

**ECONOMIC RESEARCH REPORTS**

ON COMPETITIVE CYCLES IN  
PRODUCTIVE ECONOMIES

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R.R. #86-19

September 1986

**C. V. STARR CENTER  
FOR APPLIED ECONOMICS**



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## On Competitive Cycles in Productive Economies\*

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Received August 15, 1986; revised June 25, 1987

In a model of overlapping generations with production, money, and an endogenous labor supply, general conditions are given for the existence of two different types of cyclical equilibria. The conditions are given in terms of the elasticities of demand for savings and for capital with respect to the interest rate and of the capital-consumption ratio at the golden rule steady state. Examples using CES technologies are also studied. *Journal of Economic Literature*  
Classification Numbers: 021, 022, 023, 131. © 1988 Academic Press, Inc.

In recent years there have been a number of studies looking at the equilibria and/or dynamical evolution of overlapping generations exchange economies (Balasko *et al.* [2] and Kehoe and Levine [14, 15]). One phenomenon that has drawn the attention of researchers is the possibility of cycles or chaotic behavior in equilibrium (Benhabib and Day [4] and Grandmont [9]) and their relation to sunspot equilibria (Azariadis and Guesnerie [1] and Guesnerie [11]). More recently, in the tradition initiated by Diamond [6], some work has studied overlapping generations models with production (Jullien [13], Farmer [7], and Reichlin [16]).

\* This work was supported by National Science Foundation Grant SES-83-20464 at the Institute for Mathematical studies in the Social Sciences, Stanford University, Stanford, California. Guy Laroque's research was conducted in part during a visit to New York University and Princeton University. The authors benefited from discussions with John Mather and Michael Woodford and also thank the referee for valuable comments. The authors also acknowledge the Technical support of the C.V. Starr Center for Applied Economics at New York University.

The present paper studies an overlapping generations economy with production, in which consumers live two periods, work when young, consume when old, and have separable utility functions while production takes place through a neoclassical production technology. There is a constant nominal quantity of outside money. We are interested in the possible existence of recurrent competitive cycles near the golden rule steady state equilibrium in such an economy.

Our subject of study is the set of competitive intertemporal equilibria for all  $t$ , from  $-\infty$  to  $+\infty$ . With this definition one can think of a subset of these equilibria which exhibit some properties of recurrence as limits of dynamical learning (or capital accumulation) processes (contrast with Kehoe-Levine [15]). In this framework, the emergence of cycles near a stationary equilibrium is directly related to the characterization of a subset of critical economies.

We characterize this set of critical economies. This involves three basic parameters, the elasticities of the demand for savings and of the demand of capital with respect to the interest rate and the ratio of the stock of capital to consumption, all evaluated at the golden rule steady state. We provide a complete description of the critical economies in terms of these three parameters. This allows us to confirm earlier results that were obtained in more particular, or different, setups. We see that, as in Grandmont [9], period two cycles are likely to occur when savings is a decreasing function of the interest rate. We find also that there may exist Hopf cycles, even when savings increase with the interest rate, provided that the quantity of outside money is negative at the golden rule and that there is enough complementarity in the production function. In our model, these cycles seem to be linked with a destabilizing real balance effect. This is consistent with results obtained by Reichlin [16] for a nonmonetary economy, by Farmer [7] in a model where the *real* quantity of outside money is determined by endogenous government policy, and by Benhabib [3] where a similar mechanism generates Hopf cycles in a model without production. The paper is organized as follows. The model is presented in the first section. The formal definition of critical economies is given in Section 2. Section 3 characterizes the set of critical economies of interest. An application to CES production functions and the economic interpretation of the results appear in Section 4. The main proofs are gathered in Section 5.

## I. THE MODEL

We consider an overlapping generations economy, with money, a durable asset with no intrinsic value, labor, and a good which can either be

used for consumption or be stored in the form of capital. We take the good as the numeraire. At date  $t$ , the price of money in terms of good is  $Q_t$ , and the real wage is  $W_t$ . The gross return on capital is  $R_{t+1}$ . There is a future market on which the two assets, money and capital, can be traded, and arbitrage, under perfect foresight, implies  $R_{t+1} = Q_{t+1}/Q_t$ .

The consumers live two periods. They work during the first periods of their lives and consume during the second. The representative agent born at date  $t$  supplies a quantity of labor  $L_t$  when young and consumes  $C_{t+1}$  when old.  $L_t$  is bounded above by a maximum physical bound  $\bar{L}$ ,  $\bar{L} > 0$ . The utility function of the representative agent is separable in labor and good and is equal to  $U(C_{t+1}) - V(L_t)$ .

ASSUMPTION I.1.  $U$  is a strictly increasing concave function from  $\mathbb{R}_+$  into  $\mathbb{R}$ . It is smooth on the interior of  $\mathbb{R}_+$  and

$$\lim_{C \rightarrow 0} U'(C) = +\infty.$$

$V$  is a strictly decreasing convex function from  $[0, \bar{L}]$  into  $\mathbb{R}$ . It is smooth on  $[0, \bar{L})$ , and

$$V'(0) = 0, \quad \lim_{L \rightarrow \bar{L}} V'(L) = +\infty.$$

The consumer born at date  $t$  chooses, under perfect foresight, a labor supply  $L_t$ , a consumption level  $C_{t+1}$ , and savings in form of capital  $K_{t+1}$  or money  $M_{t+1}$  by maximizing his utility function  $U(C_{t+1}) - V(L_t)$  subject to the budget constraints of the two periods of his life:

$$K_{t+1} + M_{t+1}Q_t = W_t L_t,$$

$$C_{t+1} = R_{t+1}K_{t+1} + M_{t+1}Q_{t+1}.$$

The technology is described through a (gross) production function  $F(K_t, L_t)$ . A stock of capital  $K_t$  invested at date  $(t-1)$  combined with an input of labor  $L_t$  at date  $t$  gives an overall quantity of good  $F(K_t, L_t)$  at date  $t$ .

ASSUMPTION I.2.  $F$  is a strictly increasing strictly concave function from  $\mathbb{R}_+^2$  into  $\mathbb{R}_+$ . It is homogeneous of degree 1. It is smooth on the interior of  $\mathbb{R}_+^2$  and

$$\lim_{K \rightarrow \infty} F(K, L)/K \leq 1 - \delta < 1$$

$$\lim_{K \rightarrow 0} F_K(K, L) = +\infty \quad \text{for all } L > 0$$

$$\lim_{L \rightarrow 0} F_L(K, L) = +\infty \quad \text{for all } K > 0.$$

Assumption I.2 states that the technology exhibits constant returns to scale, and that the depreciation rate  $\delta$ , in the limit for  $K$  very large, is strictly positive (this will provide a bound on the set of feasible allocations). The producers maximize their profit  $F(K_t, L_t) - R_t K_t - W_t L_t$ . Since there are constant returns to scale, at a competitive equilibrium their maximum profit is equal to zero, and the level of production adjusts to the level of demand.

Finally, for simplicity, we assume that there is a constant quantity of (positive or negative) outside money  $M$  over time.

In all the following, we consider the maximum physical quantity of labor  $\bar{L}$  as given and fixed. An *economy*  $e$  is therefore characterized by the data of  $(U, V, F, M)$ , where  $U, V$ , and  $F$  satisfy Assumptions I.1 and I.2 and  $M$  is a real number. We shall say that an economy  $e_1$  is close to an economy  $e^2$  when  $M^1$  is close to  $M^2$  in  $\mathbb{R}$  and  $(U^1, V^1, F^1)$  is close to  $(U^2, V^2, F^2)$  for the  $C^2$  topology (i.e., on any compact set of their domain of definition, the functions and their two first derivatives are close).

We now proceed by describing the competitive equilibria of an economy  $e = (U, V, F, M)$ .

**DEFINITION I.3.** An intertemporal competitive equilibrium with perfect foresight is a sequence  $X_t = (K_t, L_t, C_t, R_t, W_t, Q_t)$ ,  $X_t$  in  $\mathbb{R}_+^6$ ,  $t = -\infty, \dots, 0, 1, \dots, +\infty$ , such that

- (i)  $(L_t, C_{t+1}, K_{t+1}, M)$  is an optimal action of the consumer born at date  $t$ , given the price system  $(W_t, R_{t+1}, Q_t, Q_{t+1})$ .
- (ii)  $(K_t, L_t)$  maximizes the profit of the firm under the technological constraint, given  $(R_t, W_t)$ .
- (iii)  $F(K_t, L_t) = C_t + K_{t+1}$  for all  $t$ .

(Note that in the presence of futures markets, i.e., when there are no sign constraints on  $K_{t+1}$  or  $M_{t+1}$  in the consumer's program, the existence of an optimal action for the consumer requires  $R_{t+1} = Q_{t+1}/Q_t$ . Note also that (iii) represents the equality between demand and supply of good. The equalities between demand and supply of money and labor are satisfied because of our choice of notations.)

The following characterization of the intertemporal competitive equilibria with perfect foresight (for short: equilibria) will be very useful:

**PROPOSITION I.4.** Any equilibrium is canonically associated with a unique sequence  $Y_t = (K_t, L_t, Q_t)$ ,  $Y_t$  in  $\mathbb{R}_+^3$ ,  $t = -\infty, \dots, 0, 1, \dots, +\infty$  which satisfies

$$\begin{aligned}
 F'_k(K_{t+1}, L_{t+1}) F'_L(K_t, L_t) U'(F'_k(K_{t+1}, L_{t+1}) F'_L(K_t, L_t) L_t) - V'(L_t) &= 0 \\
 K_{t+1} + MQ_t - F'_L(K_t, L_t) L_t &= 0 \\
 Q_{t+1} - F'_k(K_{t+1}, L_{t+1}) Q_t &= 0
 \end{aligned}
 \tag{1.1}$$

and conversely.

The proof is given in Section V. Let  $\tilde{Y}$  be a sequence of  $(Y_t)$ ,  $Y_t$  in  $\mathbb{R}_+^3$ ,  $t = -\infty, \dots, 0, 1, \dots, +\infty$ , and let  $Y$  be the space of such sequences equipped with the uniform topology. Let  $Z(Y_t, Y_{t+1}; e)$  be the mapping defined by the left hand side of (1.1) for the economy  $e$ .  $Z$  is smooth with respect to  $(Y_t, Y_{t+1})$ . Given  $\tilde{Y}$ , let  $\tilde{Z}(\tilde{Y}; e)$  be the sequence  $Z(Y_t, Y_{t+1}; e)$ ,  $Z(Y_t, Y_{t+1}; e)$  in  $\mathbb{R}^3$ ,  $t = -\infty, \dots, 0, 1, \dots, +\infty$ .  $\tilde{Z}(\cdot; e)$  is a continuous mapping from  $Y$  into  $Y$ , and Proposition I.4 says that *the set of equilibria of the economy  $e$  can be described by the roots  $\tilde{Y}$  of the equation*

$$\tilde{Z}(\tilde{Y}; e) = 0$$

This is a complicated equation to study. A preliminary step is to look for stationary equilibria, along which all quantities stay constant. Proposition I.4 shows that these equilibria are characterized by a  $Y = (K, L, Q)$  in  $\mathbb{R}_+^3$  that satisfies

$$\begin{aligned}
 F'_k(K, L) F'_L(K, L) U'(F'_k(K, L) F'_L(K, L) L) &= V'(L) \\
 K + MQ &= F'_L(K, L) L \\
 F'_k(K, L) &= 1 \quad \text{or} \quad Q = 0.
 \end{aligned}
 \tag{1.2}$$

The last line of (1.2) proves that, as usual in overlapping generations models, there may exist two types of stationary equilibria, the *golden rule steady states* where  $F'_k(K, L) = 1$  and the *nonmonetary steady states* where  $Q = 0$ .

First, the golden rule steady states. The first and the third equations of (1.2) give the unique real allocation corresponding to the golden rule. In fact, they are the first order conditions associated with the concave program:

$$\begin{aligned}
 \text{Max } U(C) - V(L) \\
 C + K &= F(K, L).
 \end{aligned}$$

(The feasible set is compact under Assumptions I.1 and I.2: see Lemma V.1 in Section V.) We shall denote by  $C^*(e)$ ,  $K^*(e)$ ,  $L^*(e)$  the solution of this program. The second equation in (1.2) gives  $Q$  provided that  $M$  is of the

same sign as  $F'_L(K^*(e), L^*(e))L^*(e) - K^*(e)$  (note that under constant returns to scale, given that  $F'_K(K^*(e), L^*(e)) = 1$ , the sign of this quantity depends only on the technology). Following Gale [8], the economy may be called "Samuelson" when it is positive (requiring a positive quantity of outside money to sustain the golden rule path) and "classical" when it is negative. *We shall limit our attention from now on to the subset  $E$  of economies which possess a (unique) monetary golden rule steady state.* From the above discussion,  $E$  is the open subset of economies which satisfy

$$Q^*(e) = (F'_L(K^*(e), L^*(e))L^*(e) - K^*(e))/M > 0.$$

Second, using the last two equations of (1.2), the nonmonetary steady states are associated with capital-labor ratios  $K/L$  that satisfy  $F'_L(K, L)L = K$ . According to the technology again, there may exist none, one, or several such capital-labor ratios. The equilibrium labor supply is then uniquely determined by the consumer's program.

As mentioned earlier, we shall focus our attention in the rest of the paper mainly on non-steady state equilibria which are near the golden rule equilibrium. The reader familiar with bifurcation theory can simply note the definitions of  $\varepsilon$ ,  $\bar{U}$ ,  $\bar{V}$ , and  $\alpha$  as well as the contents of Proposition II.1 in the next section and then skip directly to Section III.

## II. EQUILIBRIA NEAR THE GOLDEN RULE

Let us denote by  $\bar{Y}^*(e)$  the golden rule equilibrium of the economy  $e$ , i.e., such that  $Y_t^* = (K^*(e), L^*(e), Q^*(e))$  for all  $t$ . We have by construction

$$\bar{Z}(\bar{Y}^*(e); e) = 0.$$

We want to look for economies which possess equilibria close to but different from  $\bar{Y}^*$ , i.e., such that there is an infinite sequence  $\tilde{y} = (y_t = (k_t, l_t, q_t))$ ,  $t = -\infty, \dots, 0, 1, \dots, +\infty$ ,  $\tilde{y} \neq 0$ , with  $\|y_t\|$  small for all  $t$  and

$$\bar{Z}(\bar{Y}^*(e) + \tilde{y}; e) = 0.$$

or equivalently

$$Z(Y^*(e) + y_t, Y^*(e) + y_{t+1}; e) = 0 \quad \text{for all } t.$$

We shall denote by  $\tilde{x} = (x_t = (k_t, l_t, c_t, r_t, w_t, q_t))$  the deviation from the golden rule of the equilibrium under consideration.

This leads us to look for a *subset of critical economies*  $C$ ,<sup>1</sup> such that, using the differentiability properties postulated in Assumptions I.1 and I.2,

$$\frac{\partial Z}{\partial Y_t} y_t + \frac{\partial Z}{\partial Y_{t+1}} y_{t+1} = 0 \quad (2.1)$$

for a sequence  $\bar{y} = (y_t)$  different from zero such that  $\|y_t\| \leq 1$  for all  $t$  from  $-\infty$  to  $+\infty$ , where the derivatives  $\partial Z/\partial Y_t$  and  $\partial Z/\partial Y_{t+1}$  are evaluated at the golden rule equilibrium  $(Y^*(e), Y^*(e))$ .

To characterize the subset of critical economies  $C$  of interest, the following notations are useful:

$$\begin{aligned} \varepsilon &= -\frac{K^*(e) K''_{KK}(K^*(e), L^*(e))}{F'_K(K^*(e), L^*(e))} \\ \bar{U} &= -\frac{C^*(e) U''(C^*(e))}{U'(C^*(e))}, \quad \bar{V} = \frac{L^*(e) V''(L^*(e))}{V'(L^*(e))} \\ \alpha &= \frac{\bar{U} + \bar{V}}{1 - \bar{U}} \quad (= +\infty \text{ when } \bar{U} = 1). \end{aligned}$$

The concavity of the utility functions and the convexity of the technology imply that  $\bar{U}$ ,  $\bar{V}$ , and  $\varepsilon$  are strictly positive.  $\bar{U}$  and  $\bar{V}$  are the measures of relative risk aversion at the golden rule, and  $\varepsilon$  is the inverse of the elasticity of the demand of capital with respect to the real interest rate, changed of sign.  $\alpha$  can take any real value outside the interval  $[0, -1]$ . The sign of  $\alpha$  has a simple economic interpretation. It is easily shown that savings, here  $WL$ , is increasing (decreasing) in the interest rate if and only if  $\alpha$  is positive (negative).<sup>2</sup> In other words, the gross substitute assumption is satisfied if and only if  $\alpha$  is positive. All the parameters  $\varepsilon$ ,  $\bar{U}$ ,  $\bar{V}$ , and  $\alpha$  depend of course continuously on the economy. To save on notations, we do not make that dependence explicit.

<sup>1</sup> The terminology is borrowed from Debreu [5]; 0 is a critical value of  $Z$  at the point  $\bar{Y}^*(e)$ . Note that there are of course critical economies outside  $C$ , where 0 is a critical value of  $Z$  at points different from  $\bar{Y}^*(e)$ .

<sup>2</sup> To prove this assertion, note that the consumer program can be written

$$\begin{aligned} \text{Max } U(C) - V(L) \\ C = RWL \end{aligned}$$

which gives the first order condition

$$RWU'(RWL) - V'(L) = 0,$$

PROPOSITION II.1. *If  $\bar{U}$  is different from 1, an economy belongs to  $\mathbf{C}$  if and only if the matrix of the linear system*

$$\begin{aligned} \frac{l_{t+1}}{L^*} &= \left( \frac{C^*}{K^*} + \alpha \right) \frac{l_t}{L^*} + \left( \frac{K^*}{\varepsilon C^*} - 1 \right) r_t + \left( 1 - \frac{C^*}{K^*} \right) \frac{q_t}{Q^*} \\ r_{t+1} &= \alpha \frac{l_t}{L^*} + \frac{K^*}{C^*} r_t \\ \frac{q_{t+1}}{Q^*} &= \alpha \frac{l_t}{L^*} + \frac{K^*}{C^*} r_t + \frac{q_t}{Q^*} \end{aligned} \quad (2.2)$$

has at least one eigenvalue of modulus 1.

When  $\bar{U}$  is equal to 1, an economy belongs to  $\mathbf{C}$  if and only if the matrix of the linear system

$$\begin{aligned} r_{t+1} &= \varepsilon r_t - \varepsilon \left( 1 - \frac{C^*}{K^*} \right) \frac{q_t}{Q^*} \\ \frac{q_{t+1}}{Q^*} &= \varepsilon r_t + \left( 1 - \varepsilon + \frac{\varepsilon C^*}{K^*} \right) \frac{q_t}{Q^*} \end{aligned} \quad (2.3)$$

has at least one eigenvalue of modulus 1.

Here  $(l_t, r_t, q_t)$  stand for the deviations from the golden rule. When there is an eigenvalue of modulus 1, one can construct a sequence along the eigenvector which does not explode when  $t$  goes either to  $+\infty$  or to  $-\infty$ . The formal proof of Proposition II.1 is given in Section V.

where  $L$  is the endogenous variable. Differentiating with respect to  $L$  and  $R$  gives

$$[R^2 W^2 U'' - V''] \frac{dL}{dR} + W U' + R W^2 L U'' = 0.$$

Rearranging terms and using the first order condition, after multiplying through by  $L/V'$ , leads to

$$-[\bar{C} + F] \frac{dL}{dR} + \frac{L}{R} (1 - \bar{C}) = 0$$

or

$$\frac{dL}{dR} = \frac{L}{R\alpha}$$

The elasticity of savings  $WL$  with respect to  $R$  is equal to  $+1/\alpha$ .

Our interest in considering the above subset  $C$  of critical economies stems from the following result, which is a consequence of the analysis developed in the next section:

**PROPOSITION II.2.** *Let  $V$  be any open neighborhood of the golden rule of an economy  $e_0$ . Then every open neighborhood of  $e_0$  in  $E$  contains an economy  $e$  which possesses an intertemporal equilibrium that differs from the golden rule of  $e$  and that lies in  $V$ , if and only if  $e_0$  lies in  $C$ .*

For proof see the arguments following the proof of Proposition III.3 in Section V.

### III. CRITICAL ECONOMIES AND THE ASSOCIATED CYCLICAL BEHAVIOR

Proposition II.1 allows us to describe precisely the subset  $C$  of critical economies which we are interested in, and bifurcation theory is the appropriate tool to analyze qualitatively the nature of the cycles that may occur at economies near  $C$ . A direct examination of (2.2) and (2.3) shows that four parameters are sufficient to describe  $C$ :  $K^*/C^*$ ,  $\varepsilon$ ,  $\alpha$ ,  $\bar{U}$ . We have seen that  $\varepsilon > 0$ ,  $\alpha > 0$  or  $\alpha < -1$ ,  $\bar{U} > 0$  and  $K^*/C^* > 0$ ,  $K^*/C^* \neq 1$  because the golden rule equilibrium is monetary. Computing the characteristic polynomials associated with (2.2) and (2.3) gives the values of  $(K^*/C^*, \varepsilon, \alpha)$  such that the economy  $e$  belongs to  $C$ . When  $\bar{U}$  is different from 1,  $e$  is in  $C$  if and only if the equation

$$P_1(\lambda, \alpha) = -\lambda^3 + \left( \frac{K^*}{C^*} + \frac{C^*}{K^*} + 1 + \frac{\alpha}{\varepsilon} \right) \lambda^2 - \left( \frac{K^*}{C^*} + \frac{C^*}{K^*} + 1 + \frac{\alpha}{\varepsilon} + \frac{\alpha C^*}{K^*} \right) \lambda + 1 + \alpha = 0 \quad (3.1)$$

has at least one root of modulus 1. When  $\bar{U} = 1$ ,  $e$  is in  $C$  if and only if the equation

$$P_2(\lambda) = \lambda^2 - \left( 1 + \frac{\varepsilon C^*}{K^*} \right) \lambda + \varepsilon = 0 \quad (3.2)$$

has at least one root of modulus 1.

One can furthermore classify the critical economies, and the associated cyclical behavior, according to the nature of the roots. First note that  $\lambda = 1$  implies  $K^*/C^* = 1$  (the golden rule equilibrium is nonmonetary), and therefore 1 is never a root for the set of economies which we consider. We can have either  $\lambda = -1$ , and we shall denote by  $C_F$  the subset of  $C$  with

this property (F for flip bifurcation), or a pair of complex conjugate roots, and we denote by  $C_H$  the corresponding subset (H for Hopf bifurcation).

PROPOSITION III.1. *An economy  $e$  belongs to  $C_F$  if and only if*

$$\alpha = -\frac{4 + 2(K^*/C^* + C^*/K^*)}{1 + 2/\varepsilon + C^*/K^*}.$$

*In any open neighborhood of an economy  $e$  in  $C_F$  there is an economy  $e'$  with an intertemporal equilibrium  $\bar{Y}$  which exhibits a cycle of period 2, i.e.,*

$$\begin{aligned} Y_t &= Y_1 && \text{for } t \text{ odd,} \\ Y_t &= Y_2 && \text{for } t \text{ even,} \end{aligned} \quad Y_1 \neq Y_2.$$

Remark III.2. An open problem is to find conditions (maybe boundary conditions on  $Y$  in order to use index theory as in Guesnerie [11]) such that the set of economies with a cycle of period two is globally characterized by the inequality

$$-1 > \alpha > -\frac{4 + 2(K^*/C^* + C^*/K^*)}{1 + 2/\varepsilon + C^*/K^*}.$$

Another related question would be to exhibit a subset of economies (defined by some general assumptions) which would have a *unique* cycle of period 2 if and only if the above (strict) inequality was satisfied. This would extend some of the results obtained by Grandmont [9] in the context of an exchange economy. In this circumstance, the inequality would mean that, to get period two cycles, one must have  $\alpha = (\bar{U} + \bar{V})(1 - \bar{U})$  negative but not too large in absolute value. Savings must be a decreasing function of the interest rate. Note that increased substitutability in production (i.e.,  $\varepsilon$  smaller and smaller) would reduce the scope for cycles. This should be the subject of further research.

We now turn to Hopf cycles.

PROPOSITION III.3.  $C_H$  is the subset of economies which satisfy

$$\frac{K^*}{C^*} > 1$$

and

$$\bar{U} = \varepsilon = 1 \quad \text{or} \quad \alpha = \frac{K^*/C^* - 1}{1 - 1/\varepsilon}, \quad \bar{U} \neq 1, \quad \varepsilon \neq 1.$$

*In any open neighborhood of an economy  $e$  in  $C_H$  there is an economy  $e'$  which possesses an intertemporal equilibrium  $\bar{Y}$ ,  $\bar{Y} \neq \bar{Y}^*$ , such that, after a suitable change of coordinates, the  $Y_t$ 's,  $t = -\infty, \dots, 0, 1, \dots, +\infty$  appear to belong to a two dimensional closed curve in  $\mathbb{R}^3$ .*

(For proof see Section V.)

#### IV. SOME DIAGRAMS AND ECONOMIC INTERPRETATION

The results that we have obtained depend heavily on the particular model under study. It would be of interest to know which part of them extend to a more general setup, for example, to economies where consumers supply labor and consume good at both periods of their lives. On the other hand, it is also useful to develop a better intuition of the empirical relevance (or absence of relevance) of our Proposition III.1 and III.3. To come closer to empirical formulations, we specialize the production function to be of the CES type in this section,

$$F(K, L) = A[\eta L^{-\rho} + (1 - \eta) K^{-\rho}]^{-1/\rho} + (1 - \delta)K,$$

where  $\rho > -1$ ,  $\sigma = 1/(1 + \rho)$  is the elasticity of substitution,  $A$  is a strictly positive constant,  $0 < \eta < 1$ , and  $\delta$ ,  $0 < \delta \leq 1$  is the rate of depreciation of the stock of capital.  $\sigma = +\infty$  corresponds the perfect substitutability between labor and capital (it violates Assumption I.2 of strict concavity of the technology, and can only be approached in the limit),  $\sigma = 1$  is the Cobb Douglas production function, and when  $\sigma$  approaches zero, one obtains the strict complementarity Leontief technology. As noted at the end of Section I, the golden rule capital-labor ratio, and therefore the value of  $\varepsilon$ , is entirely determined by the technology. Some straightforward (but tedious) calculations give

$$\varepsilon = \frac{1}{\sigma} \frac{\delta}{\delta K^*/C^* + 1}. \quad (4.1)$$

Substituting the above formula for  $\varepsilon$  in Propositions III.1 and III.3 allows a discussion in terms of  $K^*/C^*$ ,  $\sigma$ ,  $\delta$  and  $\alpha$ , where  $\sigma$  and  $\delta$ , hopefully two parameters directly related to empirical work on production functions, take the place of  $\varepsilon$ .

We analyze the shape of  $C_F$  and  $C_H$ , for fixed  $\sigma$  and  $\delta$ , in the plan  $(K^*/C^*, \alpha)$ . Letting  $x = K^*/C^*$  and using Propositions III.1 and III.3, we obtain:

PROPOSITION IV.1.  $C_V$  is characterized by the equation

$$\alpha = f(x) = -\frac{2\delta(x+1)^2}{2\sigma\delta x^2 + (\delta + 2x)x + \delta},$$

where  $x$  stands for  $K^*/C^*$  and is a positive number. We have

- (i)  $f(0) = -2$ .
- (ii)  $\lim_{x \rightarrow \infty} f(x) = -1/\sigma$ .
- (iii)  $f(x) = -2$  has a single positive root  $x = (2\sigma - \delta)/(\delta - 2\sigma\delta)$  if and only if  $\delta/2 < \sigma < 1/2$  (otherwise it has no roots).
- (iv)  $f(x) = -1/\sigma$  has a single positive root  $x = (\delta - 2\sigma\delta)/(4\delta\sigma - 2\sigma - \delta)$ , if and only if  $\delta < 1/2$  and  $\sigma > 1/2$  (otherwise it has no roots).
- (v)  $f'(x)$  has one zero (and one change of sign) for  $x = (2\sigma - \delta)/(2\sigma + \delta - 4\delta\sigma)$  positive if and only if ( $\delta < 1/2$  and  $\sigma > \delta/2$ ) or ( $\delta > 1/2$  and  $\delta/2 < \sigma < \delta/2(2\delta - 1)$ ) (otherwise  $f'(x)$  keeps the same sign for all positive  $x$ ).

PROPOSITION IV.2.  $C_H$  is characterized by

$$\alpha = h(x) = \frac{\delta(x-1)}{\delta - \sigma - \delta\sigma x} \quad \text{for } x > 1.$$

We have

- (i) for  $\sigma < \delta/(1 + \delta)$ ,  $h(x)$  increases from 0 to  $+\infty$  when  $x$  goes from 1 to  $(\delta - \sigma)/\delta\sigma$ , and from  $-\infty$  to  $-1/\sigma$  when  $x$  goes from  $(\delta - \sigma)/\delta\sigma$  to  $+\infty$ ;
- (ii) for  $\sigma = \delta/(1 + \delta)$ ,  $h(x) = -(1 + \delta)\delta = -1/\sigma$ ;
- (iii) for  $\sigma > \delta/(1 + \delta)$ ,  $h(x)$  decreases from 0 to  $-1/\sigma$  when  $x$  goes from 1 to  $+\infty$ .

The two above propositions show a structural difference between economies such that  $\sigma < \delta/(1 + \delta)$  and economies with  $\sigma > \delta/(1 + \delta)$ , i.e., between economies with little substitutabilities in the production technology and economies with large substitutabilities. These differences are shown in Figs. 1, 2, and 3 where we have represented the intersection of  $C$  with the plan  $(x, \alpha)$ , where  $x$  stands for  $K^*/C^*$ , for  $\delta = 1$ , and  $\sigma = 1/3$ ,  $1/2$ , and 1, respectively.

To understand these diagrams, one must recall first that the region  $-1 \leq \alpha \leq 0$  has no economic interpretation. Second the line  $x = 1$  corresponds to economies with a zero quantity of outside money at the golden rule stationary state, economies which we have excluded from our

analysis (recall that, for  $x=1$ , the polynomial  $P_1(\lambda)$  has  $\lambda=1$  as a root). Therefore, the image of our space of economies on the plane  $(x, \alpha)$  for fixed  $\delta$  and  $\sigma$  is given by  $\{x > 0 \text{ or } x < -1, x > 0, x \neq 1\}$ . It is the union of four open rectangles.

The diagrams confirm of course that the flip bifurcation can only occur for  $\alpha < -1$ , i.e., when savings is a decreasing function of the interest rate. Proposition IV.1(ii) and (iv) show that, for fixed  $\delta$ , the domain of economies in  $(x, \alpha)$  which possess a flip cycle shrinks when substitutability in production ( $\sigma$ ) increases (see Remark III.2).

Similarly, the drawings do show that the Hopf bifurcation concerns only economies with a negative quantity of outside money at the golden rule ( $K^*/C^* > 1$ ). Following Proposition IV.2 we see that with low values of  $\sigma$ , i.e., complementarity in production, we may get Hopf cycles for positive  $\alpha$  (Fig. 1). A tentative economic interpretation of these cycles can be made transparent in the simple case where  $\bar{U}=1$ . Using first order approximations, the deviations  $x_t = (k_t, l_t, c_t, r_t, w_t, q_t)$  of the equilibrium quantities from their golden rule values are given by (see Proposition II.1)

$$\begin{aligned} k_t &= -\frac{K^*}{\varepsilon} r_t \\ l_t &= 0 \\ c_t &= k_t - k_{t+1} = \frac{K^*}{\varepsilon} (r_{t+1} - r_t) \\ w_t &= \frac{K^*}{L^*} r_t \\ q_{t+1} &= q_t + r_t Q^* \\ r_{t+1} &= \varepsilon r_t + \frac{\varepsilon}{K^*} M q_t. \end{aligned} \tag{4.2}$$

In this circumstance, the employment level is constant (we have approximately an inelastic supply of labor). We have written the system in a way such that  $(k_t, l_t, c_t, w_t)$  are derived from the evolution of  $r_t$ , the driving force of the system being the interaction between  $r_t$  and  $q_t$ . The description of the cycle goes as follows. During the phase where  $k_t$  is positive (overaccumulation of capital), the rate of return is smaller than 1, implying  $q_{t+1}$  smaller than  $q_t$ . The money price of output ( $1/q_t$ ) increases: inflation increases the profitability of capital and explains its overaccumulation. Wages are higher than their level of reference. Looking at the first period budget constraint of the consumer

$$K_{t+1} = W_t L - M Q_t$$

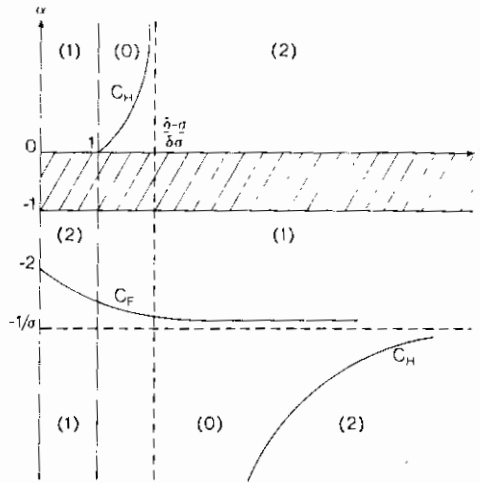


FIG. 1. (x) The number in parentheses is the number of eigenvalues of system (2.2) of modulus smaller than 1.  $\delta = 1$ ;  $\sigma = 1.3$ .

we see that while the increase in wage allows for a further increase in the stock of capital, inflation (the decrease in  $Q_t$ ), the stock of outside money being negative, counteracts this effect. Therefore, when the real balance effect becomes big enough, the stock of capital starts decreasing—the cycle originates in the interaction between income distribution and the real balance effect.

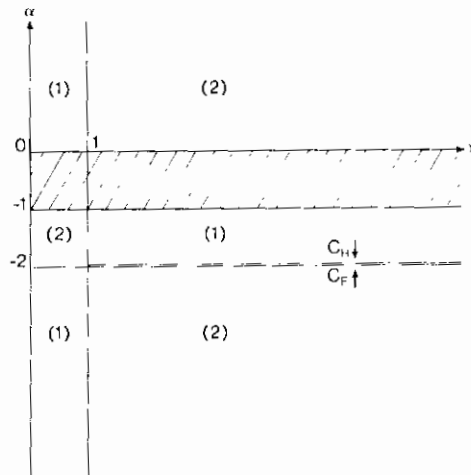


FIG. 2. (x) The number in parentheses is the number of eigenvalues of system (2.2) of modulus smaller than 1.  $\delta = 1$ ,  $\sigma = 1.2$ .

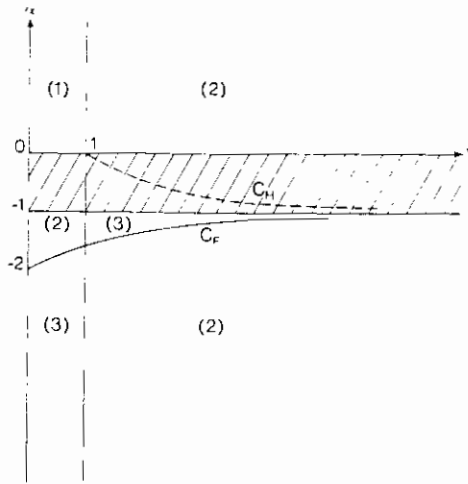


FIG. 3. ( $\lambda$ ) The number in parentheses is the number of eigenvalues of system (2.2) of modulus smaller than 1.  $\delta = 1$ ;  $\sigma = 1$  (Cobb Douglas production function).

When  $\sigma = \delta/(1 + \delta)$ , as in Fig. 2,  $C_H$  is located all in the region  $\alpha < 0$  (for  $\delta = 1$ , it coincides with a part of  $C_F$ , that corresponds to  $x > 1$ ). When  $\sigma$  becomes larger,  $C_H$  becomes smaller and smaller. It is reduced to the empty set in the Cobb Douglas case (Fig. 3).

### V. PROOFS

*Proof of Proposition I.4.* The existence of an optimal action for the consumer requires the arbitrage condition

$$Q_{t+1} = R_{t+1}Q_t$$

to hold. Furthermore, the optimal action is then characterized by the first order condition

$$R_{t+1}W_tU'(C_{t+1}) = V'(L_t)$$

and the budget constraints

$$K_{t+1} + MQ_t = W_tL_t$$

$$C_{t+1} = R_{t+1}W_tL_t.$$

The optimal decision of the firm satisfies also the first order conditions

$$R_t = F'_K(K_t, L_t)$$

$$W_t = F'_L(K_t, L_t).$$

Eliminating  $C_{t+1}$ ,  $R_t$ , and  $W_t$  among the six above relationships gives (1.1).

To prove the converse, let  $(Y_t)$  be a sequence satisfying (1.1) and define

$$R_t = F'_K(K_t, L_t)$$

$$W_t = F'_L(K_t, L_t)$$

$$C_t = R_t W_{t-1} L_{t-1}.$$

Parts (i) and (ii) of Definition 1.3 are satisfied by sufficiency of the first order conditions. To prove that (iii) holds, we use the constant returns to scale assumption. From Euler's identity,

$$F(K_t, L_t) = F'_K(K_t, K_t) K_t + F'_L(K_t, L_t) L_t,$$

$$= R_t K_t + W_t L_t,$$

$$= R_t K_t + M Q_t + K_{t+1}$$

$$= R_t K_t + M R_t Q_{t-1} + K_{t+1}$$

$$= R_t W_{t-1} L_{t-1} + K_{t+1}$$

$$= C_t + K_{t+1}.$$

Q.E.D.

*Proof of Proposition II.1.* By definition, an economy  $e$  belongs to  $\mathbf{C}$  if and only if there exists  $(y_t)$  different from zero,  $\|y_t\| \leq 1$  for all  $t$ , such that

$$\frac{\partial Z}{\partial Y_t} y_t + \frac{\partial Z}{\partial Y_{t+1}} y_{t+1} = 0.$$

Using the definition of  $Z$ , the above equation can be written as the following system of three equations:

$$\left[ \frac{F''_{KK} k_{t+1} + F''_{KL} l_{t+1}}{F'_K} + \frac{F''_{KL} k_t + F''_{LL} l_t}{F'_L} \right] (F'_K F'_L U' + (F'_K F'_L)^2 L U'')$$

$$+ (F'_K F'_L)^2 U'' l_t - V'' l_t = 0$$

$$k_{t+1} + M q_t - F'_L l_t - L(F''_{KK} k_t + F''_{LL} l_t) = 0$$

$$q_{t+1} - Q(F''_{KK} k_{t+1} + F''_{KL} l_{t+1}) - F'_K q_t = 0.$$

All the derivatives are taken at the golden rule equilibrium, and we have dropped the asterisks from the variables  $(K, L, Q, C)$ . To simplify, it is useful to eliminate  $k_t$  and use instead  $r_t$  the deviation of the rate of return on capital from its equilibrium value. One has

$$r_t = F''_{KK} k_t + F''_{KL} l_t$$

and using the homogeneity of degree zero of  $F'_K$  and  $F'_L$

$$F''_{KK} K + F''_{KL} L = F''_{Kl} K + F''_{LL} L = 0.$$

$$r_t = -\frac{L}{K} (F''_{KL} k_t + F''_{LL} l_t).$$

Using  $F'_K = 1$ ,  $C = F'_L L$ , and  $F'_L U' = V'$ , one obtains

$$\begin{aligned} \left( r_{t+1} - \frac{K}{C} r_t \right) \left( 1 + \frac{CU''}{U'} \right) + \frac{CU''}{U'} \frac{l_t}{L} - \frac{LV''}{V'} \frac{l_t}{L} &= 0 \\ K \left( \frac{l_{t+1}}{L} - \frac{r_{t+1}}{c} \right) + MQ \frac{q_t}{Q} - C \frac{l_t}{L} + Kr_t &= 0 \\ \frac{q_{t+1}}{Q} - r_{t+1} - \frac{q_t}{Q} &= 0. \end{aligned}$$

Finally

$$\begin{aligned} (1 - \bar{U}) \left[ r_{t+1} - \frac{K}{C} r_t \right] - (\bar{U} + \bar{V}) \frac{l_t}{L} &= 0 \\ \frac{l_{t+1}}{L} - \frac{r_{t+1}}{\varepsilon} + r_t + \frac{MQ}{K} \frac{q_t}{Q} - \frac{C}{KL} \frac{l_t}{L} &= 0 \\ \frac{q_{t+1}}{Q} - r_{t+1} - \frac{q_t}{Q} &= 0. \end{aligned} \tag{5.1}$$

Note that by the second equation in (1.1),  $K + MQ = C$ .

Now an economy is in  $C$  if we can find a sequence  $(l_t, r_t, q_t)$  different from zero, satisfying (5.1) and such that  $\|(l_t, r_t, q_t)\| \leq 1$  for all  $t$ . When  $\bar{U}$  is different from 1, we can divide the first equation by  $(1 - \bar{U})$  and we obtain (2.2). The determinant of the matrix of the linear system (2.2) is equal to  $(1 + \alpha)$ , which is always different from zero. Thus the system is invertible and starting from any  $(l_0, r_0, q_0)$ , one can compute  $(l_t, r_t, q_t)$  for all  $t$ . To ensure that  $\|(l_t, r_t, q_t)\|$  stays bounded, a necessary and sufficient condition is that the matrix of (2.2) has at least one eigenvalue of modulus 1.

When  $\bar{U} = 1$ , the first equation gives  $l_t = 0$ , all  $t$  (the supply of labor stays

constant). The system reduces to (2.3) whose matrix has a determinant equal to  $\varepsilon$ , different from zero. The same argument as above applies.

Q.E.D.

**LEMMA V.1.** *Given the production function  $F$ , the set of feasible stationary allocations, i.e.,  $(K, L, C)$  such that  $K \geq 0$ ,  $0 \leq L \leq \bar{L}$ ,  $C \geq 0$  and  $C + K = F(K, L)$ , is compact.*

*Proof of Lemma V.1.* By definition, it is a closed subset of  $\mathbb{R}_+^3$ . We have to show that it is bounded. We prove this fact by contraposition.

Let  $(K^k, L^k, C^k)$  be a sequence of feasible allocations with  $\|K^k, L^k, C^k\| \rightarrow \infty$  with  $k$ . Since  $L^k$  is bounded, we can assume (take a subsequence) that  $L^k$  converges to  $L$ . Because of the equality  $C^k + K^k = F(K^k, L)$ , we must then have  $K^k \rightarrow \infty$  with  $k$ . Dividing through by  $K^k$  and using Assumptions I.2 give the desired contradiction. Q.E.D.

The proofs of Propositions III.1 and III.2 split into two parts. The characterization of  $C_F$  and  $C_H$  in terms of the parameters  $(K^*/C^*, \varepsilon, \alpha)$  is easy and we shall begin with it. The use of the results on bifurcation to prove the existence of a cycle is more involved and needs a lot of steps. We deal with that later.

**LEMMA V.2.**

(i) *The polynomial  $P_1(\lambda, \alpha)$  has a root equal to  $-1$  if and only if*

$$\alpha = -\frac{4 + 2(K^*/C^* + C^*/K^*)}{1 + 2/\varepsilon + C^*/K^*}.$$

$P_2(\lambda)$  does not have a root equal to  $-1$ .

(ii) *For the polynomials  $P_1(\lambda, \alpha)$  or  $P_2(\lambda)$  to have a pair of complex conjugate roots of modulus 1, one must have*

$$\frac{K^*}{C^*} > 1.$$

*Then  $P_1(\lambda, \alpha)$  has a pair of complex roots of modulus 1 iff  $\varepsilon \neq 1$  and*

$$\alpha = \frac{K^*/C^* - 1}{1 - 1/\varepsilon}$$

*while  $P_2(\lambda)$  has a pair of complex conjugate roots of modulus 1 iff  $\varepsilon = 1$ .*

*Proof of Lemma V.2.*

(i) Comes directly from the computation of  $P_1(-1, \alpha)$  and  $P_2(-1)$ .

(ii) Is immediate as far as  $P_2(\lambda)$  is concerned. The product of the roots is  $\varepsilon$ , and their sum must be smaller than 2 which gives  $K^*/C^* > 1$ .

The product of the roots of  $P_1(\lambda, \alpha)$  is equal to  $(1 + \alpha)$ . Therefore,  $P_1(\lambda, \alpha)$  has two complex conjugate roots of modulus 1 if and only if it is of the form  $-(\lambda - 1 - \alpha)(\lambda^2 + 2a\lambda + 1)$ , for some  $a$  with  $|a| < 1$ . By identification, we obtain

$$1 + \alpha - 2a = \frac{K^*}{C^*} + \frac{C^*}{K^*} + 1 + \frac{\alpha}{\varepsilon}$$

$$-2a(1 + \alpha) + 1 = \frac{K^*}{C^*} + \frac{C^*}{K^*} + 1 + \frac{\alpha}{\varepsilon} + \frac{\alpha C^*}{K^*}.$$

By difference, we obtain  $2a = -1 - C^*/K^*$ , and therefore  $|a| < 1$  is equivalent to  $K^*/C^* > 1$ . Replacing  $a$  by its value in the first equation gives the desired result. Q.E.D.

To prove the second part of Propositions III.1 and III.2, we proceed as follows. We take an economy  $e_0$  in the critical set  $\mathbf{C}$  (except perhaps for a negligible subset in  $\mathbf{C}$ ). We construct a one parameter family of economies  $e$  near  $e_0$  and we apply to this family the bifurcation theorems.

Let  $e_0 = (U_0, V_0, F_0)$  be an economy in  $\mathbf{C}$  with

$$\bar{U}_0 = -C^*(e_0) U''(C^*(e_0))/U'(C^*(e_0)) \neq 1,$$

so that  $\alpha_0 = (\bar{U}_0 + \bar{V}_0)/(1 - \bar{U}_0)$  is well defined. We consider a one parameter family of economies  $e$  close to  $e_0$  built in the following way (there are many other possibilities).  $e$  has the same functions  $V = V_0$ ,  $F = F_0$  as  $e_0$  and the same golden rule steady state, which is implied by the equality  $U'(C^*(e_0)) = U'_0(C^*(e_0))$ . The only difference is in the concavity of  $U$ ,

$$U''(C) = \frac{\bar{U}}{\bar{U}_0} U''_0(C) \quad \text{for } C \text{ in } (0, C^*(e_0) + 1),$$

where  $\bar{U}$  and  $\alpha$  are linked through

$$\alpha = \frac{\bar{U} + \bar{V}_0}{1 - \bar{U}}.$$

It is easy to see that on  $(0, C^*(e_0) + 1)$ ,

$$U'(C) = \frac{\bar{U}}{\bar{U}_0} U'(C) + \left(1 - \frac{\bar{U}}{\bar{U}_0}\right) U'(C^*(e_0))$$

and therefore there is an open interval  $A$  around  $\alpha_0$  where  $U'(C^*(e_0) + 1)$  is strictly positive and  $U'(C)$  can be extended in a smooth way on  $(0, +\infty)$  with positive values.  $\alpha$  will be our parameter of interest. It is immediate to check that the demand function of the consumer, and therefore the mapping  $Z$ , is smooth with respect to  $\alpha$ . *In all the following, we shall replace  $e$  by  $\alpha$  as an argument of the functional relationships.*

LEMMA V.3. *Assume  $\bar{U}_0 \neq 1$ . Then there is a neighborhood  $N \times A$  of  $(Y^*, \alpha_0)$  and a smooth function  $H: N \times A \rightarrow N$  such that, for all  $Y_t$  in  $N$ ,*

$$Y_{t+1} = H(Y_t, \alpha)$$

*if and only if*

$$Z(Y_{t+1}, Y_t; \alpha) = 0.$$

*Proof of Lemma V.3.* Take  $N$  small enough such that  $C_t \leq C_0 + 1$ . Apply the implicit function theorem (in fact the proof of Proposition II.1 has shown that  $\partial Z / \partial Y_{t+1}$  is regular whenever  $\bar{U} \neq 1$ ). Q.E.D.

The eigenvalues of  $DH(Y^*, \alpha)$  are, by construction, the roots of the polynomial  $P_t(\lambda, \alpha)$ , defined in (3.1). To apply the standard results of bifurcation theory, we have to make sure that the eigenvalues of modulus 1 cross the unit circle when  $\alpha$  crosses  $\alpha_0$ . This is the purpose of the next two lemmas.

LEMMA V.4. *Assume  $\alpha_0$  satisfies (i) of Lemma V.2. Then, there exists a neighborhood  $A$  of  $\alpha_0$  and a uniquely defined smooth function  $\lambda_F(\alpha)$  defined on  $A$  such that*

- (i)  $\lambda_F(\alpha_0) = -1$ ;
- (ii)  $P_t(\lambda, \alpha) = 0$  for  $\lambda$  in a neighborhood of  $-1$ , and  $\alpha$  in  $A$  if and only if  $\lambda = \lambda_F(\alpha)$ ;
- (iii)  $\lambda'_F(\alpha_0)$  is strictly negative and  $P'_{12}(-1, \alpha_0)$  is strictly positive.

*Proof of Lemma V.4.* It is straightforward applications of the implicit function theorem:

$$\begin{aligned} P'_{12}(-1, \alpha) &= -3 - 2 \left( \frac{K}{C} + \frac{C}{K} + \frac{\alpha}{\varepsilon} + 1 \right) - \left( \frac{K}{C} + \frac{C}{K} + \frac{\alpha}{\varepsilon} + 1 + \frac{\alpha C}{K} \right) \\ &= -3 \left[ 2 + \frac{K}{C} + \frac{C}{K} + \frac{\alpha}{\varepsilon} \right] - \frac{\alpha C}{K}, \end{aligned}$$

which gives, using Lemma V.2(i)

$$P'_{1z}(-1, \alpha_0) = 3 \left( \frac{\alpha_0}{2} + \frac{\alpha_0 C}{2K} \right) - \frac{\alpha_0}{K} = \frac{\alpha_0}{2} \left( 3 + \frac{C}{K} \right) < 0.$$

We have  $\lambda'_R(\alpha_0) = -P'_{1z}(-1, \alpha_0)/P'_{1\lambda}(-1, \alpha_0)$ , and to prove (iii), we show that  $P'_{1z}(-1, \alpha_0)$  is positive:

$$P'_{1z}(\lambda, \alpha_0) = \frac{\lambda^2}{\varepsilon} - \left( \frac{1}{\varepsilon} + \frac{C}{K} \right) \lambda + 1$$

$$P'_{1z}(-1, \alpha_0) = \frac{2}{\varepsilon} + 1 + \frac{C}{K} > 0. \quad \text{Q.E.D.}$$

LEMMA V.5. Assume  $K^*/C^* > 1$  and  $\alpha_0 = (K^*/C^* - 1)/(1 - 1/\varepsilon)$ . Then there exists a neighborhood  $A$  of  $\alpha_0$  and a uniquely defined differentiable function  $|\lambda_H(\alpha)|$  defined on  $A$  such that

- (i)  $|\lambda_H(\alpha_0)| = 1$ ;
- (ii)  $|\lambda_H(\alpha)|$  is the modulus of the complex roots of  $P_1(\lambda, \alpha)$  for  $\alpha$  in  $A$ ;
- (iii)  $|\lambda_H(\alpha_0)|'$  is strictly positive.

*Proof of Lemma V.5.* Since the product of the roots of  $P_1(\lambda, \alpha)$  is equal to  $(1 + \alpha)$ , at  $\alpha_0$ ,  $P_1(\lambda, \alpha)$  has a real eigenvalue  $\lambda_R$  equal to  $(1 + \alpha)$ . An argument similar to that of Lemma V.4 gives a continuously differentiable function  $\lambda_R(\alpha)$  with

$$\lambda'_R(\alpha_0) = -\frac{P'_{1z}(1 + \alpha_0, \alpha_0)}{P'_{1\lambda}(1 + \alpha_0, \alpha_0)}.$$

Straightforward calculations, using the fact that  $\alpha_0/\varepsilon = \lambda_R(\alpha_0) - K/C$ , give

$$P'_{1\lambda}(1 + \alpha_0, \alpha_0) = -\lambda_R(\alpha_0)^2 + \left( 1 + \frac{C}{K} \right) \lambda_R(\alpha_0) - 1$$

which is always negative for  $K/C > 1$ . Now

$$|\lambda_H|^2 \lambda_R = 1 + \alpha$$

give again by the implicit function theorem

$$2\lambda_R(\alpha_0) |\lambda_H(\alpha_0)|' = 1 - \lambda'_R(\alpha_0)$$

or

$$|\lambda_H(\alpha_0)|' = \frac{1}{2\lambda_R(\alpha_0)} \frac{P'_{1z}(1 + \alpha_0, \alpha_0) + P'_{1\lambda}(1 + \alpha_0, \alpha_0)}{P'_{1\lambda}(1 + \alpha_0, \alpha_0)}.$$

Now

$$\begin{aligned} P'_{1x}(1 + \alpha_0, \alpha_0) + P'_{1\lambda}(1 + \alpha_0, \alpha_0) &= \lambda_R(\alpha_0)[\lambda_R(\alpha_0) - 1] \left( \frac{1}{\varepsilon} - 1 \right) \\ &= -\lambda_R(\alpha_0) \left( \frac{K^*}{C^*} - 1 \right). \end{aligned}$$

Therefore

$$|\lambda_H(\alpha_0)|' = - \left( \frac{K^*}{C^*} - 1 \right) \frac{1}{2P'_{1\lambda}(1 + \alpha_0, \alpha_0)} > 0. \quad \text{Q.E.D.}$$

*Proof of Proposition III.1.* The first part of the proposition is a direct consequence of Lemma V.2(i).

Let  $e_0$  be an economy in  $C_F$  and take an open neighborhood  $V$  of  $e_0$ . This implies that, using our one dimensional parametrization, there is an open neighborhood  $A$  of  $\alpha_0$ , such that for  $\alpha$  in  $A$ , the economies associated with  $\alpha$  belong to  $V$ . We shall show by contradiction that there is an economy  $\alpha$  which possesses a period two cycle.

Suppose this is not the case. The mapping  $H(Y, \alpha)$  has a unique fixed point  $Y^*$ , independent of  $\alpha$ . Let  $T(Y, \alpha)$  be the second iterate of  $H$ , i.e.,  $T(Y, \alpha) \equiv H(H(Y, \alpha), \alpha)$ . This implies  $DT(Y^*, \alpha) = (DH(Y^*, \alpha))^2$ , and the eigenvalues of  $DT(Y^*, \alpha)$  are the squares of the eigenvalues of  $DH(Y^*, \alpha)$ . Consider the map  $Y - T(Y, \alpha)$ . By our contradiction argument, it has a unique zero at  $Y^*$  for all  $\alpha$  in  $A$ . Its topological degree over a circle around  $Y^*$  is a homotopy invariant and is equal to the sign of  $\det(I - DT(Y^*, \alpha))$ . But  $\det[\lambda I - DT(Y^*, \alpha)] = -(\lambda - \lambda_F^2(\alpha))(\lambda - \lambda_1^2(\alpha))(\lambda - \lambda_2^2(\alpha))$ , where  $\lambda_F(\alpha)$  is as in Lemma V.4 and  $\lambda_1(\alpha), \lambda_2(\alpha)$  are the two other roots of  $P_1(\lambda, \alpha)$ . Since  $(\lambda - \lambda_1^2(\alpha))(\lambda - \lambda_2^2(\alpha))$  does not change sign for  $\lambda = 1$  and  $\alpha$  in a small enough neighborhood of  $\alpha_0$ , Lemma V.4 shows that  $\det(I - DT(Y^*, \alpha))$  as a function of  $\alpha$  changes sign at  $\alpha = \alpha_0$ . This implies that  $T(Y, \alpha)$  has some other fixed point than  $Y^*$ , the desired contradiction. Q.E.D.

To prove Proposition III.2, several more steps are necessary. The first one is to reduce the dimension of the problem to 2, in order to apply standard results on the Hopf bifurcation. This is done through the center manifold theorem. Let  $e_0$  be an economy in  $C_H$  and denote by  $E^c$  (resp.  $E^s$ ) the two dimensional (resp. one dimensional) eigenspace of  $DH(Y^*, \alpha_0)$  associated with the two complex conjugates eigenvalues of modulus 1 (resp. the simple real eigenvalue equal to  $(1 + \alpha_0)$ ). Any  $Y$  in  $\mathbb{R}^3$  can be written in a unique way as  $Y^c + Y^s$ ,  $Y^c$  in  $E^c$ ,  $Y^s$  in  $E^s$ , and we set, with obvious notations,

$$H(Y, \alpha) = H^c(Y^c, Y^s, \alpha) + H^s(Y^c, Y^s, \alpha),$$

where  $H^c(Y^c, Y^s, \alpha)$  belongs to  $E^c$  and  $H^s(Y^c, Y^s, \alpha)$  belongs to  $E^s$ .

LEMMA V.6 (Center Manifold). *Assume  $-1 > \alpha_0 > -2$ . There exists a smooth map  $\phi: E^c \times \mathbb{R} \rightarrow E^c$  with the following properties. Let  $F_\alpha: E^c \rightarrow E^c$  be a family of maps indexed by  $\alpha$  defined as  $F_\alpha(Y^c) = H^c(Y^c, \phi(Y^c, \alpha), \alpha)$ . Then*

(i) (invariance)  $H^c(Y^c, \phi(Y^c, \alpha), \alpha) = \phi(F_\alpha(Y^c), \alpha)$  for  $(\alpha, Y^c)$  in an open neighborhood of  $(\alpha_0, Y^{c*})$ .

(ii) (tangency)  $D_\alpha \phi(Y^{c*}, \alpha_0) = D_\alpha \phi(Y^{c*}, \alpha_0) = 0$ .

*Proof of Lemma V.6.* Since  $-1 > \alpha_0 > -2$ , the real eigenvalue of  $DH(Y^* \alpha_0)$ ,  $1 + \alpha_0$ , is smaller than 1 in absolute value. The center manifold theorem applies (see Looss [12]) to the map from  $\mathbb{R}^2$  into itself which associates to  $((Y, \alpha), \alpha)$ . The center manifold is of dimension 3 and  $\phi$  is its chart. The local invariance property is described by (i) which says that any point  $(Y^c, \phi(Y^c, \alpha))$  on the manifold is mapped by  $H$  into another point  $(F_\alpha(Y^c), \phi(F_\alpha(Y^c), \alpha))$  on the manifold. The tangency property, given our choice of coordinates, is simply expressed by (ii). Q.E.D.

*Remark V.7.* For  $\alpha_0 > 0$  or  $\alpha_0 < -2$ , the same result as Lemma V.6 holds for  $H^{-1}$  (instead of  $H$ ). This allows us to treat directly all cases where  $\alpha_0 \neq -2$ .

We are now in a position to apply the Hopf bifurcation theorem to the family  $F_\alpha$  which maps  $E^c$  into itself.

THE HOPF BIFURCATION THEOREM FOR MAPS (Guckenheimer and Holmes [10, p. 162]). *Let  $F_\alpha: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be one parameter family of mappings which has a smooth family of fixed points at which the eigenvalues of  $DF_\alpha$  are complex conjugates  $\lambda(\alpha), \bar{\lambda}(\alpha)$ . Assume*

$$|\lambda(\alpha_0)| = 1 \text{ but } \lambda^j(\alpha_0) \neq 1 \text{ for } j = 1, 2, 3, 4 \tag{SH1}$$

$$\frac{d}{d\alpha} (|\lambda(\alpha)|) = d \neq 0 \text{ at } \alpha = \alpha_0. \tag{SH2}$$

*Then there is a smooth change of coordinate  $h$  so that the expression of  $hF_\alpha h^{-1}$  in polar coordinates has the form*

$$hF_\alpha h^{-1}(r, \theta) = (r(1 + d(\alpha - \alpha_0) + ar^2), \theta + c + br^2) + \text{higher order terms.}$$

*If in addition  $\alpha \neq 0$ , there is a two dimensional surface  $\Sigma$  in  $\mathbb{R}^2 \times \mathbb{R}$  having quadratic tangency with the plane  $\mathbb{R}^2 \times \{\alpha_0\}$  which is invariant for  $F$ . If  $\Sigma \cup (\mathbb{R}^2 \times \{\alpha\})$  is larger than a point, then it is a closed curve.*

*Proof of Proposition III.3.* The first part of the proposition is a direct consequence of Lemma V.2(ii).

Take an economy  $e_0$  in  $C_H$  and an open neighborhood of  $e_0$ . Without

loss of generality, we can take  $\bar{C}_0 \neq 1$ ,  $\varepsilon_0 \neq 1$ , and  $\alpha_0 \neq -2$  (otherwise pick an economy  $e_1$  satisfying the above requirements close to  $e_0$  and a neighborhood of  $e_1$  in the given neighborhood of  $e_0$ ).

Assume  $F_x$  satisfies (SH1) and (SH2), to be checked later. The Hopf bifurcation theorem applied to  $F_x$  gives a closed trajectory in  $E^c$ . By (i) of Lemma V.6, we obtain the desired result.<sup>3</sup> By Remark V.7, this arguments works both for  $-1 > \alpha_0 > -2$  and for  $(\alpha_0 > 0$  or  $\alpha_0 < -2)$ .

These remains to be checked that  $F_x$  satisfies (SH1) and (SH2).

(i)  $F_x$  satisfies (SH1).

By construction  $DH^c(Y^{*c}, Y^{*c}, x_0)$  has the same two complex conjugate eigenvalues of modulus 1 as  $DH(Y^*, x_0)$ . Therefore  $|\lambda(x_0)| = 1$ . We check that  $\lambda^j(x_0) \neq 1$  for  $j=1, 2, 3, 4$ . In fact, we know that 1 is *not* a root of  $P_1(\lambda, x) = 0$  in the relevant domain. Given that the product of the roots is  $1 + x$ ,  $-1$  is a root iff  $\alpha_0 = -2$ , which was ruled out above. Finally

$$\operatorname{Re} P_1\left(\frac{1+i\sqrt{3}}{2}, x_0\right) = x_0 \left(1 - \frac{C}{2K}\right) \neq 0.$$

and

$$\operatorname{Re} P_1(i, x_0) = x_0 \left(1 - \frac{1}{\varepsilon}\right) - \frac{K}{C} - \frac{C}{K} = -1 - \frac{C}{K} \neq 0.$$

(ii)  $F_x$  satisfies (SH2).

By construction,  $DH(Y^*, x)$  is a matrix of current element  $h_{jk}(x)$ ,  $j, k = 1, 2, 3$  which is a continuously differentiable function of  $x$  and satisfies

$$\begin{aligned} h_{13}(x_0) &= h_{23}(x_0) = h_{31}(x_0) = h_{32}(x_0) = 0 \\ A(x_0) &= h_{11}(x_0)h_{22}(x_0) - h_{12}(x_0)h_{21}(x_0) = 1 \\ &\text{(eigenvalues of modulus 1)} \end{aligned}$$

and

$$h_{33}(x_0) = 1 + \alpha_0 \quad \text{(real eigenvalue).}$$

<sup>3</sup> In the Hopf bifurcation theorem if  $a = 0$ , the invariant circle can exist above or below  $a_0$ , depending on higher order terms of  $hF_x h^{-1}$  that appear in the statement of the theorem. If all higher order terms disappear the problem of finding an invariant circle is still open. However, in the case where the system is completely linear, invariant circles can be constructed for any  $r$  at  $x = x_0$ .

A direct computation shows that, at  $\alpha = \alpha_0$ ,

$$\frac{d \det DH(Y^*, \alpha)}{d\alpha} = (1 + \alpha_0) \frac{d\Delta(\alpha)}{d\alpha} + \frac{dh_{33}(\alpha)}{d\alpha} = 1.$$

It is easy to see that  $dh_{33}(\alpha)/d\alpha$  is equal to the derivative of the real eigenvalue of  $DH(Y^*, \alpha)$  with respect to  $\alpha$  (take the equality involving the eigenvector associated with this eigenvalue and compute derivatives). By Lemma V.5 this implies

$$\frac{d\Delta(\alpha)}{d\alpha} > 0.$$

Now by Lemma V.6(ii), the derivative of the determinant of  $DF_x(Y^{*c})$  with respect to  $\alpha$  is equal to  $d\Delta(\alpha)/d\alpha$ . This completes the proof of Proposition III.3. Q.E.D.

The "if" part of Proposition II.2 is a direct consequence of Proposition III.1 and III.3. The "only if" part follows from the following consideration. From Proposition II.1 if  $e_0$  is not in  $C$ , its golden rule equilibrium is locally unstable either in the forward or in the backward direction in time (no roots of unit modulus). Therefore any initial condition  $Y$ , with  $Y \neq Y^*$ , but in a sufficiently small neighborhood  $V$  of  $Y^*$ , will give rise to trajectories that move out of the neighborhood  $V$ , either going forward in time or going backward in time. This will also be true for economies in a sufficiently small open neighborhood of  $e_0$ . This contradicts the requirement of Proposition II.2 that every open neighborhood of  $e_0$  contain some economy  $e$  such that any open neighborhood  $V$  of the golden rule of  $e$  possesses an intertemporal equilibrium that differs from the golden rule of  $e$  but that lies in  $V$  for  $t$  varying from  $-\infty$  to  $+\infty$ .

The proofs of Propositions IV.1 and IV.2 are straightforward. They are left to the reader for the sake of brevity.

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