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Abstract

The paper provides a complete characterization of the Nash equilibrium outcomes for the War of Attrition when time is continuous and information is complete. It allows for asymmetric payoffs and an arbitrary time horizon. In addition to certain (asymmetric) pure strategy equilibria which always exist, we establish the conditions under which there is also a continuum of mixed strategy equilibria. These are most likely to exist when either the horizon is infinite or the game is symmetric.

1. Introduction

In this paper, we present a general analysis of the War of Attrition in continuous time with complete information. In this game, each of two players must choose a time in the interval [0,1] at which he plans to concede in the event that the other player has not already conceded.[1] The return to conceding decreases with time, but, at any time, a player earns a higher return if the other concedes first. The game was introduced by Maynard Smith (1974) to study the evolutionary stability of certain patterns of behavior in animal conflicts. It has subsequently been applied by economists to a variety of economic conflicts such as price wars (e.g. Fudenberg and Tirole (1986), Ghemawat and Nalebuff (1985), Kreps and Wilson (1982)), bargaining (Ordover and Rubinstein (1985), Osborne (1985)), the supply of public goods (Bliss and Nalebuff (1984)), and oil exploration (Wilson (1983)).[2]

Most authors have assumed specific functional forms for the payoffs. The only general analysis of the equilibria of this game in continuous time with complete information is by Bishop and Cannings (1978). They study a general symmetric game, but restrict the equilibrium analysis to symmetric Nash equilibria which satisfy a certain stability property.[3] This paper generalizes their model to allow for asymmetric return functions and arbitrary payoffs in the event that neither player ever concedes. We provide a complete characterization of the Nash equilibrium outcomes.

In our analysis, we distinguish between two kinds of equilibria. An equilibrium is "degenerate" if, with probability 1, one of the players moves at time 0 or both wait until time 1. There is always at least one equilibrium of this type. In addition, there is sometimes a continuum of "nondegenerate" equilibria. These are equilibria in which both players move over an interval (0,t) according to a strictly increasing, continuous distribution, after which both wait until the time 1. The indeterminacy comes at the beginning of the game where the only constraint is that only one player may move immediately with positive probability.

There are two types of restrictions on the return functions which provide the necessary and sufficient conditions for the existence of a nondegenerate equilibrium. The first is an "initial" restriction which guarantees that a distribution function can be chosen for each player to make the other player indifferent to moving in any arbitrary small interval starting at time 0. A sufficient (but not necessary) condition is that the return to leading strictly exceed the return to following at time 0. The second is a "terminal" restriction. It requires that each player be indifferent between moving an instant after time 0 and waiting until time 1. There are two ways to satisfy this restriction, each of which generally corresponds to a different class of equilibria.

The first possibility is for both players to earn returns to leading which is equal to their terminal returns (i.e. their payoffs if both players wait until time t) at exactly the same time t. In this case, both players move according to a strictly increasing, continuous distribution function from time 0 to time t, whereupon they both wait until time 1.[4] The second possibility is for at least one of the players to move before time 1 with probability one, so that the terminal return of the other player is irrelevant. For this condition to be satisfied, the difference between his return to following and his return to leading must converge to zero as time approaches time 1. In this case, both players move according to a strictly increasing, continuous distribution function from time 0 to time 1. In some instances, both conditions may be satisfied, in which case there are two distinct, one parameter families of equilibria.

Models with an infinite horizon in which players discount the returns from future actions usually satisfy the second condition. Because of the discounting, the return from any action taken at time t approaches zero as t becomes large. As a result, the present value of the gain from following also approaches 0. The condition may also hold for models with a finite horizon if the payoff from following depends on the amount of time remaining in the game. One such example is the complete information analogue to the concession game studied by Kreps and Wilson (1982) and Fudenberg and Kreps (1985). In many applications, however, the return to following is bounded away from the return to leading whenever the horizon is finite. In these cases, nondegenerate equilibria are likely to exist only if the returns are symmetric.

The paper is organized as follows. In Section 2, we introduce the game and present the assumptions which define the War of Attrition. In Section 3, we derive the properties of the equilibrium strategies on the interior of the strategy space. In Sections 4 and 5, we develop the initial and terminal conditions. In Section 6, we provide a complete characterization of the equilibria. Section 7 contains a brief discussion of the implications of subgame perfection. We conclude in Section 8 with a discussion of the results and their relation to some of the applications which have appeared in the economics literature.

2. The Game

We begin with a general description of the game. Two players, a and b, must decide when to make a single move at some time t between 0 and 1.

The payoffs are determined as soon as one player moves. In what follows, α refers to an arbitrary player and β to the other player. If player α moves first at some time t, he is called the **leader** and earns a return of $L_\alpha(t)$. If the other player moves first at time t, then player α is called the **follower** and earns a return of $F_\alpha(t)$. If both players move simultaneously at time t, the return to player α is $S_\alpha(t)$. We will refer to $S_\alpha(1)$ as the **terminal** return. None of these functions need be identical for the two players.

In the strategic form of the game, a pure strategy for player α is a time $t_\alpha \in [0,1]$ at which he plans to move given that neither player moves before that time. Given a strategy pair $(t_a,t_b) \in [0,1]\times[0,1]$, the payoff to player α is then defined as follows:

$$P_\alpha(t_a,t_b) = \begin{cases} L_\alpha(t_\alpha) & \text{if} \quad t_\alpha < t_\beta \\ S_\alpha(t_\alpha) & \text{if} \quad t_a = t_b \\ F_\alpha(t_\beta) & \text{if} \quad t_\alpha > t_\beta \end{cases}$$

2.1 Assumptions on the Payoff Functions

Our first assumption guarantees that the payoff functions are continuous everywhere but on the diagonal.

A1 L_α and F_α are continuous functions on $[0,1]$.[5]

Games with this structure are generally called "noisy" games of timing.[6] Our next assumption characterizes the generalized War of Attrition.

A2 (i) $F_\alpha(t) > L_\alpha(t)$ for $t \in (0,1)$;

(ii) $F_\alpha(t) > S_\alpha(t)$ for $t \in [0,1)$;

(iii) $L_\alpha(t)$ is strictly decreasing for $t \in [0,1)$.

Condition (i) requires that the return to following at any time $t > 0$ strictly exceeds the return to leading at time t. We do not rule out the possibility that $L_\alpha(0) = F_\alpha(0)$ nor the possibility that $F_\alpha(t) = L_\alpha(t)$. Condition (ii), however, requires that the return to following strictly exceed the return to tying at all times less than 1. Combined with condition (iii), these conditions imply that, at any time $t < 1$, each player prefers to wait if the other player plans to move but, if forced to move first, would prefer to move sooner than later. Note, however, that, since S_α is not necessarily continuous, our assumptions impose no restrictions on the relation between the terminal return, $S_\alpha(1)$, and either $L_\alpha(1)$ or $F_\alpha(1)$. Consequently, $S_\alpha(1)$ may reflect the equilibrium payoff of any continuation game which is played whenever both players wait until time 1.

In Figure 1, we have illustrated three possible relations between the return functions. In each case, the return functions are normalized so that $S_\alpha(1) = 0$. As required by Assumption A2, in each case, F_α lies above L_α over the open interval (0,1) with L_α strictly decreasing throughout. Figure 1(a) illustrates typical return functions corresponding to a game where the returns are discounted and the horizon is infinite. In this case, both the return to leading and the return to following converge to 0 as time approaches 1. Figure 1(b) illustrates the case where the return from leading is bounded above the return at time 1. This corresponds to a model of oil exploration analyzed by Hendricks (1984). Figure 1(c) illustrates the case where the return to leading actually falls below the terminal return at some time $t < 1$. It corresponds to simple bargaining models or contests in which the each player receives a compromise outcome if neither ever concedes. We will return to each of these examples again in Section 7. Although not illustrated, Assumption A2 requires that S_α must also lie below F_α over the interval $(0,1]$. However, neither function need be decreasing.

2.2 Equilibrium

It is important for our results to permit agents to randomize across pure strategies. A **mixed strategy** for player α is a probability distribution function G_α on $[0,1]$.[7] If we extend the domain of the payoff functions to the set of all pairs of mixed strategies in the obvious way, then a strategy combination (G_a^*,G_b^*) is an **equilibrium** if $P_\alpha(G_\alpha^*,G_\beta^*) \geq P_\alpha(G_\alpha,G_\beta^*)$ for all mixed strategies G_α, $\alpha = a,b$ and $\beta \neq \alpha$.

For the remainder of the paper, (G_a,G_b) will refer to an equilibrium combination, and $q_\alpha(t)$ will denote the probability with which player α moves at exactly time t. We will repeatedly use the fact that if (G_a,G_b) is a pair of equilibrium distributions, then $P_\alpha(t,G_\beta) = \sup_{v\in[0,1]}P_\alpha(v,G_\beta)$ for any t in the support of G_α.

3. Equilibrium Restrictions on the Interval (0,1)

In this section, we focus on the properties of the equilibrium strategies in the interior of the unit interval.

We begin by establishing that the supports of the strategies of both players must have the same interior over the interval of times during which neither player moves with probability 1.

LEMMA 3.1: Suppose $G_\alpha(t_1) = G_\alpha(t_2) < 1$ for $t_2 > t_1$. Then $G_\beta(t_1) = G_\beta(t_2+\delta)$ for some $\delta > 0$.

PROOF: Suppose $G_\alpha(t_1) = G_\alpha(t_2) < 1$ for $t_1 < t_2$ and choose $\epsilon \in (0,t_2-t_1)$. Then, for any $t \in (t_1+\epsilon,t_2]$, it follows from Assumption A2 that player β prefers to move at time $t_1+\epsilon$ since there is no chance that player α will move in the intervening interval:

$$P_\beta(t,G_\alpha)-P_\beta(t_1+\epsilon,G_\alpha) = [L_\beta(t)-L_\beta(t_1+\epsilon)][1-G_\alpha(t_1)] < 0.$$

Furthermore, for any $\epsilon_1 > 0$ sufficiently small, the right-continuity of G_α implies that there is an arbitrarily small $\delta > 0$ such that $G_\alpha(t_2+\delta)-G_\alpha(t_2) < \epsilon_1$. It then follows from Assumptions A1 and A2 that, for $t \in (t_2,t_2+\delta)$,

$$\begin{aligned}P_\beta(t,G_\alpha)-P_\beta(t_1+\epsilon,G_\alpha) &= \int_{t_1+\epsilon}^{t}[F_\beta(v)-L_\beta(t_1+\epsilon)]dG_\alpha(v) + [S_\beta(t)-L_\beta(t_1+\epsilon)]q_\alpha(t) \\ &\quad + [L_\beta(t)-L_\beta(t_1+\epsilon)][1-G_\alpha(t)] \\ &= o(\epsilon_1)+o(\epsilon_1)+[L_\beta(t)-L_\beta(t_1+\epsilon)][1-G_\alpha(t_2)+o(\epsilon_1)] \\ &< 0.\end{aligned}$$

Letting $\epsilon \to 0$, we may then conclude that for any $t \in (t_1,t_2+\delta]$, there is an earlier time $v \in (t_1,t)$ at which player β prefers to move. Q.E.D.

The next Lemma rules out any mass points in the interior of the interval of times which are reached with positive probability.

LEMMA 3.2: For $t \in (0,1)$, $\lim_{v\uparrow t} G_\beta(v) < 1$ implies $q_\alpha(t) = 0$.

PROOF: Suppose, for some $t \in (0,1)$, that $q_\alpha(t) > 0$. Then, for any $\epsilon > 0$, there is an (arbitrarily small) $\delta > 0$ such that (i) $L_\beta(t-\delta) - L_\beta(t+\delta) < \epsilon$, and (ii) $q_\alpha(t+\delta) = 0$ with $G_\alpha(t+\delta) - G_\alpha(t-\delta) < q_\alpha(t) + \epsilon$. It then follows from Assumptions A1 and A2 that, for ϵ and δ chosen sufficiently small,

$$\begin{aligned} P_\beta(t+\delta,G_\alpha) - P_\beta(t,G_\alpha) &= [F_\beta(t) - S_\beta(t)]q_\alpha(t) + \int_t^{t+\delta}[F_\beta(s) - L_\beta(t)]dG_\alpha(s) \\ &\quad + [L_\beta(t+\delta) - L_\beta(t)][1 - G_\alpha(t+\delta)] \\ &= [F_\beta(t) - S_\beta(t)]q_\alpha(t) + o(\epsilon) + o(\epsilon) > 0. \end{aligned}$$

Similarly, for $v \in (t-\delta,t)$,

$$\begin{aligned} P_\beta(t+\delta,G_\alpha) - P_\beta(v,G_\alpha) &= [F_\beta(v) - S_\beta(v)]q_\alpha(v) + \int_v^{t+\delta}[F_\beta(s) - L_\beta(v)]dG_\alpha(s) \\ &\quad + [L_\beta(t+\delta) - L_\beta(v)][1 - G_\alpha(t+\delta)] \\ &= o(\epsilon) + [F_\beta(t) - L_\beta(v)]q_\alpha(t) + o(\epsilon) + o(\epsilon) > 0. \end{aligned}$$

Consequently, player β will never move in the interval $(t-\delta,t]$. This implies that $G_\beta(t-\delta) = G_\beta(t)$. Then, if $\lim_{v\uparrow t} G_\beta(v) < 1$, Lemma 3.1 implies that $G_\alpha(t-\delta) = G_\alpha(t)$, contradicting the hypothesis that $q_\alpha(t) > 0$. Q.E.D.

If G_α is strictly increasing over some interval, then we may use the fact that player α must be indifferent between moving at any two times in the interval to explicitly characterize the equilibrium strategy of player β over this interval in terms of the return functions L_α and F_α.

For $0 \le t_0 < t \le 1$, define

$$I_\beta(t_0,t) = \exp[\int_{t_0}^{t}[dL_\alpha(v)/(F_\alpha(v)-L_\alpha(v))]].^{8}$$

LEMMA 3.3: Suppose G_α is strictly increasing over the interval $[t_0,t_1]$. Then, for $t_0 > 0$ and $t \in (t_0,t_1)$, $G_\beta(t) < 1$ implies

$$G_\beta(t) = 1 - [1-G_\beta(t_0)]I_\beta(t_0,t). \tag{3.1}$$

PROOF: Suppose that G_α is strictly increasing over the interval $[t_0,t_1]$. Then, since $G_\alpha(t) < 1$ for $t < t_1$, it follows from Lemma 3.2 that G_β is continuous on $(0,t_1)$. Therefore, for any $t \in [t_0,t_1)$,

$$\begin{aligned} 0 &= P_\alpha(t,G_\beta) - P_\alpha(t_0,G_\beta) \\ &= \int_{t_0}^{t}[F_\alpha(v)-L_\alpha(t_0)]dG_\beta(v) + [1-G_\beta(t)][L_\alpha(t)-L_\alpha(t_0)] \end{aligned} \tag{3.2}$$

Since G_β and L_α are both monotonic and continuous on $[t_0,t]$, we may apply the formula for integration by parts (Rudin (1964), p.122) to obtain

$$\begin{aligned} [1-G_\beta(t)][L_\alpha(t)-L_\alpha(t_0)] = &\int_{t_0}^{t}[1-G_\beta(v)]dL_\alpha(v) \\ &-\int_{t_0}^{t}[L_\alpha(v)-L_\alpha(t_0)]dG_\beta(v). \end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2) and rearranging terms then yields, for all $t \in [t_0,t_1)$,

$$\int_{t_0}^{t} [L_\alpha(v)-F_\alpha(v)][1-G_\beta(v)] \left[\frac{dG_\beta(v)}{1-G_\beta(v)} - \frac{dL_\alpha(v)}{L_\alpha(v)-F_\alpha(v)} \right] = 0. \tag{3.4}$$

But, since $[L_\alpha(v)-F_\alpha(v)][1-G_\beta(v)] < 0$ for all $v \in [t_0,t]$, equation (3.4) implies that

$$\int_{t_0}^{t} dG_\beta(v)/[1-G_\beta(v)] = -\int_{t_0}^{t} dL_\alpha(v)/[F_\alpha(v)-L_\alpha(v)].$$

Employing a change of variable (Rudin (1964), p.122–124), we may then apply the fundamental theorem of calculus to obtain:

$$\log[1-G_\beta(t)] = \log[1-G_\beta(t_0)] + \int_{t_0}^{t} dL_\alpha(v)/[F_\alpha(v)-L_\alpha(v)].$$

Taking antilogs and rearranging terms then yields equation (3.1). Q.E.D.

Note that, if L_α is continuously differentiable, equation (3.1) is simply the solution to the differential equation

$$G_\beta'(t)/[1-G_\beta(t)] = L_\alpha'(t)/[L_\alpha(t)-F_\alpha(t)]$$

with initial condition $G_\beta(t_0) \in [0,1]$. In this case, G_β has a continuous density function g_β over the interval (t_0,t_1).

We establish next that if neither player moves with probability 1 at time 0, then the game cannot end with certainty until time 1.

LEMMA 3.4: $G_a(0) < 1$ and $G_b(0) < 1$ implies $G_\beta(0) < 1$ for $\beta = a,b$.

PROOF: Let $\hat{t} = \sup\{t \geq 0: G_\alpha(t) < 1$ for $\alpha = a,b\}$ be the earliest time by which one of the players plans to move with certainty. The lemma is equivalent to the requirement that $\hat{t} \notin (0,1)$. Suppose $0 < \hat{t} < 1$.

We will show first that the strategy of at least one of the players must have a mass point at time $\hat{t}$. Suppose not. Then, for some player β, G_β is strictly increasing over an interval $(t',\hat{t})$ and $\lim_{t\uparrow\hat{t}}G_\beta(t) = 1$. Then, for any $t \in (t',\hat{t})$, Lemma 3.3 combined with Assumption A2 implies that G_α is strictly increasing over (t',t). Combining Lemma 3.3 with Assumption A2 again, we then obtain that $\lim_{t\uparrow\hat{t}}G_\beta(t) = 1 - [1-G_\beta(t')]I_\beta(t',\hat{t}) < 1$. A contradiction.

But, if $q_\beta(\hat{t}) > 0$, then Lemma 3.2 implies that $\lim_{t\uparrow\hat{t}}G_\alpha(t) = 1$. The definition of $\hat{t}$ then implies that G_α is strictly increasing over some interval $(t',\hat{t})$. It then follows from Assumption A2 and Lemma 3.3 that $\lim_{t\uparrow\hat{t}}G_\alpha(t) < 1$. This contradiction proves the lemma. Q.E.D.

Define $t^* \equiv \sup\{t \geq 0: G_\alpha$ is strictly increasing on $[0,t)$ for $\alpha = a,b\}$ to be the beginning of the first interval during which one of the players moves with probability 0. Combining Lemmata 3.1 to 3.4, we can show that, unless one of the players moves with probability 1 at time 0, neither player ever moves in the interval $(t^*,1)$.

LEMMA 3.5: Suppose $G_\alpha(0) < 1$. Then

(i) $G_\beta(t) = 1 - [1-q_\beta(0)]I_\beta(0,t)$ for $0 \leq t < t^*$, and

(ii) $G_\beta(t) = G_\beta(t^*)$ for $t^* \leq t < 1$.

PROOF: Suppose that $G_\alpha(0) < 1$. Then Lemma 3.3 and the right-continuity of G_β imply that $G_\beta(t) = 1 - [1-q_\beta(0)]I_\beta(0,t)$ for $0 \le t < t^*$. Therefore, the lemma will be proved if we can establish part (ii). Suppose $q_\beta(0) < 1$ and $t^* < 1$. Then Lemma 3.4 implies that $G_\beta(t^*) < 1$. Let

$$t' = \sup\{v \ge 0: G_\beta(v) = G_\beta(t^*)\}.$$

Since $G_\beta(t^*) < 1$, it follows that $t' \le 1$. We need to show that $t' = 1$.

Suppose first that $t' = t^*$. Then the definition of t^* implies that, for some $t'' > t'$, $G_\alpha(t'') = G_\alpha(t^*)$. Since $G_\alpha(t^*) < 1$, it then follows from Lemma 3.1 that $G_\beta(t'') = G_\beta(t^*)$, contradicting our assumption that $t' = t^*$.

Suppose next that $t^* < t' < 1$. Then, since Lemma 3.4 implies that $G_\alpha(t') < 1$, it follows from Lemma 3.2 that $G_\beta(t^*) = G_\beta(t') < 1$. But then Lemma 3.1 implies that, for some $\delta > 0$, $G_\alpha(t^*) = G_\alpha(t'+\delta) < 1$. Applying Lemma 3.1 again then yields $G_\beta(t^*) = G_\beta(t'+\delta)$, contradicting the definition of t'. Q.E.D.

Lemma 3.5 implies that the support of the equilibrium strategies is composed of at most an interval $[0,t^*]$ and $\{1\}$. Furthermore, any differences among the equilibrium stategies of player β must occur in the values of either $q_\beta(0)$ or t^*.

4. Equilibrium Restrictions on the Strategies at Time 0

In this section, we establish the restrictions imposed on the equlibrium strategies at time 0.

LEMMA 4.1: $q_a(0)q_b0) = 0.$

PROOF: Suppose that $q_\alpha(0) > 0$. Then, for any $\epsilon > 0$, there is an (arbitrarily small) $\delta > 0$ such that (i) $L_\beta(0)-L_\beta(\delta) < \epsilon$, and (ii) $q_\alpha(\delta) = 0$ with $G_\alpha(\delta)-G_\alpha(0) < \epsilon$. It then follows from Assumptions A1 and A2 that, for ϵ and δ chosen sufficiently small,

$$\begin{aligned} P_\beta(\delta,G_\alpha)-P_\beta(0,G_\alpha) &= [F_\beta(0)-S_\beta(0)]q_\alpha(0) + \int_0^\delta [F_\beta(s)-L_\beta(0)]dG_\alpha(s) \\ &\quad + [L_\beta(\delta)-L_\beta(0)][1-G_\alpha(\delta)] \\ &= [F_\beta(0)-S_\beta(0)]q_\alpha(0) + o(\epsilon) + o(\epsilon) > 0 \end{aligned}$$

which implies that $q_\beta(0) = 0$. Q.E.D.

The argument is similar to the argument behind Lemma 3.2. If player α plans to move at time 0 with positive probability, player β can increase his expected return by waiting an instant longer before moving. In this case, however, we cannot eliminate the possibility that one of the players moves with positive probability at time 0. The reason is that the player no longer has the option of moving at some earlier time.

5. Equilibrium Restrictions on the Value of t^*

In this section, we consider the equilibrium restrictions implied by the value of t^*. These are determined by comparing the payoff to a player from moving at or before time t^* with his payoff from waiting until time 1. There are three cases to consider as determined by the value of t^*.

LEMMA 5.1: Suppose $t^* = 0$.

(i) $q_\alpha(0) = 0$ and $q_\beta(0) < 1$ implies $L_\alpha(0) \le S_\alpha(1)$.

(ii) $0 < q_\alpha(0) < 1$ implies $L_\alpha(0) = S_\alpha(1)$.

(iii) $q_\alpha(0) = 1$ implies either $L_\alpha(0) \ge S_\alpha(1)$ or $L_\alpha(0) \ge F_\alpha(t)$ for some $t \in [0,1)$.

PROOF: Suppose $t^* = 0$.

(i) If $q_\alpha(0) = 0$, then it follows from Lemma 3.5 that $q_\alpha(1) = 1$. Therefore, G_α is an optimal response only if

$$0 \le P_\alpha(1,G_\beta) - \lim_{t \downarrow 0} P_\alpha(t,G_\beta) = q_\beta(1)[S_\alpha(1) - L_\alpha(0)].$$

But if $q_\beta(0) < 1$, then Lemma 3.5 implies that $q_\beta(1) = 1 - q_\beta(0) > 0$ which in turn implies that $S_\alpha(1) \ge L_\alpha(0)$.

(ii) If $0 < q_\alpha(0) < 1$, then Lemma 4.1 implies that $q_\beta(0) = 0$. It then follows from Lemma 3.5 that $q_\beta(1) = 1$ and $q_\alpha(1) = 1 - q_\alpha(0) > 0$. Therefore, if G_α is an optimal response, then

$$0 = P_\alpha(1,G_\beta) - P_\alpha(0,G_\beta) = S_\alpha(1) - L_\alpha(0).$$

(iii) If $L_\alpha(0) < S_\alpha(1)$ and $L_\alpha(0) < F_\alpha(t)$ for all $t \in [0,1)$. Then Assumption A2 implies that

$$P_\alpha(1,G_\beta) - P_\alpha(0,G_\beta) = [F_\alpha(0) - S_\alpha(0)]q_\beta(0) + \int_0^1 [F_\alpha(t) - L_\alpha(0)]dG_\beta(t)$$
$$+ [S_\alpha(1) - L_\alpha(0)]q_\beta(1) > 0.$$

Consequently, $q_\alpha(0) > 0$ cannot be an optimal response. Q.E.D.

The implications of Lemma 5.1 are summarized in Table 1. Typical elements represent $(q_a(0),q_b(0))$. The value of x is any number in the interval [0,1].

TABLE 1. Possible Initial Mass Points When $t^* = 0$

	$L_b(0) > S_b(1)$	$L_b(0) = S_b(1)$	$L_b(0) < S_b(1)$
$L_a(0) > S_a(1)$	(1,0) (0,1)	(1,0) (0,1)	(1,0) (0,1)[b]
$L_a(0) = S_a(1)$	(x,0) (0,1)	(x,0) (0,x)	(x,0) (0,1)[b]
$L_a(0) < S_a(1)$	(0,1) (1,0)[a]	(0,x) (1,0)[a]	(0,1)[b] (0,0) (1,0)[a]

[a] If $L_a(0) \geq F_a(t)$ for some $t \in [0,1)$.

[b] If $L_b(0) \geq F_b(t)$ for some $t \in [0,1)$.

When $t^* > 0$, there is an additional restriction to consider besides the tradeoff between moving at time t^* and waiting until time 1. It must also be possible to construct a strategy which makes the other player indifferent between moving at any time near 0. This requires that an "integral" condition be satisfied.

Define $I_\beta(0,0) = \lim_{t\downarrow 0} I_\beta(0,t)$.[9]

LEMMA 5.2: If $0 < t^* < 1$, then $I_\beta(0,0) = 1$ and $L_\alpha(t^*) = S_\alpha(1)$.

PROOF: The requirement that $I_\beta(0,0) = 1$ follows from Lemma 5.5 and the requirement that G_α be right–continuous. To establish that $L_\alpha(t^*) = 0$, note that Lemmata 3.4 and 3.5 imply that $q_\alpha(1) > 0$. Therefore, G_α is an optimal response only if

$$0 = P_\alpha(1,G_\beta) - \lim_{t\uparrow t^*} P_\alpha(t^*,G_\beta) = [S_\alpha(1) - L_\alpha(t^*)]q_\beta(1) \tag{5.1}$$

But since Lemmata 3.4 and 3.5 also imply that $q_\beta(1) > 0$, it follows that $L_\alpha(t^*) = 0$. Q.E.D.

Define $I_\beta(1,1) \equiv \lim_{t\uparrow 1} I_\beta(t,1)$.[10] If $t^* = 1$, and $I_\beta(1,1) = 0$, then Lemma 3.5 implies that player β moves with probability 1 before time 1. In this case, the value of $S_\alpha(1)$ is irrelevant. Consequently, when $t^* = 1$, there are no additional restrictions at time t^* unless $I_\beta(1,1) > 0$ for some player β.

LEMMA 5.3: Suppose $t^* = 1$. Then

(i) $I_\beta(0,0) = 1$ and

(ii) $I_\beta(1,1) > 0$ implies $L_\alpha(1) \geq S_\alpha(1)$ and $I_\alpha(1,1)[L_\alpha(1) - S_\alpha(1)] = 0$.

PROOF: The proof of part (i) follows again from Lemma 3.5 and the requirement that G_β be right–continuous. To establish (ii), note that G_α is an optimal response only if

$$0 \leq P_\alpha(1,G_\beta) - \lim_{t\uparrow 1} P_\alpha(1,G_\beta) = [S_\alpha(1) - L_\alpha(1)]q_\beta(1) \tag{5.2}$$

Lemma 3.5 implies that $q_\beta(1) > 0$ whenever $I_\beta(1,1) > 0$. Therefore, (5.2) implies that $L_\alpha(1) \geq S_\alpha(1)$. Furthermore, if $I_\alpha(1,1) > 0$, then it again follows from Lemma 3.5 that $q_\alpha(0) > 0$, in which case equation (5.1) must be satisfied for $t^* = 1$. Q.E.D.

6. A Characterization of the Equilibrium Outcomes

Using the restrictions derived in Sections 3 to 5, we now provide a complete characterization of the equilibrium outcomes. We retain the convention that α refers to either a or b and β to the other player.

We begin with a characterization of the equilibria which are **degenerate** in the sense that, in equilibrium, neither player ever moves in the interval (0,1). In Lemma 5.1, we demonstrated the necessity of the conditions summarized in Table 1 for any pair of strategies to form a degenerate equilibrium. Theorem 6.1 establishes that these conditions are sufficient as well.

THEOREM 6.1: The following conditions characterize the set of equilibrium outcomes for which $q_\alpha(0)+q_\alpha(1) = 1$, $\alpha = a,b$.

(i) (a) $L_\alpha(0) \geq S_\alpha(1)$ or $L_\alpha(0) \geq F_\alpha(t)$ for some $t \in [0,1]$, and

(b) $q_\alpha(0) = 1$ and $q_\beta(0) = 0$.

(ii) $L_\alpha(0) \leq S_\alpha(1)$ and $q_\alpha(1) = 1$, $\alpha = a,b$.

(iii) $L_\alpha(0) = S_\alpha(1)$, $L_\beta(0) \leq S_\beta(1)$, $q_\alpha(0) \in [0,1]$, and $q_\beta(1) = 1$.

PROOF: The necessity of these conditions follow from Lemmata 4.1 and 5.1. All that remains is to show that they are sufficient.

(i) If $L_\alpha(0) \geq F_\alpha(t)$ for some $t \in [0,1)$, then choose the strategies so that $q_\alpha(0) = 1$ and $q_\beta(t) = 1$. Then, for any $v \in (0,1]$,

$$P_\beta(v,G_\alpha) - P_\beta(0,G_\alpha) = [F_\beta(0) - S_\beta(0)] > 0$$

which implies that G_β is an optimal response. And

$$P_\alpha(j,G_\beta) - P_\alpha(0,G_\beta) = \begin{cases} [L_\alpha(j) - L_\alpha(0)] \le 0 & \text{for } j < t \\ [S_\alpha(t) - L_\alpha(0)] < [F_\alpha(t) - L_\alpha(0)] \le 0 & \text{for } j = t \\ [F_\alpha(t) - L_\alpha(0)] \le 0 & \text{for } j > t \end{cases}$$

which implies that G_α is an optimal response.

If $L_\alpha(0) \ge S_\alpha(1)$, then a similar argument establishes that $q_\alpha(0) = 1$ and $q_\beta(t) = 1$ form a pair of best responses.

(ii) If $L_\alpha(0) \le S_\alpha(1)$ and $q_\alpha(1) = 1$ for $\alpha = a,b$, then, for any $t < 1$,

$$P_\beta(1,G_\alpha) - P_\beta(t,G_\alpha) = S_\beta(1) - L_\beta(t) \ge 0$$

which implies that $q_\beta(1) = 1$ is an optimal response. This establishes (ii).

(iii) If $L_\alpha(0) = S_\alpha(1)$ and $q_\beta(1) = 1$, then

$$P_\alpha(1,G_\beta) - P_\alpha(0,G_\beta) = S_\alpha(1) - L_\alpha(0) = 0$$

and, for any $t \in (0,1)$, Assumption A2(iii) implies

$$P_\alpha(1,G_\beta) - P_\alpha(t,G_\beta) = S_\alpha(1) - L_\alpha(t) > 0.$$

Therefore, for any $q_\alpha(0) \in [0,1]$, $q_\alpha(1) = 1-q_\alpha(0)$ is an optimal response.

Finally, if $L_\beta(0) \leq S_\beta(1)$, and $q_\alpha(0)+q_\alpha(1) = 1$, then Assumption A2(ii) implies that

$$P_\beta(1,G_\alpha)-P_\beta(0,G_\alpha) = q_\alpha(0)[F_\beta(0)-S_\beta(0)] + q_\alpha(1)[S_\beta(1)-L_\beta(1)] \geq 0,$$

and, for any $t \in (0,1)$,

$$P_\beta(1,G_\alpha)-P_\beta(t,G_\alpha) = q_\alpha(1)[S_\beta(1)-L_\beta(t)] \geq 0$$

which implies that $q_\beta(1) = 1$ is an optimal response. Q.E.D.

We turn next to the possibility of equilibria which are **nondegenerate** in the sense that both players move with positive probability in the interval (0,1).

THEOREM 6.2: (a) There is an equilibrium such that $q_\alpha(0)+q_\alpha(1) < 1$ for some player α only if $I_a(0,0) = I_b(0,0) = 1$.

(b) If $I_a(0,0) = I_b(0,0) = 1$, these equilibria are characterized by the strategy combinations

$$G_\beta(t) = 1 - [1-q_\beta(0)]I_\beta(0,t) \quad \text{for } t \in [0,t_1), \text{ and}$$
$$q_\beta(1) = [1-q_\beta(0)]I_\beta(0,t_1)$$

for $\beta = a,b$, subject to the following restrictions on $t_1 \in (0,1]$ and $(q_a(0),q_b(0)) \in [0,1)\times[0,1)$: (i) $q_a(0)q_b(0) = 0$, and

(ii) either (a) $L_\alpha(t_1) = S_\alpha(1)$, $\alpha = a,b$, or (b) $t_1 = 1$ and either

(1) $I_a(1,1) = I_b(1,1) = 0$, or (2) $I_\alpha(1) = 0$ and $L_\alpha(1) \geq S_\alpha(1)$ for some α.

PROOF: The necessity of the requirement that $I_a(0,0) = I_b(0,0) = 1$ together with the conditions of part (b) follows from Lemmata 3.5, 4.1, 5.2, and 5.3. The theorem then follows upon verifying that each of the strategy pairs defined in part (b) form a pair of best responses. Q.E.D.

Theorem 6.1 is not a characterization of degenerate equilibria, but a characterization of degenerate equilibrium outcomes. Whenever there is a degenerate equilibrium in which one of the players moves immediately, there is generally a large class of equilibrium strategies which support this outcome. The only restriction is that the player who waits must adopt a strategy in which he threatens not to move for a sufficiently long period of time that the other player finds it optimal to move immediately.

At least one of the degenerate equilibria described in Theorem 6.1 always exists. If the return to one of the players from leading at time 0 is at least as great as either his return at time 1 or his return to following at some time t before time 1, then there is an equilibrium in which he moves at time 0 with probability 1.[11] On the other hand, if both players earn a return at time 1 which is at least as great as their returns to leading at time 0, then it is an equilibrium for both players to wait until time 1 with probability 1. If, in addition, one of the players earns the same return to leading at time 0 as his return at time 1, then, so long as the other player waits until time 1 with probability 1, it is an equilibrium response for him to move at time 0 with any probability between 0 and 1.

Theorem 6.2 describes the nondegenerate equilibria which are possible when equation (3.1) defines a right continuous distribution function for each player at time 0. Given this condition, the existence of such equilibria depends on whether it is possible to make both players indifferent between moving an instant after time 0 and waiting until time 1. There are two ways this relation might be satisfied. One possibility is that there is a time t at which both players earn a return to leading which is equal to their terminal returns at the same time. In this case, there is a continuum of equilibria in which both players move over the interval (0,t) according to the continuous probability function described by equation (3.1), after which both wait until time 1.[12] This case arises when the return functions appear as in Figure 1(c) and t^* has the same value for both players. The equilibria are indexed by the probability with which one of the players moves at time 0.

Alternatively, there may be a continuum of equilibria when equation (3.1) implies that time 1 is never reached. If the return to following converges to the return to leading as t goes to 1, then equation (3.1) may imply a zero mass point at time 1 (i.e. $I_\alpha(1,1) = 0$). When the return functions for both players have this property, then the returns at time 1 have no bearing on the optimal responses of either player. In this case, the equilibrium strategies satisfy the equation (3.1) throughout the interval (0,1). This family of equilibria may also exist even if $I_\beta(1,1) = 1$ for some player β, provided $I_\alpha(1,1) = 0$ and the return to player α to leading evenually falls below his terminal return. Then neither player has any incentive to deviate from the strategies described by equation (3.1) since player α intends to move with probability 1 by time 1.

Finally, we note that it is possible for both classes of nondegenerate equilibria to exist if the players strictly prefer their return at time 1 to leading an instant before this time.

7. Subgame Perfection

Employing the natural analogue of Selten's definition of subgame perfection to continuous time, one can show that nondegenerate equilibria are always subgame perfect. The argument is based on Lemma 3.4, which implies that every information set is reached with positive probability. The same argument can be used to establish that degenerate equilibria in which both players wait until time 1 with positive probability are also subgame perfect. However, degenerate equilibria in which one of the players moves immediately are not always subgame perfect. In fact, as Fudenberg **et. al.** and Ghemawat and Nalebuff have shown, some degenerate equilibrium **outcomes** may not be subgame perfect. In Hendricks and Wilson (1985b), we demonstrate that in the discrete time version of the war of attrition it is subgame perfect for player β to move immediately if and only if (i) $L_\beta(0) \geq S_\beta(1)$ and (ii) $L_\beta(t^*) = S_\beta(1)$ implies $L_\alpha(t^*) \leq S_\alpha(1)$, $t \in [0,1)$. These same conditions are also necessary and sufficient in continuous time.

The sufficiency of these conditions can be verified by noting that the following pair of strategies are subgame perfect. Upon reaching any time $t \leq t^*$, player β plans to move immediately and player α plans to wait until time 1, and, upon reaching any later time, both players plan to wait until time 1. The argument for the necessity of these conditions when $F_\alpha(t) < S_\alpha(1)$ for some time t is a bit complicated. If $F_\alpha(t) \geq S_\alpha(1)$ for all t, a rough argument for the necessity of these conditions goes as follows. Condition (i) is necessary to make moving immediately is an dominated strategy for player β. So suppose condition (ii) is violated. Then it cannot be optimal for player β

to move in the interval $(t^*,1)$. Consequently, upon reaching time t^*, the optimal response of player α is to move with probability 1. This implies in turn that player α should choose to wait in an interval just before time t^* to obtain the benefits from following at time t^*. But in that case, player α should move with probability 1 upon reaching the the **beginning** of that interval. Successive application of this argument then implies that player α must move at time 0 with probability 1 which implies that player β should wait.

When nondegenerate equilibria do not exist, the subgame perfect strategies are unique in the discrete time formulation. Player β moves with probabilty 1 upon reaching any period and player α always waits. In continuous time, however, there is always an infinity of equilibrium strategies. Any distribution function for player α which increases at a sufficiently low rate to induce player β to move immediately will work.

8. Concluding Remarks

We conclude with a brief overview of our results. Under the assumptions of Section 2, there is always a degenerate equilibrium in which either one of the players moves immediately or both wait until time 1. Under more restrictive conditions, there is also a class of nondegenerate equilibria in which both players move according to a continuous density function over some interval. These conditions are generally satisfied only in games with symmetric return functions or games derived from an infinite horizon model.

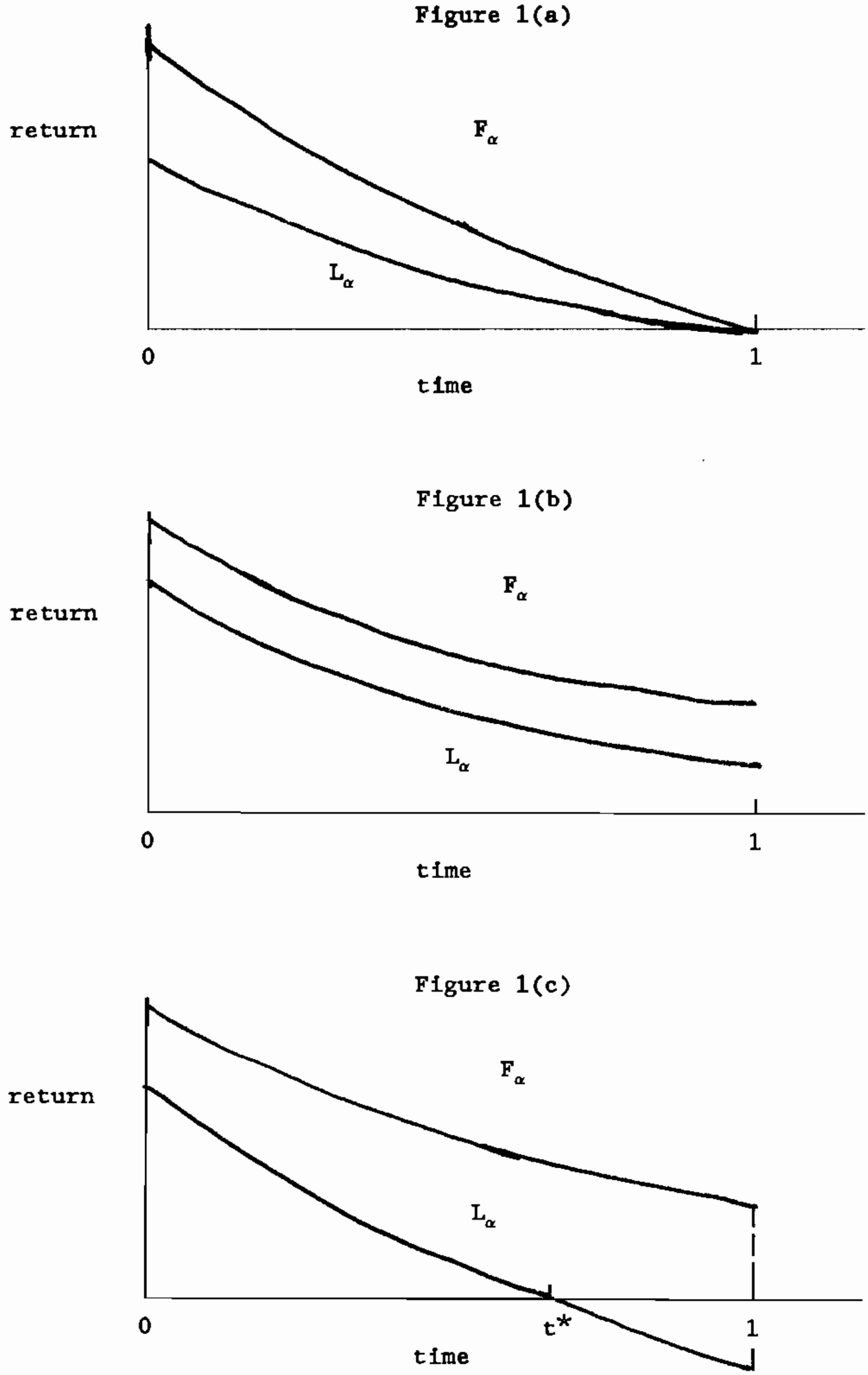
For many economic applications which require a finite horizon, the gain from following is bounded above the return from leading throughout the interval [0,1]. Thus, for any pair of strategies satisfying equation (3.1), there is a positive probability of reaching time 1. This implication frequently eliminates the possibility of nondegenerate equilibria. In games where the return to the players from leading is positive at any finite t, but the returns at time 1 are zero, they are eliminated because truncating the horizon introduces a downward discontinuity in the payoffs at time 1 as in Figure 1(b).[13] (See Hendricks and Wilson (1985a) for a detailed discussion of this issue.) In these games, the only equilibria are the degenerate asymmetric equilibria in which one of the players moves immediately.

In games where the return to leading eventually falls below the terminal return as in Figure 1(c), nondegenerate equilibria are generally eliminated for a different reason. The difficulty in this case is that the return to leading must equal the terminal return at exactly the same time for both players. This property is likely to hold only for symmetric games. Consequently, unless there is a special reason to assume that returns are symmetric, such as in the biology models, nondegenerate equilibria do not exist generically. Two economic applications where this result is of interest is in the patent race model of Fudenberg et. al. (1983) and the exit model by Ghemawat and Nalebuff (1985). In both of these models, returns are assumed to be asymmetric and, as a result, the nondegenerate equilibria of Theorem 6.2 are eliminated. Nondegenerate equilibria generally exist only in the symmetric versions of these models.

For applications which require an infinite horizon, however, the terminal conditions are less restrictive. Suppose, for instance, that all of the return functions decline at an exponential rate δ. Then, if we transform time, z, according to the formula $t = z/[z+1]$, simple calculations reveal that not only does $F_\alpha(1) = L_\alpha(1) = S_\alpha(1) = 0$ as in Figure 1(a), but $I_\beta(1,1) = 0$ as well. In this case, Theorem 6.2 guarantees the existence of a one parameter family of equilibria in which both players always move over the entire interval according to a continuous density function.

These terminal conditions may also be satisfied for some applications with a finite horizon if the net return to following depends on the amount of time remaining in the game. In their continuous time version of the "chain store" paradox, for example, Kreps and Wilson (1982)) assume that a contest for a market between two firms takes place over a predetermined interval (0,T). They also assume that the benefit from following over leading is proportional to T−t and that the cost of leading (in their case leaving the market) is proportional to t. Simple calculations again reveal that $I_\beta(1,1) = 0$. Consequently, under complete information, their model possesses a one parameter family of nondegenerate equilibria in which one of the players is certain to concede by time T. Furthermore, since the probability of ever reaching time T is zero, this result holds for any specification of $S_\alpha(1)$.[14]

FIGURE 1. Three Possible Patterns for the Return Functions



Footnotes

[1] Games with an infinite horizon can be incorporated into this framework by simply renormalizing the time variable. See Section 7.

[2] Most of these models incorporate some degree of incomplete information. The method of analysis is similar, but the set of equilibria is sometimes substantially reduced. For a more complete review of this literature, see Hendricks and Wilson (1985b).

[3] The equilibrium concept is known as evolutionary stable strategies (ESS). Following Selten (1980), a strategy r is said to be evolutionary stable if (i) r is a best reply to itself and (ii) for any alternative best reply r′ to r, r is a better reply to r′ than r′ is to itself.

[4] We permit the terminal returns to exceed their returns to leading just before time 1.

[5] It is not necessary to define $F_\alpha(1)$ and $L_\alpha(1)$ since the only return which can be realized at time 1 is $S_\alpha(1)$. Defining $L_\alpha(t)$ to be continuous at 1 merely allows us to identify $\lim_{t\uparrow 1} L_\alpha(t)$ with $L_\alpha(1)$ and $\lim_{t\uparrow 1} F_\alpha(t)$ with $F_\alpha(1)$.

[6] The games are called "noisy" because the payoff to the follower depends only on when the other player moves. This reflects the assumption that a player who plans to wait until time t to move does not have to **commit** himself to moving until he has observed the history of the game up to time t. Consequently, if the other player moves before time t, the first player can react optimally, independently of what he had planned to do had the other player not moved at time t. A "silent" game of timing is one in which each player must commit himself to a time at which he will move independently of the action of the other at the outset of the game. Silent games of timing have been also been used in economic models (e.g. Reinganum (1981a),(1981b)).

[7] By a probability distribution on [t,1], we mean any right−continuous nondecreasing function G from $(-\infty,\infty]$ to [0,1] with $G(t) = 0$ for $t < 0$ and $G(1) = 1$. Throughout this paper, we will adopt the convention that

$$\int_{t_0}^{t_1} f(v)dG(v) = \lim_{t\uparrow t_1} \int_{t_0}^{t} f(v)dG(v).$$

That is, the integral does not include any mass point at time t_1.

[8] If $\int_{t_0}^{t_1}[dL_\alpha(v)/(F_\alpha(v)-L_\alpha(v))]dG_\alpha(v)$ does not exist, then define

$\int_{t_0}^{t_1}[dL_\alpha/(F_\alpha-L_\alpha]dG_\alpha = \lim_{t\downarrow t_0}\int_{t}^{t_1}[dL_\alpha/(F_\alpha-L_\alpha]dG_\alpha$. See footnote 7.

[9] Note that $I_\beta(0,0)$ can only take on the values 0 and 1. Furthermore, if $F_\alpha(0) > L_\alpha(0)$, then $I_\beta(0,0) = 1$.

[10] Note that $I_\beta(1,1)$ can only take on the values 0 and 1. Furthermore, if $F_\alpha(1) > L_\alpha(1)$, then $I_\beta(1,1) = 1$.

[11] If we extend the concept of a strategy to define a strategy starting at time $t \in [0,1]$ and impose the continuous time analogue of subgame perfection, then there is an equilibrium in which player α moves immediately with probability 1 if and only if $L_\alpha(0) \geq S_\alpha(1)$. See Section 7.

[12] If $t = 0$, then these equilibria correspond to the equilibria described in Theorem 6.1 when $L_a(0) = L_b(0) = 0$.

[13] For instance, in the oil exploration example considered by Hendricks (1984), if the firm has not drilled its lease by a certain time, it loses the lease.

[14] This game may also possess another family of nondegenerate equilibria. If the firms were to play a sequence of n such contests (see Fudenberg and Kreps (1985)), the terminal payoff to each firm at the end of the kth contest would represent its equilibrium payoff from playing the remaining $n-k$ contests. If this is nonzero and returns are symmetric, then the family of nondegenerate equilibria described in condition (i) of Theorem 6.2 also exists. Notice that in this equilibrium, the behavior of the firms in the kth contest depends on their behavior in subsequent contests through the value of the terminal payoffs.

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