377-1385.

Takayama, A., Mathematical Economics, Dryden Press, Hinsdale, Ill., 1974.

Townsend, R.M., "Forecasting the Forecasts of Others," J. Political Econ. 91 (1983a), 546-588.

———, "Equilibrium theory with learning and disparate expectations: Some issues and methods," in Frydman and Phelps (1983b).

Woodford, M., "Indeterminacy of equilibrium in the overlapping generations model: A survey," mimeo, Columbia Univ., May 1984.

———, "Stationary sunspot equilibria in a finance constrained economy," mimeo, Columbia Univ., January, 1986a. Forthcoming in J. Econ. Theory.

———, "Expectations, finance, and aggregate instability," C.V. Starr Center w.p. no. 86-15, N.Y.U., March 1986b.