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THE DYNAMICS OF EFFICIENT INTERTEMPORAL ALLOCATIONS WITH MANY AGENTS, RECURSIVE PREFERENCES AND PRODUCTION*

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ABSTRACT

The local dynamics of efficient allocations are studied for heterogeneous-agent economies with production where preferences are recursive but not time-separable. In the case of a one-sector neoclassical technology, it is shown that optimal trajectories locally converge to a unique steady state, provided that every agent's utility satisfies normality and increasing marginal impatience conditions. In the two-sector case, with the above restrictions on preferences, it is shown that oscillations can result only if the consumption goods sector is sufficiently capital-intensive.

1. Introduction

Infinitely-lived representative agent models have been extensively used as microfoundations for macroeconomic theory. At the same time, some criticism has been directed towards these models for ignoring the complications that can arise when many agents are introduced. Recently the dynamics of efficient intertemporal allocations have been studied by Becker [1980], Bewley [1982], Yano [1984] and Coles [1983] under the assumptions of time-additively separable preferences for the agents and stationary production technologies. This paper, however, is based on the model of Lucas and Stokey [1984] who used the Koopmans [1960] framework of recursive intertemporal preferences for the agents in the economy. This framework allows the marginal rates of substitution between current consumption and future utility of different agents to adjust to each other. More recently, Epstein [1985a] studied the local dynamics of the Lucas-Stokey model in a continuous time one-sector economy. Using Uzawa [1968] utility functions, where the discount factors of agents are allowed to depend upon consumption and aggregate utility levels, and imposing the condition of increasing marginal impatience, he showed that the stability results for the continuous time, one-sector model carry over to the case of heterogeneous agents. In a later paper (Epstein [1985b]) he generalized the Uzawa functionals to a class of preferences having a recursive structure in continuous time, and obtained global stability results for multi-sector models under restrictive assumptions, termed q-conditions in the turnpike literature (see McKenzie [1981]).

It is well known that the stability results for one-sector models are not generally obtainable in multi-sector models in continuous time (see Benhabib and Nishimura [1979a]). Moreover, in the discrete time framework, even a two-sector model can lead to oscillations and "flip" cycles (see Benhabib and

Nishimura [1985]). The question, therefore, arises as to whether such instability can occur due to the heterogeneity of agents (or even whether this heterogeneity can contribute to stability) in multi-agent models in discrete time.

In this paper we study the local dynamics of efficient accumulation and allocations of agents' utilities for general specifications of recursive preferences in one-sector and two-sector models, respectively. For the one-sector model, we show in section 3 (Theorem 1) that if future utility is a normal "good" for all consumers, the local dynamics are nonoscillatory. Furthermore, if all consumers display increasing marginal impatience, then the steady-state is unique and locally stable. By contrast, in the cases where agents have additively-separable preferences over time, only those agents who share the highest discount factor own capital in the long run and the aggregate steady-state stock adjusts to this discount factor. Moreover, insofar as the highest discount factor is shared by more than one agent, the distribution of the steady-state capital stock as well as the steady-state utility levels of these agents cannot be determined independently of initial conditions.

In section 4 (Theorem 2), we show for a two-agent, two-sector economy, that if future utility is normal for both consumers, oscillations arise only if the consumption goods sector is sufficiently capital-intensive. The uniqueness and local stability of the steady-state follow from the imposition of increasing marginal impatience upon both agents and an additional restriction on technology. Thus, oscillations require either inferiority of future utility, or the relative capital-intensity of the consumption goods sector, or both (see Theorem 2 and the remarks that follow).

2. Technology and Preferences

In this section, the per-capita net output of the economy will be characterized by a nonnegative production function of capital, $f(k)$. At any time t, the output is divided between current consumption c_t and the capital stock for the next period k_{t+1}

(1) $$c_t + k_{t+1} = f(k_t) \qquad t = 0,1,\ldots$$

where we assume

(F.1) f is twice continously differentiable on $(0,\infty)$, with $f' > 0$, $f'' < 0$, and $\lim_{k\to 0} f'(k) = \infty$

(F.2) There exists a point $\hat{k} > 0$ such that $f(k) > k$ if $0 < k < \hat{k}$ and $f(k) < k$ if $k > \hat{k}$.

Let $K = [0,\hat{k}]$, $K^\infty = K \times K \times \ldots$. For $k_0 < \hat{k}$, the feasible correspondence $F:K \to K^\infty$ is defined by

(2) $$F(k_0) = \{(k_0,k_1,\ldots) \in K^\infty \mid 0 \le k_{t+1} \le f(k_t),\ t \ge 0\}$$

which gives the set of feasible capital accumulation paths attainable from the initial stock level k_0. We shall write ${}_0k = (k_0,k_1,\ldots)$.

Let $R_+^\infty = R_+ \times R_+ \ldots$ and ${}_0C = (c_0,c_1,\ldots)$. Let $G:K^\infty \to R_+^\infty$ be defined by

(3) $$G({}_0k) = \{(c_0,c_1,\ldots) \mid 0 \le c_t \le f(k_t),\ \ t \ge 0\}$$

where ${}_0k \in K^\infty$. Then the correspondence $C:K \to R_+^\infty$ defined by

(4) $$C(k_0) = \{{}_0C \in R_+^\infty, {}_0C = G({}_0k) \mid {}_0k = (k_0,k_1,\ldots) \in F(k_0)\}$$

gives the set of feasible consumption paths attainable from k_0. We impose the product topology on R^∞_+. It can be shown that C is upper semi-continuous in k_0. As K is a compact subset of R_+, $C(K) = \bigcup_{k_0 \in K} C(k_0)$ is a compact set (Hildebrand and Kirman [1976, p. 192]).

Following Lucas and Stokey [1984] we focus upon utility functions that can be expressed as

(5) $$U\,({}_0c) = V(c_0, U\,{}_1c))$$

where $V\colon R \times R \rightarrow R$ is an aggregator function satisfying:

(V.1) V is continuous and bounded.

(V.2) V is strictly concave.[1]

(V.3) $V(0,0) = 0$

(V.4) $(c,z) < (c',z')$ implies $V(c,z) < V(c',z')$

(V.5) $|V(c,z) - V(c,z')| \le \delta(z-z')$ for all $c \in R_+$; z, $z' \in R^+$; for some $0 \le \delta \le 1$.

Lucas and Stokey [1984, Theorem 1] have established that, under (V.1)-(V.5), every aggregator function V has associated with it a unique utility function U, which satisfies (5) and meets other regularity conditions. For the purposes of the local dynamics studied in this paper, we further assume

(V.6) $V(c,z)$ is twice continuously differentiable, with $V_1 = \partial V/\partial c > 0$, $V_2 = \partial V/\partial z > 0$, and $V_{11}, V_{22} < 0$.

Due to the nature of the optimization over heterogeneous agents, discussed in the next section, we wish to "invert" the aggregator function in order to specify current consumption as a function of current lifetime and next-period's lifetime utility.

For the j^{th} agent, let ${}_0c_j = c_{j,0}, c_{j,1}, c_{j,2}, \ldots$ denote a sequence of consumption allocations, and define

$$Z_{j,0} = \{z_{j,0} = U({}_0c_j) \mid \sum_{j=1}^{n} c_{j,t} \le c_t, \quad t \ge 0, \text{ and } {}_0c \in C(k)\}$$

as the set of initial lifetime utility levels attainable from the set of initial capital stocks.

Let $\kappa = \{k_1 | k_1 \in [0,f(k_0)], \ k_0 \in K\}$, be the set of second-period capital stocks feasible from initial stocks, and define

$$Z_{j,1} = \{z_{j,1} = U({}_1c_j) | \sum_{j=1}^{n} c_{j,t} \le c_t, \quad t \ge 1 \text{ and } {}_1c \in C(\kappa)\}$$

as the set of second-period lifetime utility levels attainable from the set of feasible second-period capital stocks. Then given monotonicity of V in c, we define the consumption mapping

$$(6) \qquad c_{j,0} = G^j(z_{j,0}, z_{j,1}) : Z_{j,0} \times Z_{j,1} \to R^+$$

as an inversion of the aggregator function. It is easily shown that the function G has the following partial derivatives.

<u>Lemma 1</u>:

(i) $G_1 = V_1^{-1} > 0$

(ii) $G_2 = -V_2 V_1^{-1} < 0$

(iii) $G_{11} = -V_{11} V_1^{-3} > 0$

(iv) $G_{22} = V_1^{-3}[2V_1 V_2 V_{12} - V_{22} V_1^2 - V_{11} V_2^2] \ge 0$

(v) $G_{12} = V_1^{-3}[V_2 V_{11} - V_1 V_{12}]$

Note that the sign of G_{12} determines precisely whether future utility is a normal good $(G_{12} < 0)$ or an inferior good $(G_{12} > 0)$. As such, it is not determined <u>a priori</u> in the context of this paper.[2]

For a separable utility function $\Sigma_{t=0}^{\infty} \delta^t u(c_t) = u(c_0) + \delta U_1$, the discount factor $0 < \delta < 1$ is independent of the levels of consumption. We shall define the consumer's subjective time discount factor from the aggregator function by $\delta(c) = V_2(c, U(c,c,\ldots,c))$. It can be shown that (V.6) implies $0 < \delta(c) < 1$ (see Beals and Koopmans [1969]). This is the impatience condition. If it fails, utilities will be unbounded along constant consumption paths. The following is the increasing marginal impatience condition:

(M) $\delta(c)$ is decreasing in c.

Under (V.6) we have the following result:

<u>Lemma 2</u>: (M) holds if and only if $V_{21}(1 - V_2) + V_{22}V_1 < 0$ where partial derivatives are evaluated at the constant consumption stream c.

<u>Proof</u>: Consider the derivative

$$\delta'(c) = \partial V_2(c, V(c,\ldots))/\partial c = V_{21} + V_{22}V_1(1+V_2+V_2^2+\ldots) = [(1-V_2)V_{21} + V_{22}V_1](1-V_2)^{-1}$$

since $0 < V_2 < 1$, $\delta'(c) < 0 \Leftrightarrow (1 - V_2)V_{21} + V_{22}V_1 < 0$. Q.E.D.

3. An n-agent Economy with Production

In this section, we provide a complete characterization of the local behavior of optimal paths in an n-agent economy with production. One of the main conclusions of this section is that a steady state is locally saddle-point stable if the agents' utilities satisfy the normality and increasing marginal impatience conditions. These assumptions imply the uniqueness of the steady state as well. The more detailed results are given in Theorem 1 below.

Consider maximizing over an infinite horizon the utility of agent 1, subject to technological constraints (given the initial capital stock) as well as utility constraints for the other n-1 agents. It can be rigorously shown that the solution to a planning problem of this type corresponds to a competitive eqilibrium for a production economy with many infinitely lived agents, where the utility constraints upon agents have been appropriately chosen to reflect endowments. (See Epstein [1985a] or Kehoe, Levine and Romer [1986]). The constraints upon the optimization problem are:

$$\begin{aligned} k_0 &\le \bar{k}_0 & & \\ u^i({}_0c_i) &\ge \bar{z}_{i,0} & & i=2,\ldots n. \\ c_{1,t} &\le f(k) - k - \Sigma_{i=2}^{n} G^i(z_{i,t},\ z_{i,t+1}) & & t=0,1\ldots \end{aligned}$$

In the following, all constraints will be assumed to hold with equality. Let $\underset{\sim}{z}_t = (z_{2,t},\ldots z_{n,t})$. Define $X = \{(k_0,\underset{\sim}{z}_0) \in K \times R_+^{n-1}\colon (z_{1,0},\underset{\sim}{z}_0) \in U(k_0)$ for some $z_{1,0} \in R_+\}$ and $U(k_0) = \{Z_{1,0},\ldots,Z_{n,0}\} \subset R_+^n$. $U(k_0)$ represents the vector of agents' utility possibility sets that are feasible from a given capital stock. Given that $(\bar{k}_0,\bar{\underset{\sim}{z}}_0) \in X$, the maximization problem may be expressed as

$$(7) \qquad w(k_0,\underset{\sim}{z}_0) = \max_{\{k_1,\underset{\sim}{z}_1\}} V[f(k_0) - k_1 - \Sigma_{i=2}^{n} G^i(z_{i,0},z_{i,1});\ w(k_1,\underset{\sim}{z}_1)]$$

where $k_0=\bar{k}_0$, $z_0=\bar{z}_0$ and optimal choices of k_1, $\underset{\sim}{z}_1$ must be such that $k_1 \in \kappa$ and $z_{i,1} \in Z_{i,1}$, $i=2,\ldots n$. The function V denotes the first agent's utility from the aggregator function.

Following Lucas and Stokey [1984] it can be shown that there exists a unique function $w\colon X \to R_+$ satisfying (7). This function is bounded, continuous, concave, differentiable, increasing in k and decreasing in $\underset{\sim}{z}$. The corresponding policy function is single valued and continuous, and generates all Pareto-optimal sequences of allocations.

The first order conditions for optimal choices of "next period" capital stock and agent 2,...n's utilities are

$$(8)\qquad k_{t+1}:\quad -V_1(c_{1,t},z_{1,t+1}) + V_2(c_{1,t},z_{1,t+1})w_1(k_{t+1},\underset{\sim}{z}_{t+1}) = 0$$

$$(9)\qquad z_{i,t+1}:\quad -V_1(c_{1,t},z_{1,t+1})G^i_2(z_{i,t},z_{i,t+1}) + V_2(c_{1,t},z_{1,t+1})w_i(k_{t+1},\underset{\sim}{z}_{t+1}) = 0, \quad i=2,\dots n; \quad t=0,1,$$

where

$$c_{1,t} = f(k_t) - k_{t+1} - \Sigma^n_{i=2}G^i(z_{i,t},z_{i,t+1})$$

$$z_{1,t+1} = w(k_{t+1},\ \underset{\sim}{z}_{t+1})$$

The partial derivatives of the value function satisfy the following envelope relations

$$(10)\qquad w_1(k_t,\underset{\sim}{z}_t) = V_1(c_{1,t},z_{1,t+1})\ f'(k_t)$$

$$(11)\qquad w_i(k_t,\underset{\sim}{z}_t) = -V_1(c_{1,t},z_{1,t+1})G^i_1(z_{i,t},z_{i,t+1}) \qquad t=0,1\dots, \quad i=2,\dots n;$$

Substituting from (10) and (11) into (8) and (9) respectively, and combining the resulting expressions yields

$$(12)\quad -V_1(c_{1,t},z_{1,t+1}) + V_2(c_{1,t},z_{1,t+1})V_1(c_{1,t+1},z_{1,t+2})f'(k_{t+1}) = 0$$

$$(13)\quad f'(k_{t+1})G^i_2(z_{1,t},z_{i,t+1}) + G^i_1(z_{i,t+1},z_{i,t+2}) = 0 \qquad t=0,1,\dots;$$

A steady state for the optimal program is an n-tuple $(\bar{k},\bar{z})$ that solves the following equations

$$(14)\quad f'(\bar{k}) = 1/(V_2(\bar{c}_1,\bar{z}_1))$$

$$(15)\quad V_2(\bar{c}_1,\bar{z}_1) = -G^i_2(\bar{z}_i,\bar{z}_i)/G^i_1(\bar{z}_i,\bar{z}_i) = V^i_2(\bar{c}_i,\bar{z}_i) \quad i=2,\dots n$$

where the superscript on the first agent's aggregator function remains suppressed. Theorem 4 in Lucas and Stokey [1984] implies that, with the one-sector neoclassical production technology and the imposition of (M) on all agents, the resulting steady state is unique. In order to see whether the optimal sequence in fact converges to this steady state, at least locally, we linearize the system around the latter and study its behavior for arbitrary deviations in initial conditions.

Totally differentiating (12) and evaluating variables at the steady state (where $f' = (V_2)^{-1} = (V_2^i)^{-1} = -G_1^i/G_2^i, \quad i=2,\dots n$):

$$(16) \qquad (V_1V_2^{-1}V_{12} - V_{11})Dc_{1,t} + (V_1V_2^{-1}V_{22} - V_{12})Dz_{1,t+1} + V_{11}Dc_{1,t+1} + V_{12}Dz_{1,t+2} + V_2V_1f''dk_{t+1} = 0$$

$$(17) \qquad Dc_{1,t} = f'dk_t - dk_{t+1} - \Sigma_{t=2}^n G_1^i dz_{i,t} - \Sigma_{i=2} G_2^i dz_{i,t+1} \qquad \forall t$$

$$(18) \qquad Dz_{1,t} = w_1 dk_t + \Sigma_{i=2}^n w_i dz_{i,t} = V_1[f'dk_t - \Sigma_{i=2}^n G^i dz_{i,t}\} \ \forall t$$

Substituting from (17) and (18) into (16) and using Lemma 1:

$$(19) \qquad f'G_{12}^1 dk_t + [G_{11}^1 + f'G_{22}^1 - (G_2^1)^2 f'']dk_{t+1} + G_{12}^1 dk_{t+2} + f'G_{12}^1 \Sigma_{i=2}^n G_2^i dz_{i,t} + [G_{11}^1 + f'G_{22}^1]\Sigma_{i=2}^n G_2^i dz_{i,t+1} + G_{12}^1 \Sigma_{i=2}^n G_2^i dz_{i,t+2} = 0$$

Differentiating the set of equations implied by (15) and solving for $dz_{i,t+2}$ as a function of dz_{it}, $dz_{i,t+1}$, dk_{t+1},

$$(20) \quad dz_{i,t+2} = -f'dz_{i,t} - (G_{11}^i + f'G_{22}^i)(G_{12}^i)^{-1}dz_{i,t+1} - G_2^i/G_{12}^i dk_{t+1} \qquad i=2,\dots n.$$

Plugging (20) into (19) and solving for dk_{t+2}:

$$(21) \quad dk_{t+2} = -f'dk_t + [(-G_{11}^1 + f'G_{22}^1)(G_{12}^1)^{-1} + \Sigma_{j=1}^n (G_z^j)^2 f''/G_{12}^j]dk_{t+1} + \Sigma_{i=2}^n[(f'G_{22}^i + G_{11}^i)(G_{12}^i)^{-1} - (f'G_{22}^1 + G_{11}^1)(G_{12}^1)^{-1}]G_2^i dz_{i,t+1}$$

(20) and (21) represent a second-order dynamic process in n variables. This can be transformed into an equivalent first-order process in $2n$ variables by defining the variables $x_{t+1} = k_t$, $y_{i,t+1} = z_{i,t}$, $i=2,\ldots n$. In addition let $R^j = (f'G_{22}^j + G_{11}^j)(G_{12}^j)^{-1}$ and $Q^j = (G_2^j)^2(G_{12}^j)^{-1}$, $j=1,\ldots n$.

Letting $a_t' = [dk_t, dz_{2,t}, \ldots, dz_{n,t}, dx_t, dy_{2,t}, \ldots dy_{n,t}]$ be a $2n$-dimensional vector, we can express the local dynamics implied by (20), (21) as

$$(22) \qquad a_{t+2} = J \, . \, a_{t+1} \qquad \text{where} \quad J = \left[\begin{array}{c|c} A & -f'I \\ \hline I & 0 \end{array}\right]$$

In the partitioning of the Jacobian J, I and 0 represent $n \times n$ identity and null matrices respectively, while A is an $n \times n$ matrix defined as

$$A = \begin{bmatrix} -R^1 + f''\Sigma_{j=1}^n Q^j & (R^2-R^1)G_2^2 \ldots\ldots (R^n-R^1)G_2^n \\ -f''Q^2(G_2^2)^{-1} & -R^2 \quad \ldots\ldots \quad 0 \\ \vdots & \\ -f''Q^n(G_2^n)^{-1} & 0 \quad \ldots\ldots \quad -R^n \end{bmatrix}$$

For local stability, the dynamic system (22) must have n roots inside and n roots outside the unit circle, and there must exist an n-dimensional stable manifold such that for any initial $(x_1, \underset{\sim}{y}_1)$ in the vicinity of the steady state, we can uniquely choose $(k_1, \underset{\sim}{z}_1)$ so that the path of capital stock and utilities converges to the steady state. The latter requires (as discussed in Scheinkman [1976]; see also Epstein [1985a]) that the stable manifold not be "vertical" at the steady state, for in that case the appropriate $(k_1, \underset{\sim}{z}_1)$ cannot be chosen as functions of $(x_1, \underset{\sim}{y}_1)$. At this point, we assume the following

(V.7) There exists an $|R^j| < |R^i|$ for all $i \neq j$.

Without loss of generality, we let agent j be agent 1, so that $|R^1| < |R^i|$ for all $i \neq 1$. This enables us to ensure that $(k_1, \underline{z}_1)$ can be chosen as functions of $(x_1, \underline{y}_1)$ near the steady state (for a proof, see Appendix 2).[3] The following theorem characterizes the roots of J when all agents have normal future utility:

<u>Theorem 1</u>: Let (F.1), (F.2); (V.1) - (V.7) hold, along with normality of preferences for all agents. Then the Jacobian in (22) has $2n$ characteristic roots which satisfy the following:

(i) For each root λ_j, there is a root f'/λ_j.

(ii) All roots are real and positive.

If, in addition, assumption (M) (increasing marginal impatience) holds, then:

(iii) For every λ_j inside (outside) the unit circle, its conjugate f'/λ_j is outside (inside) the unit circle.

<u>Proof</u>: See Appendix 1.

<u>Remark 1</u>: When (M) fails locally, saddle point instability will occur due to the emergence of multiple steady states, with each unstable steady state bracketed between a pair of stable ones.

<u>Remark 2</u>: Normality assumptions on preferences assure the nonoscillatory behavior of capital and utilities around the steady state. This follows because the roots of the matrix J are positive. When normality assumptions are relaxed we can expect oscillatory behavior. If future utility is inferior for one or more agents, the accumulation of capital will lead to higher current consumption and a reduction in the capital stock, giving rise to

oscillations. These oscillations get amplified the stronger is the inferiority, leading in some cases to the loss of saddle-point stability in the steady state (via a root crossing -1), and to the bifurcation of periodic cycles. (For a discussion in the representative agent case, see Iwai [1972] and Benhabib, Majumdar, and Nishumura [1985].) Indeed, the proof of Theorem 1 can easily be generalized to show that, for each agent having inferiority in future utility, J would have a pair of negative roots.

4. Extension to the Two-Sector Model

In this section, we consider the special case of a two-agent economy, but allow for a two-sector technology. The main results of this section are that when agents have concave, non-separable preferences, normality of future utility can lead to oscillations only if the consumption goods sector is sufficiently capital-intensive.

With the two-sector assumption, the consumption constraint upon the dynamic programming problem has to be modified as follows:

$$c_{1,t} \le T(k_t,k_{t+1}) - G^2(z_{2,t},z_{2,t+1})$$

where $T(k_t,k_{t+1})$ denotes the production possibility frontier of an economy producing consumption (c_t) and investment (k_{t+1}) goods with given per-capita stocks of capital (k_t). Full depreciation of capital is assumed in the above formulation. It can be shown that: $T_1 > 0$, $T_2 < 0$, $T_{11},T_{22} < 0$, and $T_{12} > (<)\ 0$ as the consumption goods sector is relatively labor (capital) intensive (see Benhabib and Nishimura [1985]).

Plugging the above constraint into the value function

$$(23)\quad w(k_t,z_{2,t}) = \underset{(k_{t+1},z_{2,t+1})}{\text{Max}} V[T(k_t,k_{t+1}) - G^2(z_{2,t},z_{2,t+1});\ w(k_{t+1},z_{2,t+1})]$$

which has envelope relations and first-order conditions

(24) $$w_1(k_t, z_{2,t}) = V_1(c_{1t}, z_{1,t+1})T_1(k_t, k_{t+1}) \qquad t=0,1,\ldots$$

(25) $$w_2(k_t, z_{2,t}) = -V_1(c_{1,t}, Z_{1,t+1})G_1^2(z_{2t}, z_{2,t+1}) \qquad t=0,1,\ldots$$

(26) $$V_1(c_{1,t}, z_{1,t+1})T_2(k_t, k_{t+1}) + V_2(c_{1,t}, z_{1,t+1})w_1(k_{t+1}, z_{2,t+1}) = 0$$

(27) $$V_1(c_{1,t}, z_{1,t+1})G_2^2(z_{2,t}, z_{2,t+1}) + V_2(c_{1,t}, z_{1,t+1})w_2(k_{t+1}, z_{2,t+1}) = 0$$

The steady state in this problem is a pair $(\bar{k}, \bar{z}_2)$ that solves

(28) $$V_2(\bar{c}_1, \bar{z}_1) = -(T_2(\bar{k})/T_1(\bar{k})) = -(G_2^2(\bar{z}_2)/G_1^2(\bar{z}_2))$$

which implies

(29) $$0 < V_2^1(\bar{c}_1, \bar{z}_1) = V_2^2(\bar{c}_2, \bar{z}_2) = -T_2(\bar{k})/T_1(\bar{k}) < 1$$

Due to the interaction between technology and preferences in the two-sector case, assuming (M) on preferences is no longer sufficient to guarantee the uniqueness of the steady state. In addition, a restriction has to be placed on technology:

(N) $$T_{11} - (T_1/T_2)T_{22} + (1 - (T_1/T_2))T_{12} < 0$$

Local analysis around the steady state is carried out by differentiating (26) and (27) around their stationary values and solving the resulting equations for $dk_{t+2}, dz_{2,t+2}$ in terms of dk_t, $dz_{2,t}$, dk_{t+1}, $dz_{2,t+1}$. Defining $x_{t+1} = k_t$ and $y_{t+1} = z_{2,t}$, this second order system is transformed into the first order system below.

(30) $$\begin{bmatrix} dk_{t+2} \\ dz_{2,t+2} \\ dx_{t+2} \\ dy_{t+2} \end{bmatrix} = \begin{bmatrix} \underset{(2\times2)}{A} & \underset{(2\times2)}{V_2^{-1}I} \\ \underset{(2\times2)}{I} & \underset{(2\times2)}{0} \end{bmatrix} \cdot \begin{bmatrix} dk_{t+1} \\ dz_{2,t+1} \\ dx_{t+1} \\ dy_{t+1} \end{bmatrix}$$

where

$$A = (1-K)^{-2} \begin{bmatrix} K(T_{11} + V_2^{-1}T_{22})(T_{12})^{-1} - R^1 & -(R^2-R^1)G_2^2(T_2)^{-1} \\ [T_{11}+V_2^{-1}T_{22}-R^1T_{12}]Q^2(G_2^2T_2)^{-1} & [R^1Q^2+R^2Q^1]T_{12}(T_2)^{-2}-R^2 \end{bmatrix}$$

and R^1, R^2, Q^1, Q^2 and G_2^2 are as in the previous section and

$$K = [Q^1 + Q^2]\, T_{12}/(T_2)^2$$

The Jacobian of (30) has four roots which come in pairs $(\lambda_j, V_2\lambda_j)^{-1}$. [This result is entirely analogous to the previous section and can be seen from the proof of Theorem 1]. Locally nonoscillatory dynamics for (30) require all roots of the Jacobian to be positive. The following theorem characterizes these roots.

Theorem 2: Let all assumptions on preferences hold, and let both agents have normal utilities. Then

(i) All roots of the Jacobian in (30) are real.

If in addition, technology and preferences are such that $T_{12}(Q^1 + Q^2)(T_2)^{-2} < 1$, then

(ii-a) All roots of the Jacobian in (30) are positive.

Whereas if $T_{12}(Q^1 + Q^2)(T_2)^{-2} > 1$, then

(ii-b) Two roots of the Jacobian in (30) are negative and two roots are positive.

In addition, if (M) and (N) are both imposed:

(iii) For each pair of positive roots, one lies inside and the other outside the unit circle.

Proof: See Appendix.

<u>Remark</u>: Part (ii) of Theorem 1 suggest that the relative capital intensity of the consumption goods sector is by itself not sufficient to generate oscillations. Intuitively, with concavity and normality of preferences, agents may have Engels curves which slope in favor of future utility. Hence, even if the accumulation of capital shifts the technological frontier out in favor of greater current consumption, this shift, if small, may be offset by the bias in preferences for greater future consumption and the path to optimal capital accumulation may behave monotonically. The shift in the technological frontier has, therefore, to overcome the normality in future utility for oscillations to occur. The condition that $T_{12}(Q^1 + Q^2)(T_2)^{-2} < 1$ (>1) implies nonoscillatory (oscillatory) paths reflects this. With normality in both agents' utilities, we have $Q^1 + Q^2 < 0$. When $T_{12} > 0$ (i.e., the consumption sector is relatively labor-intensive), $T_{12}(Q^1 + Q^2)(T_2)^{-2} < 0$ $(\therefore < 1)$ and the condition for no oscillations is always met. However, when $T_{12} < 0$ the degree of normality in future utility has to be taken into account. The stronger is the normality, the smaller are Q^1 and Q^2 <u>in magnitude</u>, and the greater T_{12} has to be <u>in magnitude</u> to make $T_{12}(Q^1 + Q^2)(T_2)^{-2} > 1$ and cause oscillations.

When pairs of negative roots occur, both can be less than -1 if the consumption good is sufficiently capital-intensive (or if future utility is sufficiently inferior for at least one agent). In that case, it can be easily shown that period two cycles can emerge via a flip bifurcation (see Benhabib and Nishimura [1985]). The existence of cycles on the Von Neuman Facet has been pointed out by McKenzie [1983]. Futhermore, under some symmetry assumptions, necessary and sufficient conditions for the local stability of the steady state were given, in the single-agent case, by Dasgupta and McKenzie [1985].

APPENDIX 1

Proof of Theorem 1

(i) The Jacobian in (22) has a characteristic equation which may be expressed as

$$|-\lambda I| \cdot |A - \lambda I + f'I|-\lambda I|^{-1} I| = 0$$

or (A1) $$|-\lambda I| \cdot |A - (\lambda + f'\lambda^{-1})I| = 0$$

where $\lambda \neq 0$.[4] Let λ_j be a root of (A1). Then substituting $\lambda = f'/\lambda_j$ into (A1) we get

$$|-(f'\lambda_j^{-1})I| \cdot | A - (f'\lambda_j^{-1} + \lambda_j)I| = 0$$

which implies that if λ_j is a root, then f'/λ_j is a root.

(ii) From the proof of (i), (A1) reduces to

(A2) $$A - \gamma I = 0 \text{ where } \gamma = \lambda + f'\lambda^{-1}.$$

Hence, the n roots of A can be related to 2n roots of J via the quadratic relation

(A3) $$(\lambda_j)^2 - \gamma_j\lambda_j + f' = 0 \qquad \text{or} \qquad \lambda_j = \tfrac{1}{2}[\gamma_j \pm \sqrt{((\gamma_j)^2 - 4f')}]$$

where γ_j is the j^th^ root of A. This enables us to state that each pair $(\lambda_j, f'\lambda_j^{-1})$ of the roots of J is real if and only if the associated

(a) γ_j is real, and

(b) $|\gamma_j| \geq 2\sqrt{f'}$

and both λ_j and $f'\lambda_j^{-1}$ are positive if and only if

(c) γ_j is positive.

The proof of (ii) therefore requires proving (a), (b) and (c) for the roots of A.

(a) Define P as a real, nonsingular, diagonal $n \times n$ matrix with

$$P_{11} = 1 \quad \text{and} \quad P_{ii} = [-f''Q^i/(R^i-R^1)(G_2^i)^2]^{1/2} \qquad i=2,..n.$$

Note that under (V.7) all elements of P are finite and real. Pre-and post-multiplying A by P^{-1} and P respectively, we get a similarity transformation to the following real, symmetric matrix

$$A^* = P^{-1}AP = \begin{bmatrix} -R^1 + \Sigma_{j=1}^n f''Q^j & [-(R^2-R^1)f''Q^2]^{\frac{1}{2}} \ldots & [-(R^n-R^1)f''Q^n]^{\frac{1}{2}} \\ [-(R^2-R^1)f''Q^2]^{\frac{1}{2}} & -R^2 \quad \ldots & 0 \\ \vdots & & \\ [-(R^n-R^1)f''Q^n]^{\frac{1}{2}} & 0 \quad \ldots & -R^n \end{bmatrix} .$$

Because it is real and symmetric, the matrix A^* has real roots. Since it was obtained from A by means of a similarity transformation, the roots of A and A^* are identical. Hence, the roots of A are real as well.

It helps in the flow of the argument to prove (c) before (b).

(c) Since a real, symmetric matrix is positive definite if and only if it has all of its roots positive, we show that A^* is positive definite in order to prove that it has all of its roots positive.

(I) Under normality assumptions, all diagonal elements of A^* are positive (see definition of R^j and Q^j on page (11) in text and note that both are negative under normality).

(II) All principal minors formed by excluding the first row and column are contained in the product $\Pi_{i=2}^k(-R^i)$ $k = 2,\ldots n$. Since $-R^i > 0$ for all agents, this product is always positive.

(III) Principal minors of order m, $(m = 2,\ldots,n)$ which include the first row and column can be expressed as follows, when expanded sequentially along the diagonal:

$$\text{(A9)}\quad A^*_m = [\Pi^m_{s=2}(-R^s)][-R^1 + f''\Sigma^n_{k=1,\neq s} Q^k + f''\Sigma^m_{s=2}Q^s\,(-R^1/-R^s)] > 0,\qquad m=2,\ldots n$$

where $A^*_n = |A^*|$. (I), (II) and (III) establish positive definiteness for A^*.

(b) Since all roots of A^* are positive, we need to show $\gamma > 2\sqrt{f'}$. Consider the matrix $\hat{A} = A^* - 2\sqrt{f'}.I$ which has roots equal to $\gamma - 2\sqrt{f'}$. A^* and $\hat{A}$ differ only in their diagonal elements. Let $\hat{R}^j = R^j + 2\sqrt{f'} = R^j + 2V_2^{-½}$. Therefore, it can be shown that

$$\hat{R}^j = \frac{2(1-(V^j_2)^{-½})\dfrac{V^j_1}{V^j_2}V^{\,j}_{12} - (\dfrac{V^j_1}{V^j_2})^2 V^{\,j}_{22} - (1-(V^j_2)^{-½})^2 V^{\,j}_{11}}{V^j_{11} - \dfrac{V^j_1}{V^j_2}V^j_{12}}\qquad j=1,\ldots n$$

Quasiconcavity of preferences ensures that the numerator of $\hat{R}^j$ is positive. Therefore $\hat{R}^j < 0$ due to the sign of its denominator (implied by normality). By definition of $\hat{A}$, and noting that $R^2 - R^1 = \hat{R}^2 - \hat{R}^1$:

$$\hat{A} = \begin{bmatrix} -\hat{R}_1 + f''\Sigma^n_{j=1}Q^j & [-(\hat{R}^2-\hat{R}^1)f''Q^2]^{½} & \ldots & [-(\hat{R}^n-\hat{R}^1)f''Q^n]^{½} \\ [-(\hat{R}^2-\hat{R}^1)f''Q^2]^{½} & -\hat{R}^2 & \ldots\ldots & 0 \\ \cdot & & & \\ [-(\hat{R}^n-\hat{R}^1)f''Q^n]^{½} & 0 & \ldots\ldots & -\hat{R}^n \end{bmatrix}$$

Using the same arguments by which A^* was shown to be positive definite in (a), we can show that $\hat{A}$ is also positive definite. Hence its roots are positive, i.e. $\gamma - 2\sqrt{f'} > 0$ or $\gamma > 2\sqrt{f'}$.

(iii) $(\lambda, f'\lambda^{-1})$ lie respectively inside and outside the unit circle if and only if, at $\lambda=1$, the LHS of (A3) is negative, i.e. $1-\gamma+f'<0$ or $\gamma > 1+f'$.

Consider the matrix $\tilde{A} = A^* - (1+f')I$. The roots of $\tilde{A}$ are equal to $\gamma - (1+f')$. Positive definiteness of $\tilde{A}$ would imply that $\gamma > 1+f'$. To show that this positive definiteness holds, define $\tilde{R}^j = R^j + (1+f') = R^j + (1+1/V_2^j)$. Therefore,

$$\tilde{R}^j = \frac{V_1^j}{(V_2^j)^2}\left[\frac{-(1-V_2^j)V_{12}^j - V_1^jV_{22}^j}{V_{11}^j - V_1^j(V_2^j)^{-1}V_{12}^j}\right] \qquad j = 1,\ldots n.$$

Assumption (M) implies that the numerator of $\tilde{R}^j$ is positive. So $\tilde{R}^j$ is made negative by the sign of its denominator. The definition of $\tilde{A}$ implies

$$\tilde{A} = \begin{bmatrix} -\tilde{R}^1 + \Sigma_{j=1}^n f''Q^j & [-(\tilde{R}^2-\tilde{R}^1)f''Q^2]^{\frac{1}{2}} & \ldots & [-(\tilde{R}^n-\tilde{R}^1)f''Q^n]^{\frac{1}{2}} \\ [-(\tilde{R}^2-\tilde{R}^1)f''Q^2]^{\frac{1}{2}} & -\tilde{R}^2 & \ldots & 0 \\ \vdots & & & \\ [-(\tilde{R}^n-\tilde{R}^1)f''Q^n]^{\frac{1}{2}} & 0 & \ldots & -\tilde{R}^n \end{bmatrix}$$

Comparison of $\tilde{A}$ with A^* shows that the same arguments used to establish positive definiteness of A^* can be used (with $\tilde{R}^j$ replacing R^j) to establish positive definiteness of $\tilde{A}$. Q.E.D.

APPENDIX 2

Existence of a local stable manifold

Let B be a $2n \times 2n$ matrix of eigenvectors for J. Partitioning

$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ where partitions are $n \times n$, we have from Theorem 1(i):

$$\begin{bmatrix} A & -f'I \\ I & 0 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & f'\Lambda^{-1} \end{bmatrix}$$

where we choose Λ as a nonsingular (see footnote 4) $n \times n$ diagonal matrix of those eigenvalues of J that lie inside the unit circle. Let

$$E_s = \{(e_1,e_2) \mid \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = B\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \quad \text{where} \quad (d_1,d_2) \in R^n \times \{0\}\}$$

be the eigenspace associated with the roots of J inside the unit circle. If B_{11} is nonsingular, then for the linear system the R^n dimensional stable manifold in R^{2n} will be given by $S = \{(e_1,e_2) \mid e_2 = B_{12}B_{11}^{-1}e_1\}$ (see Scheinkman [1976], Lemma 14). Since the stable manifold for the nonlinear system near the steady state given by equations (12),(13) is tangent to the stable manifold of the linear system at the steady state, we can express $(k_1,\underset{\sim}{z}_1)$ as functions of $(x_1,\underset{\sim}{y}_1)$ along this stable manifold where, for small deviations around the steady state $(dk_1,d\underset{\sim}{z}_1)'$ is approximated by $B_{12}B^{-1}_{11}(dk_0,d\underset{\sim}{z}_0)'$.

Multiplying the left hand matrix into the first-column of the right hand matrix, we obtain equations

$$\text{(A.4)} \qquad AB_{11} - f'B_{21} = B_{11}\Lambda$$

and

$$\text{(A.5)} \qquad B_{11} = B_{21}\Lambda$$

Combining (A.4) and (A.5): $AB_{11} - f'B_{11}\Lambda^{-1} = B_{11}\Lambda$ or $AB_{11} = B_{11}(\Lambda+f'\Lambda^{-1}) = B_{11}\Gamma$. Hence B_{11} is the eigenspace of A. From Appendix 1, under assumption (V.7) the nonsingular matrix P performs a similarity transform from A to a real, symmetric matrix A^*. Since a symmetric matrix has a nonsingular eigenspace, the eigenspace of A^* is nonsingular. Let this eigenspace be H. Hence $A^*H = H\Gamma$ or, substituting for A^*: $P^{-1}APH = H\Gamma$. Premultiplying both sides by P we obtain the equality $APH = PH\Gamma$. By inspection, $B_{11} = P.H$. Since both P and H are nonsingular, their product B_{11} is also nonsingular. This concludes the proof. Q.E.D.

APPENDIX 3

Proof of Theorem 2:

(i) Letting $P = T_{11} + V_2^{-1}T_{22}$, we can partition the Jacobian of (30) as

$$\begin{bmatrix} A & -V_2^{-1}I \\ I & 0 \end{bmatrix} \quad \text{where} \quad A = (1-K)^{-2}B$$

and

$$B = \begin{bmatrix} KP(T_{12})^{-1}-R^1 & -(R^2-R^1)G_2^2(T_2)^{-1} \\ (P-R^1T_{12})Q^2(G_2^2T_2)^{-1} & (R^1Q^2 + R^2Q^1)T_{12}(T_2)^{-2} - R^2 \end{bmatrix}$$

If γ is a root of A and δ a root of B, then $\gamma = \delta(1-K)^{-1}$ so that if δ is real so is γ. Checking the discriminant of the characteristic equation a sufficient condition for the roots of B to be real is $b_{12}b_{21} \geq 0$,

where $b_{12} = -(R^2-R^1)G_2^2(T_2)^{-1}$

and $b_{21} = (P-R^1T_{12})Q^2(G_2^2T_2)^{-1}$, where R^1, R^2, G_2^2, T_2, Q^2, $P < 0$.

So $b_{12}b_{21} \geq 0$ if $P-R^1T_{12} \leq 0$ and $R^2-R^1 < 0$

or if $P-R^1T_{12} \geq 0$ and $R^2-R^1 > 0$

If, on the other hand $P-R^1T_{12} \geq 0$ (≤ 0) while $R^2-R^1 < 0$ (>0),[5] it can be shown that by rearranging terms and completing squares, the discriminant of B can be expressed as

$$\left[R^2 - R^1 + \frac{(P-R^2T_{12})Q^1 - (P-R^1T_{12})Q^2}{(T_2)^2}\right]^2 + \frac{4(P-R^2T_{12})(P-R^1T_{12})Q^1Q^2}{(T_2)^4}$$

Thus, a sufficient condition for the discriminant to be nonnegative is that

$$\text{(A6)} \qquad (P-R^1T_{12})(P-R^2T_{12}) \geq 0$$

But $R^2 - R^1 < 0$ (>0) together with $P-R^1T_{12} \geq 0$ (≤ 0) imply $PT_{12}^{-1} - R^2 > 0$ (<0) and (A6) holds. This establishes that the roots of B, and therefore of A, are real. To prove that the roots of the Jacobian in (30) are real, the roots of A must further satisfy $|\gamma| > |2V_2^{-\frac{1}{2}}|$. It has been established from the proof of Theorem 1 that

$$\text{(A7)} \qquad R^i + 2V_2^{-\frac{1}{2}} < 0 \qquad i=1,2$$

Furthermore, it can be seen that

$$\text{(A8)} \qquad R^i - 2V_2^{-\frac{1}{2}} = \frac{2(1+(V_2^i)^{-\frac{1}{2}})\frac{V_1^i}{V_2^i}V_{12}^i - (\frac{V_1^i}{V_2^i})^2V_{22}^i - (1+(V_2^i)^{-\frac{1}{2}})^2V_{11}^i}{V_{11}^i - \frac{V_1^i}{V_2^i}V_{12}^i} < 0$$

by quasiconcavity and normality of preferences, while concavity of technology implies that

$$\text{(A9)} \qquad P \pm 2V_2^{-\frac{1}{2}}T_{12} = T_{11} + V_2^{-1}T_{22} \pm 2V_2^{-\frac{1}{2}}T_{12} < 0.$$

We shall use (A7), (A8) and (A9) to show that the characteristic equation of A has no roots in the interval $[-2V_2^{-\frac{1}{2}}, 2V_2^{-\frac{1}{2}}]$. The LHS of this equation has the form:

$$\text{(A10)}\quad \gamma^2 - \frac{1}{1-K}\left[-(R^1+R^2) + \frac{KP}{T_{12}} + \frac{(R^1Q^2+R^2Q^1)T_{12}}{(T_2)^2}\right]\gamma + \frac{1}{1-K}\left[R^1R^2 - \frac{PR^1Q^2 - PR^2Q^1}{(T_2)^2}\right]$$

which simplifies to:

$$\text{(A11)}\quad \frac{1}{1-K}\left[(-R^1-\gamma)(-R^2-\gamma) - \frac{[Q^1(-R^2-\gamma) + Q^2(-R^1-\gamma)][-P-T_{12}\gamma]}{(T_2)^2}\right]$$

Let Z be the closed interval $[-2V_2^{-\frac{1}{2}}, 2V_2^{-\frac{1}{2}}]$. Then from (A7), (A8) and (A9) it is obvious that (A11) > 0 (< 0) for $1-K > 0$ (<0) over all $\gamma \in Z$. Hence, (A11) $\neq 0$ for any $\gamma \in Z$, and A has no roots in Z.

(ii-a) Suppose that $1 - (Q^1+Q^2)T_{12}/(T_2)^2 = 1-K > 0$. From (A10), the determinant of A is

$$\text{(A12)}\quad \text{Det}(A) = (1-K)^{-1}\left[R^1R^2 - PT_2^{-2}[Q^1R^2 + Q^2R^1]\right] > 0$$

Since the determinant equals the product of the roots, both roots of A have the same sign. To determine this sign consider

$$\text{Tr}(A) = [-R^1(1-Q^2T_{12}) - R^2(1-Q^1T_{12}) + P(Q^1 + Q^2)](1-K)^{-1}T_2^{-2}$$

If $T_{12} > 0$, $1 - Q^iT_{12}(T_2)^{-2} > 1$ ($>>0$) $i=1,2$. Also, when $T_{12} < 0$ but $1-K > 0$ as assumed, it must be true that $1 - (Q^1+Q^2)T_{12}/(T_2)^2 > 0$, i.e., $1-Q^iT_{12}/(T_2)^2 > Q^jT_{12}/(T_2)^2 > 0$ $i,j=2$. In either case, it can be seen that $\text{Tr}(A) > 0$. This implies that both roots of A (and all four roots of the Jacobian in (30)) are positive.

(ii-b) Suppose $1 - (Q^1+Q^2)T_{12}/(T_2)^2 = 1 - K < 0$. From (A12), $\det A < 0$. This proves that the roots of A bear opposite signs. Hence, the roots of the Jacobian in (30) come in pairs with opposite signs.

(iii) From the proof of Theorem 1(iii) it follows that pairs of roots $(\lambda,(V_2\lambda))^{-1}$ will lie inside and outside the unit circle respectively if and only if the roots of A satisfy $\gamma > 1+V_2^{-1}$. Also from the proof of Theorem 1 (iii) we know that, under assumption (M), $R^i + (1+V_2^{-1}) < 0 \quad i = 1, 2$. Similarly, assumption (N) on technology implies that

$$P + (1+V_2^{-1})T_{12} = T_{11} - (T_1/T_2)T_{22} + (1-T_1/T_2)T_{12} < 0.$$

Defining the interval $\Omega = [0,1+V_2^{-1}]$, it follows from assumptions (M) and (N) that for any $\gamma \in \Omega$, the characteristic equation (A11) > 0 (<0) as $1-K > 0$ (< 0). Hence, A has no roots in Ω. Q.E.D.

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FOOTNOTES

[1] Although strict concavity of the aggregator functions is assumed to ensure the differentiability and concavity of the value function , it is not needed for the proofs of Theorems 1 and 2. In those proofs, only the quasiconcavity of the aggregator functions is used.

[2] Normality and inferiority are defined in terms of the aggregator function. The interpretation for the stream of consumption in a general utility function is as follows. Normality (inferiority) of future utility implies a monotonic (oscillatory) response of consumption in future periods to a perturbation in initial wealth. Under normality (inferiority), an increase in wealth at $t=0$ increases (decreases) next period's utility z_1, in addition to increasing initial utility z_0. The latter implies an increase (a large increase) in initial consumption c_0, while the former effect implies an increase (a decrease) in next period's wealth. This in turn implies an increase (a decrease) in next period's consumption c_1, and an increase (increase) in the subsequent period's utility z_2. By induction, it follows that normality leads to an increase in the consumption stream for every future period, while

inferiority leads to an oscillatory pattern.

[3] Although assumption (V.7) is also used in Appendix 1 to show that A is similiar to a symmetric, positive definite matrix, a perturbation argument would allow us to dispense of this assumption. If some arbitrarily small perturbation of A where (V.7) holds has positive and real roots so will A, although A may have repeated roots. The latter possibility is not a problem for the proof of Appendix 1.

[4] To rule out $\lambda = 0$, (J) has to be nonsingular. This requires that $f' \neq 0$ and $|A| \neq 0$. (Because $|J| = |A|.|[f'I]A^{-1}I|$). The former holds due to assumption (F.2). To establish the latter, it will be shown that if all agents have normal preferences, A is positive definite.

[5] Note that if $R^2-R^1 \neq 0$, as implied by (V.7), the discriminant of the characteristic equation of B is not equal to zero. This ensures the distinctness of the eigenvalues, and thus the nonsingularity of the eigenspace, of B. While the proof that the roots of B are real does not depend on (V.7), we continue to maintain it for convenience.