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1. Introduction

In this paper we will study the global dynamics of capital accumulation for a two-sector model of technology which is not necessarily convex and where the preferences of an infinitely-lived agent are stationary but not additively separable.[1] In the last section of the paper, we will also study the global dynamics of efficient allocations over time in a pure exchange model of two agents that have additively non-separable preferences.

One-sector optimal growth models with non-convex technologies have been studied by Majumdar and Mitra (1982), (1983) and Dechert and Nishimura (1983). Benhabib and Nishimura (1985) gave a characterization of the monotonic and oscillatory dynamics of a two-sector model of growth with a representative agent having additively separable preferences. The dynamics of optimal growth models with stationary but non-separable preferences have been studied by Beals and Koopmans (1969) and Iwai (1971), among others, for one-sector convex technologies. Recently, Lucas and Stokey (1984) and Epstein (1985) also studied the dynamics of capital accumulation in models of many agents that have non-separable utility functions.

The paper is organized as follows. The general model is set up in section 2. In section 3 we obtain monotonicity and convergence results for the capital stock under "normality" assumptions on preferences and factor intensity assumptions on technology. In section 4 we obtain results on oscillatory dynamics under alternative factor-intensity conditions or under the assumption of inferiority of "future utilities." Section 5 considers a model where the stock of capital enters the utility function and gives conditions under which it converges to a non-zero steady-state, and also we consider a pure exchange model with two agents and show that utilities over time will be monotonic or oscillatory depending on the normality or inferiority properties of the agent's preferences.

2. Technology, Preferences

Let $c = T(y,k)$ be a social production function where k is the amount of capital stock available, and y and c are the maximal amounts of the capital stock and the consumption good producible. $c = T(y,k)$ may be understood to represent the joint production of outputs c and y from the input k or it may be viewed as an aggregated social production function in a two-sector economy, in which one industry produces a pure consumption good c and the other produces a pure capital good y.[2] In either case, all the variables are normalized by labor. We assume that

(P.1) There exists $\bar{k} > 0$ such that for $0 < k < \bar{k}$, $T(y,k) = 0$ implies $y > k$, and for $k > \bar{k}$, $T(y,k) = 0$ implies $y < k$.

(P.2) $T(y,k)$ is continuous on $[0,k] \times [0,\bar{k}]$ and of C^2 class on $(0,\bar{k}) \times (0,\bar{k})$ and $T_1 < 0$, $T_2 > 0$, $T_{11} < 0$, $T_{22} < 0$, where $T_1 = \partial T/\partial y$, $T_2 = \partial T/\partial k$, $T_{11} = \partial^2 T/\partial y^2$, $T_{22} = \partial^2 T/\partial k^2$ and $T_{12} = \partial^2 T/\partial y \partial k$.

(P.3) For $k = 0$, $c = T(y,k)$ implies $c = y = 0$.

Let $k = [0,\bar{k}]$, $K^\infty = K \times K \times \ldots$. Given (P.2) we can obtain a function $y(k)$ as a solution of $T(y,k) = 0$. Then the correspondence $F: K \to K^\infty$ is defined by

(1) $$F(k_0) = \{(k_0,k_1,\ldots) \in K^\infty \mid 0 \le k_{t+1} \le y(k_t),\ t \ge 0\},$$

which gives the set of feasible capital paths attainable from the initial stock k_0. We shall write ${}_0k = (k_0,k_1,\ldots)$.

Let $R_+^\infty = R_+ \times R_+ \ldots$ and ${}_0c = (c_0,c_1,\ldots)$. Let $G: K^\infty \to R_+^\infty$ be defined by

(2) $G({}_0k) = (T(k_1,k_0),\ T(k_2,k_1),\ldots)$

where ${}_0k \in K^\infty$. Then the correspondence of $C:K \to R_+^\infty$ defined by

(3) $C(k_0) = \{{}_0c \in R_+^\infty | {}_0c = G({}_0k),\ {}_0k = (k_0,k_1,\ldots) \in F(k_0)\}$

gives the set of feasible consumption paths attainable from k_0. We impose the product topology on R_+^∞. It can be shown that C is upper semi-continuous in k_0. As K is a compact subset of R_+, $C(K) = \bigcup_{k_0 \in K} C(k_0)$ is a compact set (Hildebrand and Kirman (1976, p. 192)).

We assume that choice among consumption paths in C(K) can be represented by a preference ordering which is complete, transitive and continuous in the product topology, or equivalently we assume that there exists a continuous utility function $U:C(K) \to R$ such that ${}_0c' > {}_0c$ if and only if $U({}_0c') > U({}_0c)$. Below we explicitly state assumptions on this intertemporal utility function.

(U.1) U is continuous on C(K) with respect to the product topology.

(U.2) Sensitivity: There exists c_0, c_0' and ${}_1c$ such that $U(c_0', {}_1c) > U(c_0, {}_1c)$.

(U.3) Limited non-complementarity: For all ${}_0c,\ {}_0c' \in C(K)$,
$U(c_0, {}_1c) \geq U(c_0', {}_1c)$ implies $U(c_0, {}_1c') \geq U(c_0', {}_1c')$,
$U(c_0, {}_1c) \geq U(c_0, {}_1c')$ implies $U(c_0', {}_1c) \geq U(c_0', {}_1c')$.

(U.4) Stationarity: For some c_0 and all ${}_1c$, $U(c_0, {}_1c) \geq U(c_0, {}_1c')$ if and only if $U({}_1c) \geq U({}_1c')$.

Since C(K) is a compact, there exists ${}_0\bar{c}$ and ${}_0\underline{c}$ in C(K) such that

(4) $a = U({}_0\underline{c}) \leq U({}_0c) \leq U({}_0\bar{c}) = b$ for all ${}_0c \in C(K)$.

Let $\bar{c} = \max\{c_0 | (c_0, c_1, \ldots) \in C(K)\}$. Then it follows from Koopmans (1960, pp. 287-295) that on C(K) there exists a function $V:[0,\bar{c}] \times [a,b] \to R_+$ such that

$$U({}_0c) = V(c_0, U({}_1c)) \quad \text{for} \quad {}_1c \in C(K). \tag{5}$$

This function V is called the aggregator. We further assume the following properties for this function.

(U.5) $V(c,U)$ has positive continuous derivatives V_1 and V_2 on $(0,\bar{c}) \times (a,b)$ where $V_1 = \partial V/\partial c$ and $V_2 = \partial V/\partial u$, and $\lim_{c_0 \to 0} V_1(c_0,u) = \infty$, $\lim_{c_0 \to 0} V_2(c_0,u)$ are bounded.

Given (5), this assumption implies that $U_1 = \partial U/\partial c_0$ and $U_2 = \partial U/\partial c_1$ exist on $(0,\bar{c})^\infty$ and $U_1 = V_1$, $U_2 = V_2 \times V_1$. The relation

$$U({}_0c) = V(c_0,\ V(c_1, \ldots, V(c_t, U({}_{t+1}c)) \ldots)$$

can then be used to prove that $U_{t+1} = \partial U({}_0c)/\partial C_t$ exists.

3. Capital Accumulation

Let $W(k_0) = \max_{{}_0k \in F(k_0)} U(G({}_0k))$. Then the optimal capital path correspondence is defined by

$$\Phi(k_0) = \{{}_0k \in F(k_0) | U(G({}_0k)) = W(k_0)\}$$

and the optimal consumption path correspondence $\Psi: K \to K^\infty$ is given by

$$\Psi(k_0) = G(\Phi(k_0)). \tag{7}$$

Let $\Phi_t(k_0)$ and $\Psi_t(k_0)$, $t \geq 0$, be the set of capital stocks in the t^{th} period of optimal capital paths and optimal consumption paths arising from k_0. The existence and the upper semi-continuity of optimal paths are easy consequences of the compactness of K^∞ and the continuity of U.

Lemma 3.1:

(i) $\Phi(k_0) \neq \emptyset$ for any $k_0 \in K$.

(ii) Φ is upper semi-continuous on K.

The value function $W: K \to R_+$ is defined by

$$W(k_0) = U(\Phi(k_0)). \tag{8}$$

It satisfies the following function equation:

$$\begin{aligned} W(k_0) &= \max_{{}_0c \in G(k_0)} U({}_0c) \\ &= \max_{{}_0c \in G(k_0)} V(c_0, U({}_1c)) \end{aligned} \tag{9}$$

$$= \max_{c_0=T(k_1,k_0)} V(c_0, \max_{_1c\in G(k_1)} U(_1c))$$

$$(10) \qquad W(k_0) = \max_{c_0=T(k_1,k_0)} V(c_0,W(k_1))$$

Lemma 3.2: If $_0\hat{k} \in \Phi(\hat{k}_0)$ and $_0\hat{c} \in \Psi(k_0)$ for $\hat{k}_0 > 0$, then $\hat{k}_t > 0$ and $\hat{c}_t > 0$ for all $t \geq 0$, that is, optimal paths from $\hat{k}_0 > 0$ are interior paths.

Proof: (i) Suppose that $\hat{k}_1 = 0$. Then given (P.3), $\hat{c}_1 = \hat{k}_2 = 0$ and $V(\hat{c}_0,V(0,a))$ is the maximum of $V(c_0,V(c_1,U(_1c)))$ on the set of feasible consumption paths where $a = U(0,0,\ldots)$. Let $g(k_1) = V(T(k_1,k_0), V(T(0,k_1),a))$. Since $k_1 > 0$, if $\hat{k}_1$ maximizes $g(k_1)$, we must have

$$\lim_{k_1\to 0} g'(k_1) \leq 0.$$

However,

$$(11) \qquad g'(k_1) = V_1(c_0,V(T(0,k_1),a))\cdot T_1(k_1,\hat{k}_0) + V_2(c_0,U(c_1,0,\ldots)V_1(c_1,a)\cdot T_2(0,k_1).$$

The derivatives are evaluated at $(\hat{k}_0,k_1,0,\ldots)$, $(c_0,c_1,0,0,\ldots)$ and $a = U(0,0,\ldots)$, where $c_0 = T(k_1,\hat{k}_0)$ and $c_1 = T(0,k_1)$.

Let $k_1 \to 0$. Then

$$(12) \qquad \lim_{k_1\to 0} g'(k_1) = V_1(\hat{c}_0,a)\cdot T_1(0,\hat{k}_0) + V_2(\hat{c}_0,a)\lim_{\substack{c_1\to 0\\ k_1\to 0}}[V_1(c_1,a)\cdot T_2(0,k_1)]$$

$V_1(\hat{c}_0,a)$, $V_2(\hat{c}_0,a)$ and $T_1(0,k_0)$ are bounded by (U.5) and (P.2). Also, $T_{22} < 0$ implies that for $k_1 > \epsilon > 0$,

$$T_2(0,k_1 - \epsilon) > T_2(0,k_1) > 0.$$

Hence

$$(13) \qquad \lim_{k_1\to 0} T_2(0,k_1) > 0.$$

Also (U.5) implies $\lim_{c_1\to 0} V_1(c_1,a) = \infty$ and the second term in the parenthesis goes to ∞. Hence $\lim_{k\to 0} g'(k_1) > 0$, and this is a contradiction.

Therefore $\hat{k}_0 > 0$ implies $\hat{k}_1 > 0$. By induction $\hat{k}_t > 0$ for all $t \geq 0$.

(ii) Suppose that $\hat{c}_0 = 0$ and $\hat{c}_1 > 0$. Let $h(k_1) = V(T(k_1,\hat{k}_0), V(T(\hat{k}_2,k_1),U(_2\hat{c})))$. Note that given k_0, k_1 can be bounded above by the solution to $0 = T(k_1,k_0)$ (which is unique if it exists), since $c_0 = T(k_1,k_0)$ and $c_0 \geq 0$. Therefore, if $\hat{k}_1$ maximizes $h(k_1)$, $\lim_{k_1\to\hat{k}_1} h'(k_1) \geq 0$ should hold. However,

(14) $$h'(k_1) = V_1(c_0,V(c_1,\hat{u}))\cdot T_1(k_1,\hat{k}_0) + V_2(c_0,V(c_1,\hat{u}))\cdot V_1(c_1,\hat{u})\cdot T_2(\hat{k}_2,k_1).$$

The derivatives are evaluated at $(\hat{k}_0,k_1,\hat{k}_2,\ldots)$, $(c_0,c_1,\hat{c}_2,\hat{c}_3,\ldots)$, and $\hat{u} = u(_2\hat{c})$ where $c_0 = T(k_1,\hat{k}_0)$ and $c_1 = T(\hat{k}_2,k_1)$. Let $k_1 \to \hat{k}_1$. Then

(15) $$\lim_{k_1\to\hat{k}_1} h'(k_1) = [\lim_{\substack{c_0\to 0\\ c_1\to\hat{c}_1}} V_1(c_0,V(c_1,\hat{u}))]\cdot T_1(\hat{k}_1,\hat{k}_0) + [\lim_{\substack{c_0\to 0\\ c_1\to\hat{c}_1}} V_2(c_0,V(c_1,\hat{u}))]\cdot V_1(\hat{c}_1,\hat{u})\cdot T_2(\hat{k}_2,k_1).$$

Since $\hat{k}_1 > k_1$, $T_{11} < 0$ implies

(16) $$T_1(\hat{k}_1,\hat{k}_0) < T_1(k_1,\hat{k}_0) < 0.$$

$T_{22} < 0$ implies

(17) $$0 \leq T_2(\hat{k}_2,\hat{k}_1) < T_2(\hat{k}_2,k_1).$$

As $V_1(\hat{c}_1,\hat{u})$ is also bounded, (U.5) implies $\lim_{k_1\to\hat{k}_1} h'(k_1) = -\infty$. This is a contradiction. Hence if $\hat{k}_0 > 0$ and $\hat{c}_1 > 0$, then $\hat{c}_0 > 0$ must be true. Next suppose that $\hat{k}_0 > 0$ and $\hat{c}_0 = \hat{c}_1 = 0$. It is clear that there exist some t such that $\hat{c}_t > 0$. Choose the smallest such t, say t′. Then $\hat{k}_{t'-1} > 0$, $\hat{c}_{t'-1} = 0$ and $\hat{c}_{t'} > 0$. Otherwise, since $c_t > 0$ is feasible for some $t > 1$ on account of $k_0 > 0$, any program with some $c_t > 0$ would dominate a program with $c_t = 0$ for all t. We can apply the same argument as above and reach the contradiction. Hence $\hat{k}_0 > 0$ must imply $\hat{c}_t > 0$ for all $t \geq 0$. Q.E.D.

<u>Corollary 3.1</u>: If ${}_0\hat{k} \in \Phi(k_0)$, ${}_0\hat{c} \in \Psi(k_0)$ for $k_0 > 0$,

$$V_1(\hat{c}_0,V(\hat{c}_1,\hat{u}))T_1(\hat{k}_1,\hat{k}_0) + V_2(\hat{c}_0,V(\hat{c}_1,\hat{u}))\cdot V_1(\hat{c}_1,\hat{u})\cdot T_2(\hat{k}_2,\hat{k}_1) = 0, \tag{18}$$

where $\hat{u} = u({}_2\hat{c})$.

<u>Proof</u>: $V(T(k_1,\hat{k}_0)$, $V(T(\hat{k}_2,k_1),\hat{u})$ is maximized at $\hat{k}_1$ and its first order condition is satisfied with equality by Lemma 3.2. Q.E.D.

We will refer to (18) as the Euler equation.

By <u>the principle of optimality</u>, if ${}_0k \in \Phi(k_0)$, then ${}_1k \in \Phi(k_1)$ or ${}_tk \in \Phi(k_t)$, $t \geq 1$. We write ${}_{con}(k) = (k,k,\dots)$ and call it a constant path. A <u>steady state</u> is defined to be $k^* \in [0,\bar{k}]$ such that ${}_{con}(k^*) \in \Phi\ (k^*)$. Zero is called a <u>trivial steady state</u>. All the other steady states are called non-trivial steady states. Let K^* be the set of steady states. Since it is easy to show that $k^* < \bar{k}$, every non-trivial steady state k^* satisfies

$$V_1(c^*,V(c^*,u^*))T_1(k^*,k^*) + V_2(c^*,V(c^*,u^*)\cdot V_1(c^*.u^*)\cdot T_2(k^*,k^*) = 0, \tag{19}$$

or

$$V_2(c^*,u^*) = -\ T_1(k^*,k^*)/T_2(k^*,k^*) \tag{19a}$$

where $c^* = T(k^*,k^*)$, $u^* = u({}_{con}(c^*))$.

It should be noted that we so far did not assume even the quasi-concavity of the utility function. There may be many optimal paths from the single initial stock k_0 (see Majumdar and Mitra (1982)). The following assumption assures the convergence of the optimal paths to the steady state:

(M) $V_1(T(k_1,k_0),W(k_1))\cdot T_2(k_1,k_0)$ is strictly increasing in k_1.

If V is twice differentiable and W is differentiable, this condition is easily interpreted. We have $\partial V_1T_2/\partial k_1 = V_{11}T_1T_2 + T_2V_{12}W' + V_1T_{12}$. From the first order conditions in the maximization problem given by equation (10) we also have $W' = -V_1T_1/V_2$ so that $\partial V_1T_2/\partial k_2 = - T_1T_2(- V_{11} + (V_1/V_2)V_{12}) + V_1T_{12}$.[3] The first term on the right hand side is positive if future utility $V({}_1c))$ is normal. This follows because $T_1 < 0$, $T_2 > 0$ and $(-V_{11} + (V_1/V_2)V_{12})$ is positive under the normality assumption. T_{12} is positive in a two-sector model if the consumption good is labor intensive (for a proof, see Benhabib and Nishimura (1985)) and is always positive in a one-sector non-joint production model given by $U(c) = U(T(k_1,k_0)) = U(f(k_0) - (1-g)k_0 - k_1)$, where g is the depreciation rate, f is the concave production function and U is a concave and increasing utility function. We now can give an informal diagrammatic exposition to show the role of assumption M. For expositional simplicity, we assume the concavity of the aggregator function $V(c_0,w(k_1))$ in c_0 and k_1. In figure 1 we draw indifference curves showing the trade-off between current consumption c_0 and future utility $W(k_1)$. These isoquants can then be drawn in the c_0 - k_1 space since $w(k_1)$ is concave and increasing in k_1. Similarly, we can also draw the production possibility surface giving the tradeoff in c_0 and k_1 (assuming full depreciation) for k_0 fixed. In Figure 1 assume that the optimal choice of k_1, $\hat{k}_1 > k_0$. This implies that the production possiblity curve shifts out. In a one-sector model this shift is a parallel one since $c_0 + k_1 = f(k_0)$. Note that whether $\hat{k}_2$ is greater than or smaller than $\hat{k}_1$ depends on whether the current consumption is a "luxury good"; $\hat{\underline{k}}_2$ depicts such a situation. On the other hand, $\hat{k}_2$ is possible if "future consumption" is not inferior and $\hat{k}_2$ is greater than k_1, implying a monotonic trajectory.

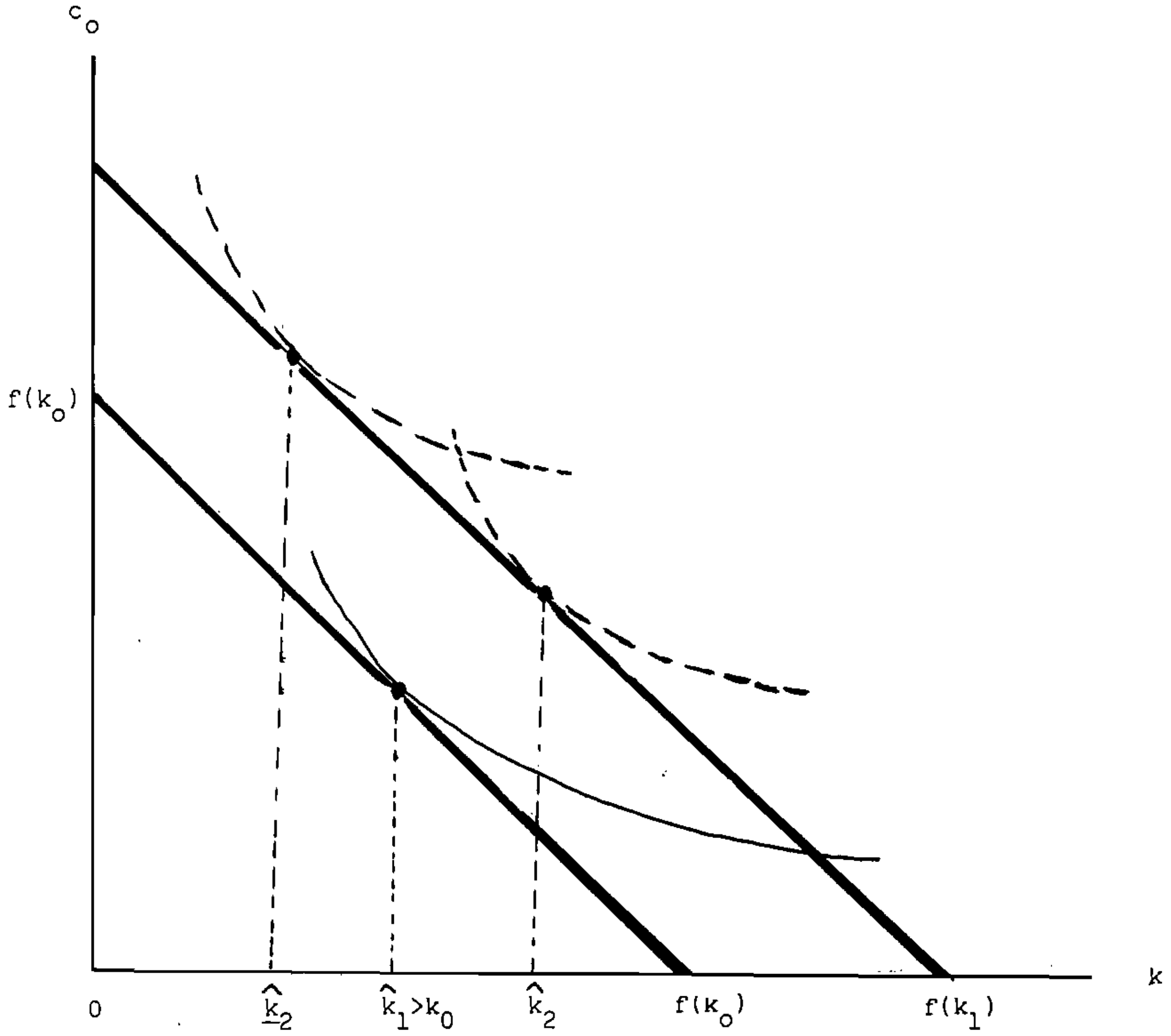


Figure 1

If $V(c_0,W(k_1)) = \tilde{u}(c_0)) + \beta W(k_1)$, where β is the discount factor, we have an additivity separable form in c_0 and k_1, which implies that the trajectory is monotonic because both c_0 and k_1 will be "normal goods." Even in the additively seperable case, however, if we have a two-sector model where the consumption good is captial intensive, oscillatory behavior is possible if T_{12} is negative. This is shown in Figure 2. Therefore, whether $V_1 \cdot T_2$ is monotonic in k_1 or not depends on whether or not future consumption is a normal good or not, as well as on the factor intensities of the two-sector model. Note that our assumption on the concavity of the aggregator function V was for the convenience of diagrammatic exposition only. We will not use it for the proofs in this section.

We shall prove several lemmas before we formally prove convergence of the optimal trajectory.

Lemma 3.3: If $k^* \in K^*$, the $\Phi(k^*)$ is a singleton.

Proof: If ${}_0k^* = 0$, it is clear that $\{{}_{con}(0)\} = \Phi(k^*)$. Let $0 < k^* < \bar{k}$. Suppose Lemma 3.3 is wrong. Then if $k' \in \Phi(k^*)$ and $k_1' \neq k^*$, then for arbitrary $T \geq 1$ any path ${}_0k$ defined by

$$k_t = k^* \qquad 0 \leq t \leq T$$

$$k_t = k'_{t-T} \qquad t \geq T + 1$$

is optimal, since k^* is also optimal. Consider then the triplet $(k_{T-1},k_T,k_{T+1}) = (k^*,k^*,k_1')$, giving rise to utility $V(T(k^*,k^*),V(T(k_1',k^*),\hat{u})) = V(T(k^*,k^*),u^*)$ for $\hat{u} = W(k_1')$. The last equality follows since (k^*,k^*,k^*) is also optimal. Thus we obtain $U^* = V(T(k_1',k^*),\hat{u})$. Using (18) yields

$$V_1(c^*,u^*) \cdot T_1(k^*,k^*) + V_2(c^*,u^*) \cdot V_1(T(k_1',k^*),W(k_1')) \cdot T_2(k_1',k^*) = 0. \qquad (20)$$

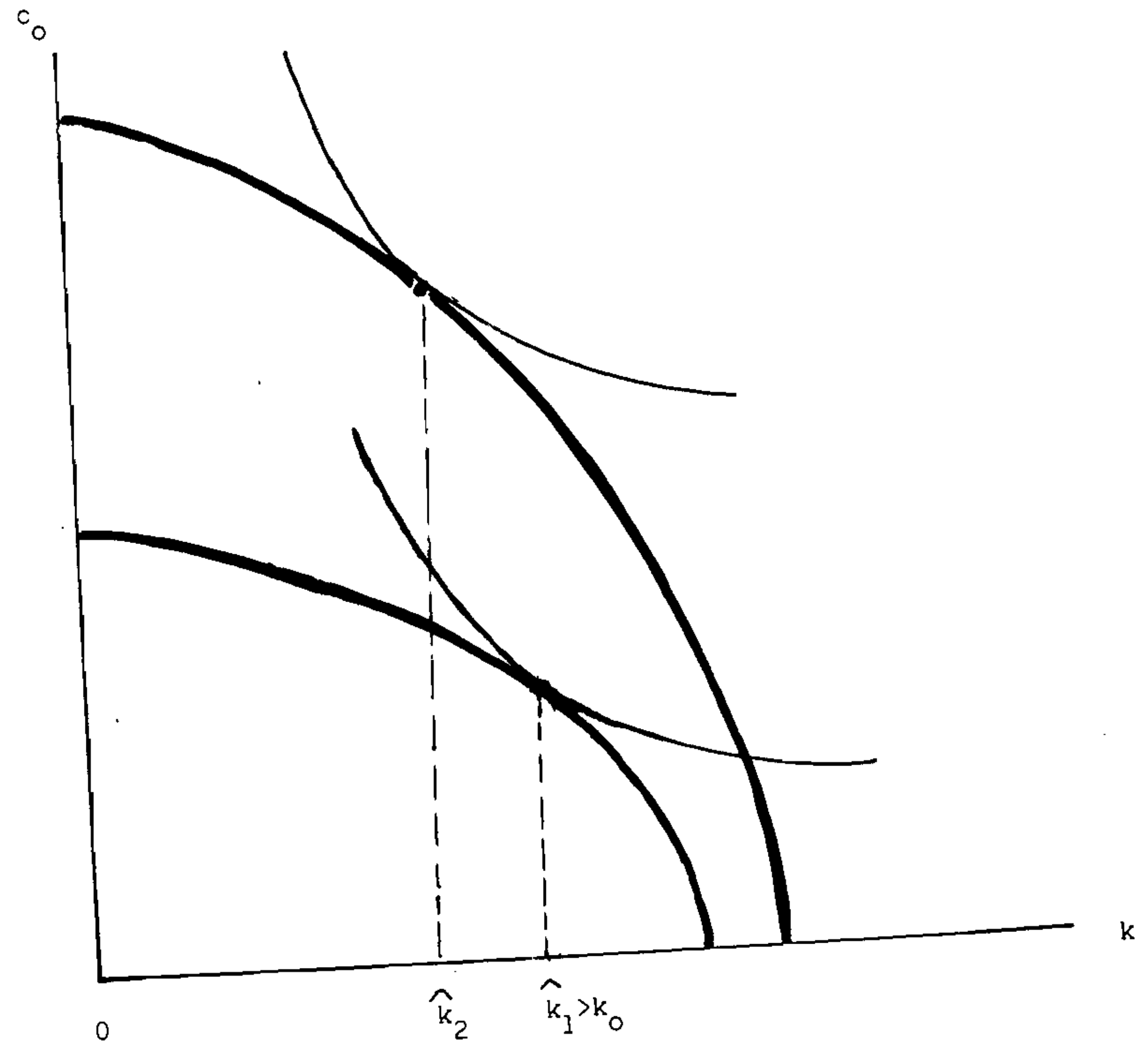


Figure 2

However, (M) implies that (20) has a unique solution $k_1' = k^*$ because it assumes that V_1T_2 is strictly increasing in k_1'. Hence $\Phi(k^*)$ consists of a unique path ${}_{con}(k^*)$. Q.E.D.

__Lemma 3.4__: If $k_1 = k_0$ for ${}_0k \in \Phi(k_0)$, then $k_0 \in K^*$ and ${}_0k = {}_{con}(k_0)$.

__Proof__: If $k_0 = k_1$ and ${}_0k \in \Phi(k_0)$, then ${}_0k \in \Phi(k_1)$ by the principle of optimality. It follows by induction, using the uniqueness implied by (M) as in Lemma (3.3), that ${}_{con}(k_0) \in \Phi(k_0)$, and k_0 is a steady state. Q.E.D.

__Corollary 3.2__: If $k_T = k_{T+1}$, for ${}_0k \in \Phi(k_0)$, then $k_t = k^*$, $t \geq T$, where k^* is a steady state.

__Lemma 3.5__: Let $k_0', k_0 \in K$. Given ${}_0k' \in \Phi(k_0')$ and ${}_0k \in \Phi(k_0)$,

(i) $k_0' > k_0$ implies $k_1' \geq k_1$

(ii) $k_0 \underset{(<)}{>} k_1$ implies $k_1 \underset{(\leq)}{\geq} k_2$.

__Proof__: (i) Suppose that $k_1' < k_1$ holds. Then $T(k_1,k_0) \geq 0$ and $T_1 < 0$ implies that $T(k_1',k_0) > 0$. Hence k_1' is attainable from k_0. The optimality of ${}_0k$ implies

$$(21) \qquad V(T(k_1,k_0),W(k_1)) \geq V(T(k_1',k_0),W(k_1')).$$

Also, $T(k_1,k_0) \geq 0$, $\bar{k} \geq k_0' > k_0 \geq 0$ and $T_2 > 0$ implies $T(k_1,k_0') > 0$. Hence k_1 is attainable from k_0'. The optimality of ${}_0k'$ implies

$$(22) \qquad V(T(k_1',k_0'),W(k_1')) \geq V(T(k_1,k_0'),W(k_1)).$$

Adding (21) and (22), we get

$$V(T(k_1,k_0),W(k_1)) + V(T(k_1',k_0')),W(k_1')) \geq V(T(k_1',k_0),W(k_1')) + V(T(k_1,k_0'),W(k_1)).$$

That is,

$$V(T(k_1',k_0'),W(k_1')) - V(T(k_1',k_0),W(k_1')) + V(T(k_1,k_0),W(k_1)) - V(T(k_1,k_0'),W(k_1)) \ge 0$$

or,

$$(23) \qquad \int_{k_0}^{k_0'}[V_1(T(k_1',\alpha),W(k_1'))T_2(k_1',\alpha) - V_1(T(k_1,\alpha),W(k_1))T_2(k_1,\alpha)]d\alpha \ge 0.$$

Note that both k_1' and k_1 are attainable from $\alpha \in [k_0,k_0']$. (M) and the hypothesis $k_1' < k_1$ imply that

$$[V_1(T(k_1',\alpha),W(k_1'))T_2(k_1',\alpha) - V_1(T(k_1,\alpha),W(k_1))T_2(k,\alpha)] < 0.$$

This together with $k_0' > k_0$ makes the value of the integral in (23) negative. This is a contradiction. Hence $k_1' \ge k_1$ must be the case.

(ii) If $k_0 < k_1$, then we consider an optimal path ${}_0k'$ from $k_0' = k_1$. By applying the result of (i), we get $k_1 \le k_1' = k_2$. Q.E.D.

<u>Theorem 3.1</u>: (Monotonicity) If ${}_0k \in \Phi(k_0)$, and $k_0 \underset{(<)}{>} k_1$, then $k_t \underset{(\le)}{\ge} k_{t+1}, t \ge 1$. The equality holds only for $k_t \in K^*$.

<u>Proof</u>: Suppose that $k_0 > k_1$. Then $k_1 \ge k_2$ by Lemma 3.5. If $k_1 = k_2$, then $k_1 \in K^*$ by Corollary 3.2. If $k_1 > k_2$, then $k_2 \ge k_3$. It is clear that induction leads to the theorem. Q.E.D.

<u>Theorem 3.2</u>: Assume that (P.1)-(P.3), (U.1)-(U.5) and (M). For any $k_0 \in K$ an optimal path monotonically converges to a steady state.

<u>Proof</u>: As $\{k_t\}$ is a monotone bounded sequence, it has a limit. Say $\tilde{k} = \lim_{t\to\infty} k_t$. Consider a sequence of optimal paths. That is, ${}_0k^0, {}_0k^1, {}_0k^2, \ldots$ where $k_0^0 = k_0, k_0^1 = k_1, \ldots, k_0^t = k_t, \ldots$. This sequence converges to

${}_{con}(\tilde{k})$ in the product topology. ${}_{con}(\tilde{k})$ is an optimal path by the upper semi-continuity of Φ. Hence $\tilde{k} \in K^{*}$. Q.E.D.

If we assume that K^{*} contains a finite number of elements (steady states only), we can write $K^{*} = \{k_0^*, \ldots, k_n^*\}$, $0 = k_0^* < k_1^* < \ldots < k_n^*$.

$$\text{Let} \quad K_i = (k_i^*, k_{i+1}^*) \qquad i = 0, \ldots, n-1$$
$$K_n = (k_n^*, k]$$

Theorem 3.3: If K^* has a finite number of elements and $k_0 \in K_i$, ${}_0k \in \Phi(k_0)$, then $\{k_t\}$ converges to k_i^* or k_{i+1}^*.

Proof: By Theorem 3.2 the optimal path converges to a steady state monotonically. Let $\tilde{k} = \lim k_t$. By lemma 3.5, $k_{i+1}^* > k_0$ implies $k_{i+1}^* \geq k_1$. If $k_{i+1}^* = k_1$, the $\tilde{k} = k_{i+1}^*$. If $k_{i+1}^* > k_1$, then $k_{i+1}^* \geq k_2$. By continuing this argument, $k_{i+1}^* \geq k_t$, $t \geq 0$. Similarly, $k_t \geq k_i^*$, $t \geq 0$, must hold. Since $\tilde{k} \in K^*$, $\tilde{k}$ must be either k_i^* or k_{i+1}^*. Q.E.D.

Suppose that one optimal path from $k_c \in K$ converges to a high steady state $k_j^* > k_c$ while some other optimal path from k_c converges to a lower steady state $k_i^* < k_c$. Then k_c is called a critical level of capital stock. We can have the following generalization of Dechert and Nishimura (1983).

Corollary 3.3: Let $k_c \in (k_i^*, k_{i+1}^*)$ be a critical level of the capital stock. Then all strictly increasing optimal paths converge to k_{i+1}^*, while all of strictly decreasing optimal paths converge to k_i^*. Moreover the critical level in (k_i^*, k_{i+1}^*) is unique (if it exists).

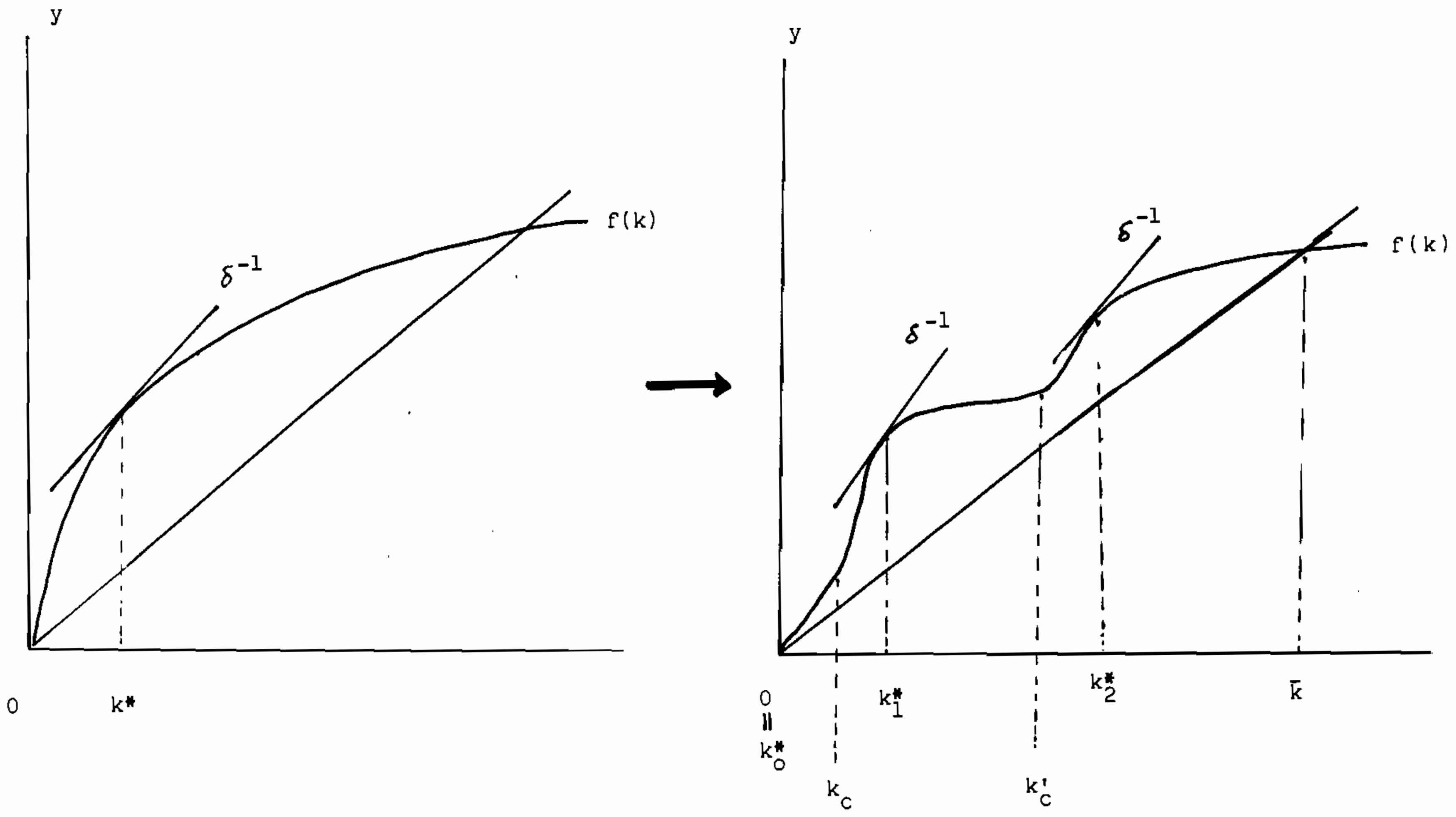
Proof: Let ${}_t\hat{k} \in \Phi(k_c)$ be a strictly increasing path. Since $k_c < k_{i+1}^*$, lemma 3.5 (i) implies $k_t < k_{i+1}^*$ for all t. Hence $k_t \to k_{i+1}^*$. Similarly any strictly decreasing path in $\Phi(k_c)$ converges to k_i^*.

Next we show the uniqueness. Suppose that k_c, k'_c are both critical levels in (k^*_i, k^*_{i+1}). Let $k_c < k'_c$. Then there is a strictly decreasing path is $\Phi(k'_c)$, say ${}_t\hat{k}'$. And there is a strictly increasing path in $\Phi(k_c)$, say ${}_t\hat{k}$. Since ${}_t\hat{k}$ converges to k^*_{i+1} and ${}_t\hat{k}'$ converge to k^*_i. There is T such that $\hat{k}'_T < \hat{k}_T$. This is a contradiction to $\hat{k}_0 = k_c < \hat{k}'_0 = k'_c$ and Lemma 3.5 (i). Hence $k_c = k'_c$ must be the case. Q.E.D.

Remark 3.1: We proved the monotone convergence of the optimal capital stocks to a steady state in a more general framework than Beals and Koopmans (1969) or Iwai (1970). Their function f, resulting in $f(k_t) - k_{t+1}$, is a special case of our $T(k_{t+1}, k_t)$. The significant difference here lies in the fact that we do not assume the concavity of the production function or the quasi-concavity of the intertemporal utility function.

Remark 3.2: As we lose the concavity of the production function or the quasi concavity of the utility function, we see the bifurcation of the multiple steady states and the convergence of critical levels of capital stock. Figure 3 shows this for the model with the separable intertemporal utility function $\Sigma^{\infty}_{t=0}\delta^t U(c_t)$ and the separable production function $c_t = f(k_t) - k_{t+1}$. In diagram (ii) of figure 3 the production curve $y = f(k)$ lacks the concavity. This is the case considered in Dechert and Nishimura (1984).

Remark 3.3: Non-separability of utility or production function could cause the multiplicity of steady states. Diagram (i) of the figure 4 shows this case. The optimal policy function, that is the optimal choice of the next period capital stock level, is indicated by the upward sloping curve. Steady states are points of intersection between the optimal policy curve and 45 degree line. Note that unstable steady states and stable steady states



(i) (ii)

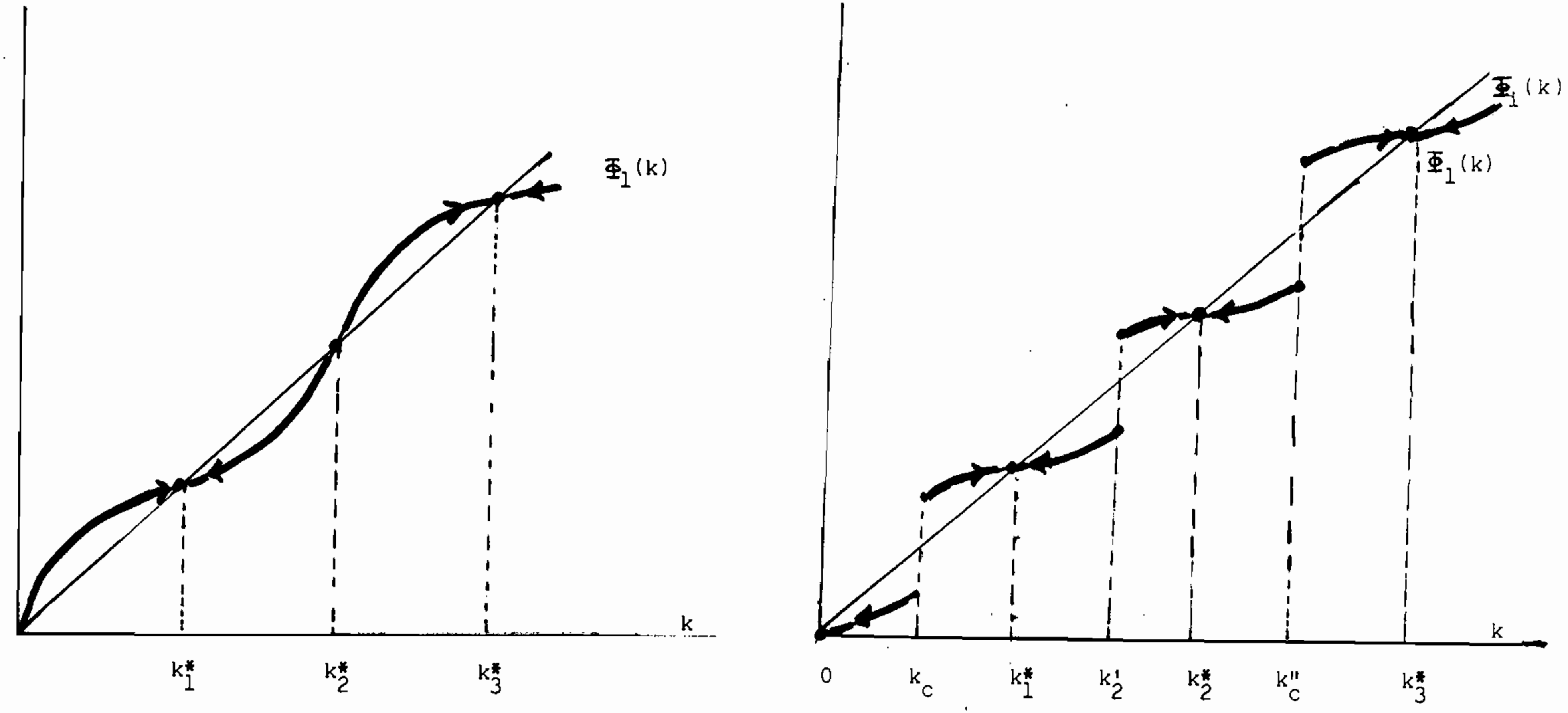
Figure 3 Bifurcation

alternate. This may be compared with the diagram (ii) of figure 4, which shows the optimal policy curve of the model with non-concave functions. Note that all optimal steady states may be stable contrary to the diagram (i).

Remark 3.4: An advantage of our method is that we can easily apply the above arguments to derive the oscillatory property of optimal paths under the alternative condition to (M).

$$(\bar{M}) \qquad V_1(T(k_1,k_0),W(k_1))\cdot T_2(k_1,k_0)$$

is strictly decreasing in k_1. Again we do not need to use the differentiability of the value function $W(k)$. We do not repeat the proofs of the results under this alternative condition. Rather, we discuss the oscillatory case under the stronger condition in the next chapter.



(i) Bifurcation due to loss of separability

(ii) Bifurcation due to loss of concavity

Figure 4

4. Normality assures convergence and inferiority produces oscillations

Though we so far did not assume the concavity of the utility function or the production function (except for the diagrammatic exposition in the previous section), we proved the monotone convergence of the optimal paths. Below we obtain slightly stronger results in terms of strict monotonicity and asymptotic convergence under concavity assumptions. In addition, when the converse of the assumption (M) holds, we show that the optimal trajectory is oscillatory, as suggested in the diagrammatic exposition in section 3. Below we assume the following:

(P.4) $T(y,k)$ is concave in y and k.

(U.6) U is strictly quasi-concave on $C(K)$.

(U.7) $V(c,u)$ has continuous second derivatives V_{11}, V_{12}, and V_{22} on $(0,\bar{c}) \times (a,b)$, where $V_{11} = \partial^2 V/\partial c^2$, $V_{12} = \partial^2 V/\partial u \partial c$ and $V_{22} = \partial^2 V/\partial u^2$.

It is easy to show that the set of feasible capital paths $F(k_0)$ and the set of feasible consumption paths become convex sets. It should be noted that (U.6) does not imply the quasi-concavity of the aggregator $V(c,u)$ with respect to (c,u).

Lemma 4.1: An optimal path from $k_0 \in K$ is unique.

Proof: Let ${}_0k'$, ${}_0k \in \Phi(k_0)$ and let ${}_0c'$, ${}_0c \in \Psi(k_0)$ be the corresponding optimal consumption paths. Suppose that ${}_0k' \neq {}_0k$. Then we assume $k_1' \neq k_1$ without loss of generality. But this with $T_1 < 0$ implies

(24) $$c_0' = T(k_1',k_0) \neq T(k_1,k_0) = c_0.$$

But for ${}_0c' \neq {}_0c$, (U.6) implies that a feasible consumption path $\theta_0c' + (1-\theta)_0c$

from k_0 yields a higher utlity. This is a contradiction. Hence ${}_0k' = {}_0k$. Q.E.D.

Now the optimal path $\Phi(k_0)$ becomes a continuous function of k_0. It can be shown that $W(k_0)$ is strictly increasing in k_0. We assume that

(U.8) $\quad W(k_0)$ is differentiable at $k_0 \in (0,\bar{k}]$.

The condition (M) assumed in the previous section is replaced by

$$(M') \qquad T_1T_2(V_{11}V_2 - V_1V_{12}) + V_2V_1T_{12} > 0.$$

$V_{11}V_2 - V_1V_{12} \leq 0$ implies that for the aggregator function $V(c,u)$ u is a non-inferior good. Note that $T_{21} \geq 0$ is trivially satisfied if $T(k_{t+1},k_t) = U(f(k_t)-k_{t+1})$, where U is the one-period utility function. On the other hand, if $T(y,k)$ is a social production function in a two-sector economy, T_1 is always non-positive and $T_{12} \geq 0$ as long as the pure consumption good sector is not capital intensive (see Benhabib and Nishimura (1985), section 5, and also the diagrammatic exposition following assumption (M) in the previous section).

<u>Theorem 4.1</u>: Under the assumptions (P.1) - (P.4), (U.1) - (U.8) and (M'), the optimal paths from $k_0 \in K$ are strictly monotonic and converge to some steady state.

<u>Proof</u>: Let $W' = \partial W(k_1)/\partial k_1$. Using (U.8) and the fact that if ${}_0k \in \Phi(k_0)$ for $k_0 \in (0,k)$, $V(T(x,k_0),W(x))$ is maximized at k_1, we obtain

$$(25) \qquad V_1(T(k_1,k_0),W(k_1))T_1(k_0,k_1) + V_2(T(k_1,k_0),W(k_1))W'(k_1) = 0.$$

Let $H(k_0,k_1) = V_1(T(k_1,k_0),\ W(k_1))T_2(k_1,k_0)$. Then $\partial H/\partial k_1 = (V_{11}T_1 + V_{12}W')T_2 + V_1T_{21}$. By substituting for $W'(k)$ from (25), (M') implies

$$(26) \qquad \partial H/\partial k_1 = 1/V_2\ [V_{11}V_2 - V_{12}V_1]T_1T_2 + V_1T_{21} > 0$$

Assume that ${}_0k' \in \Phi(k_0')$, ${}_0k \in \Phi(k_0)$ and suppose $k_0' > k_0$. Let $k_1' = \Phi_1(k_0')$. First consider the case where $k_1' \neq k_1$. By the uniqueness of optimal paths we have

$$(28) \qquad \int_{k_0}^{k_0'} [V_1(T(k_1',\alpha),W(k_1'))T_2(k_1',\alpha) - V_1(T(k_1,\alpha),W(k_1))T_2(k_1,\alpha)]d\alpha > 0.$$

The strict inequality comes from the uniqueness of the optimal path and the hypothesis $k_1' \neq k_1$. Using double integral, we can rewrite (28).

$$(29) \qquad \int_{k_0}^{k_0'} \int_{k_1}^{k_1'} [(V_{11}T_1 + V_{12}W')T_2 + V_1T_{21}]d\beta d\alpha > 0$$

where $V = V(T(\beta,\alpha),W(\beta))$, $T = T(\beta,\alpha)$. If k_0' is sufficiently close to k_0, the integrand of (29) has the same sign as (26) for any (β,α) contained in $[k_0,k_0'] \times [k_1,k_1']$. This is a result of (25) and the continuity of $k_1 \equiv \Phi_1(k_0)$. Hence for k_0 sufficiently close to k_0' and $k_0' > k_0$ (29) implies $k_1' > k_1$. This implies that $k_1 \equiv \Phi_1(k_0)$ is increasing in k_0 for $k_0 \in (0,\bar{k})$. The rest of the argument is identical to that of (ii) in lemma 3.5.

To complete the proof we have to rule out the case where $k_0' > k_0$ implies $k_1' = k_1$. First note that if $k_0' > k_0 = 0$, then $k_1' > 0 = k_1$. If $k_0' > k_0 > 0$, then ${}_0k'$ and ${}_0k$ satisfy the Euler equation (18).

Assuming that $k_1' = k_1$ in (18), we have

$$(30) \qquad \frac{V_1(T(k_1,x),W(k_1))}{V_2(T(k_1,x),W(k_1))} T_1(k_1,x) + V_1(T(k_2.k_1),W(k_2)) \cdot T_2(k_2,k_1) = 0.$$

(30) is satisfied by $x = k_0', k_0$. However, (M′) implies that the first term on the left hand side of (30) is increasing in x. Therefore, if $k_0' > k_0$, we cannot have $k_1 = k_1'$ without violating (18). This rules out the case where $k_1 = k_1'$ and completes the proof. Q.E.D.

To prove our results for the oscillation case, we maintain all the assumptions except that we reverse the assumption (M′). ($\overline{M}'$) below requires either the consumption good to be capital intensive ($T_{12} < 0$) as discussed in Benhabib and Nishimura [1985] or future utility to be an inferior good ($V_{11}V_2 - V_1V_{12} > 0$). If both of these conditions holds, whether $\overline{M}'$ is satisfied or not depends on their weighted sum of the two effects.

($\overline{M}'$) $$(V_{11}V_2 - V_1V_{12})T_1T_2 + V_2V_1T_{12} < 0.$$

However, it may not be reasonable to assume that ($\overline{M}'$) holds globally. So we restrict ourselves to the rectangular region $H = [c_0,d_0] \times [c_1,d_1]$ where $0 < c_j < d_j < \overline{k}, \quad j = 1,2.$

<u>Theorem 4.3</u>: Let (P.1) - (P.4), (U.1) - (U.8) and ($\overline{M}'$) hold. Let ${}_0k \in \Phi(k_0)$, ${}_0k' \in \Phi(k_0')$.

(i) For (k_0,k_1) and $(k_0',k_1') \in H$, if $k_0 > k_0'$, then $k_1 < k_1'$.

(ii) For (k_0,k_1) and $(k_1,k_2) \in H$, if $k_0 > k_1$, then $k_1 < k_2$.

<u>Proof</u>: (i) Since either $(V_{11}V_2 - V_1V_{12})$ or $-T_{21}$ is positive in condition ($\overline{M}'$), the same proof as in theorem 4.1 is applied to show that $k_0 > k_0'$ implies $k_1' \neq k_1$ for ${}_0k \in \Phi(k_0)$, ${}_0k' \in \Phi(k_0')$. Choose k_0 and k_0' such that $((k_0',k_1),(k_0,k_1)) \in H$ and k_0' is sufficiently close to k_0. We suppose that $k_0 > k_0'$ and $k_1 > k_1'$ holds. Then $((k_0',k_1'),(k_0,k_1')) \in H$ follows from the fact that $((k_0,k_1),(k_0',k_1'))$ is in H. Also, $T(k_1,k_0) \geq 0$ and $T_1 > 0$ implies that $T(k_1',k_0') > 0$. Hence, k_1' is attainable from k_0. Also, since an optimal path from $k_0 > 0$ is an interior path, $c_0 = T(k_1,k_0) > 0$. If k_0' is sufficiently close to k_0, we have $T(k_1,k_0') > 0$ as well. This is due to the continuity of $\Phi(k_0)$. Then the principle of optimality implies

(31) $$V(T(k_1,k_0),W(k_1)) > V(T(k_1',k_0),W(k_1'))$$

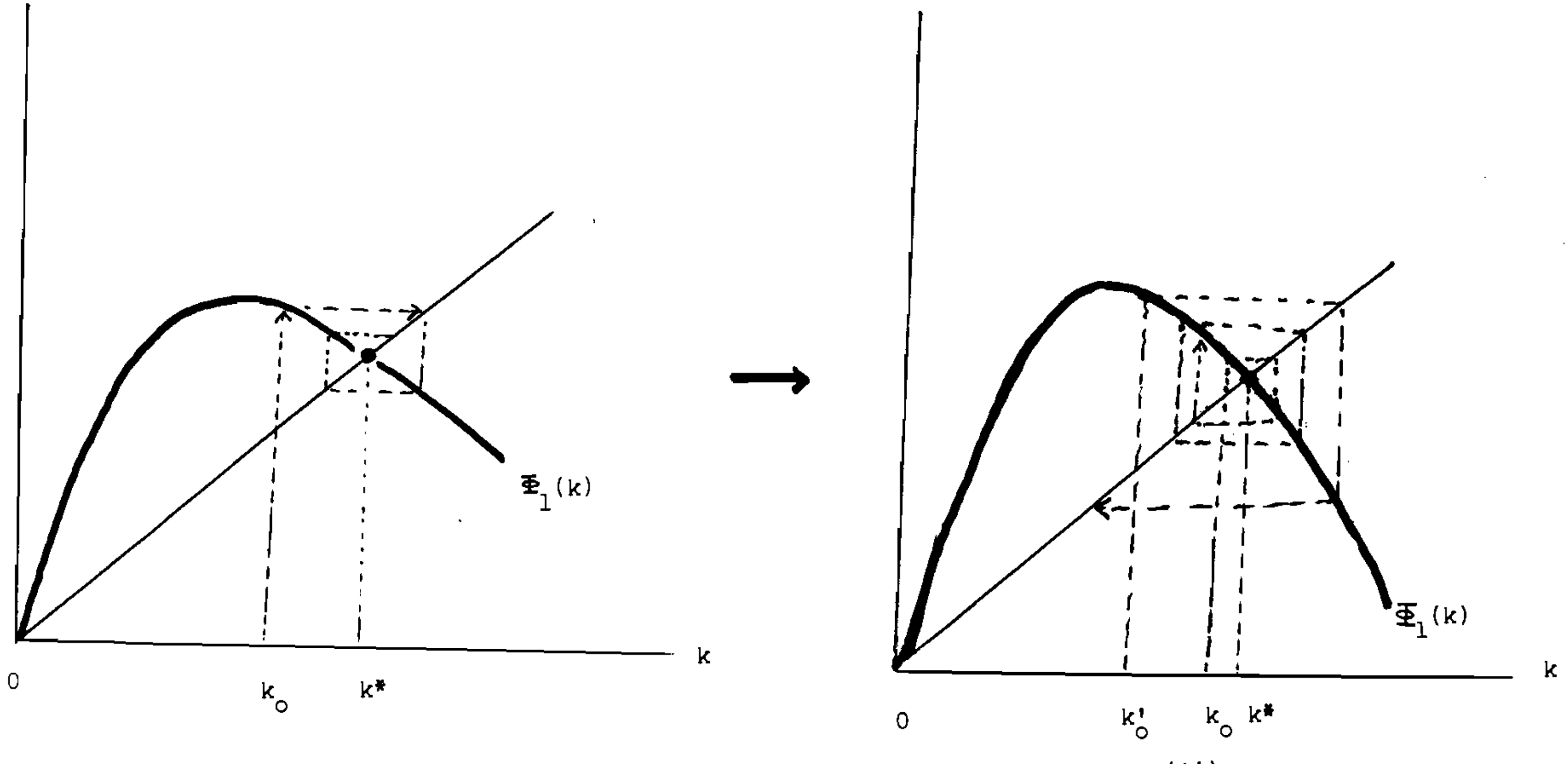
$$V(T(k_1',k_0'),W(k_1')) > V(T(k_1,k_0'),W(k_1))$$

From (31) we can derive the same double integral as given by (29) above. But for k_0' sufficiently close to k_1 the sign of the term under the double integral is negative under $(\bar{M})$, as in proof of Theorem 4.1. Hence we have the contradiction and $k_1' > k_1$ must hold. This shows that $\Phi_1(k_0) = k_1$ is strictly decreasing in k_0 as long as $(k_0,\Phi_1(k_0)) \in H$. Hence for any $(k_0,\Phi_1(k_0))$, $(k_0',\Phi_1(k_0')) \in H$, $k_0 > k_0'$ implies $\Phi_1(k_0) = k_1 < \Phi_1(k_0') = k_1'$.

(ii) This follows immediately from part (i) if we let $k_0' = k_1$ and $k_1' = k_2$.

Remark 4.1: When $(\bar{M}')$ holds over the appropriate domain, $\Phi_1(k)$ will be monotonic. Then it is easily shown that both odd and even iterates of k_t will be monotonic. This implies that both odd and even iterates converge. If the limits of odd and even iterates are the same we get convergence to a steady state; otherwise we get convergence to a two-period cycle. The proof in the case of additively separable utility also applies to the non-separable case. (See Benhabib and Nishimura (1985), Theorem 3.) Of course, if $(\bar{M}')$ does not hold and $\Phi_1(k)$ is not monotonic, unstable and chaotic behavior can occur (see Bolrin and Montrucchio [1986]).

Remark 4.2: Under the assumption $(\bar{M}')$ over the appropriate region birfurcations of periodic optimal paths are possible. This is so even when production and utility functions share concavity properties provided δ is small enough. Figure 5 shows that as δ decreases the steady state becomes unstable and a periodic path bifurcates. This is the case whose existence is shown in Benhabib and Nishimura [1985].



(i) (ii)

Figure 5 Bifurcation due to decrease in δ

5. Generalizations

(I)

We have so far discussed the Koopmans type intertemporal utility function so that it can be transformed into an aggregator function. However, it is clear from proofs given above that we can deal with any intertemporal utility function $\mathfrak{R} = \mathfrak{R}(k_0, k_1, \ldots)$ which may be transformed into the aggregator form

$$(32) \qquad \mathfrak{R}(k_0, k_1, W(k_1)).$$

This is more general than Koopmans type utility function. For example, consider the following example.

$$(33) \qquad \mathfrak{R} = \textstyle\sum_{t=1}^{\infty} \delta^t U(c_{t-1}, k_t), \quad 0 < \delta < 1$$

subject to

$$f(k_t) - k_{t+1} = c_t \geq 0 \ , \quad t \geq 0, \ k_0 \ \text{ given.}$$

Let $W(k_0)$ the maximized value of utility, then it satisfies

$$(34) \qquad W(k_0) = \max[U(f(k_0) - k_1, k_1) + \delta W(k_1)]$$

This differs from the "Koopmans" type intertemporal utility function, since the current utility does not only depend on current consumption, $f(k_0) - k_1$, but also depend on k_1.

Rather than discussing the problem (33) in the general context, we shall briefly sketch how problem (33) is handled by a method similar to that above. We assume that

(p.1) The production function $f(k)$ is continuous at $k \geq 0$ and continuously differentiable at $k > 0$.

(p.2) $f(0) = 0, \quad f'(k) > 0, \quad f(\bar{k}) = \bar{k} > 0.$

(u.1) The utility function is continuous at $c, k \geq 0$.

(u.2) $U_1(c,k) > 0$ exists and is continuous at $c > 0, k > 0$

$U_2(c,k) > 0$ exists and is continuous at $c \geq 0, k > 0$.

(u.3) $\lim_{c \to 0} U_1(c,k) = \infty$ for $k > 0$

$\lim_{k \to 0} U_2(c,k) = \infty$ for $c > 0$

The monotonicity of optimal path is implied by the following assumption.

(m) $U_1(f(k_0) - k_1, k_1)$ is strictly increasing in k_1.

Under the assumptions above we can show that the optimal paths are interior and converge to steady states. Again we do not require the concavity of the utility function or the production function.

Lemma 5.1: Let ${}_0\hat{k}$ and ${}_0\hat{c}$ be optimal capital and consumption paths from $\hat{k}_0$ in problem (33). If $\hat{k}_0 > 0$, $\hat{k}_t > 0$ and $c_t > 0$ for $t \geq 0$.

Proof: (i) Suppose that $\hat{k}_1 = 0$. Then $\hat{c}_1 = \hat{k}_2 = 0$. Consider

$$g(k_1,\hat{k}_2) = U(f(\hat{k}_0) - k_1,k_1) + \delta U(f(k_1) - \hat{k}_2,\hat{k}_2). \tag{36}$$

$g(k_1,\hat{k}_2)$ is maximized at $k_1 = \hat{k}_1 = 0$. However,

$$\lim_{\hat{k}_2 \to 0} \partial g(k_1,\hat{k}_2)/\partial k_1 = - U_1(f(\hat{k}_0) - k_1,k_1) + U_2(f(\hat{k}_0) - k_1,k_1)$$

$$+ \delta \lim_{x \to 0} U_1(f(k_1) - x,x) f'(k_1)$$

Since $0 \le \lim_{x\to 0} U_1(f(\hat{k}_0) - x, x) < U_1(f(\hat{k}_0) - k_1, k_1)$ for $k_1 > 0$ by assumption (m), and since the other terms are positive, $\lim_{k_1\to 0} U_2(f(k_0) - k_1, k_1) = \infty$ implies that the $\lim_{k_1\to 0}\ \lim_{k_2\to 0} \partial g(k_1,k_2)/\partial k_1$ is positive. This is contradiction. Hence $\hat{k}_1 > 0$. It is obvious that $\hat{k}_t > 0$ for $t \ge 0$ must follow.

(ii) Let $\hat{c}_0 = 0$, $\hat{c}_1 > 0$. Consider

$$h(k_1) = U(f(\hat{k}_0) - k_1, k_1) + \delta U(f(k_1) - \hat{k}_2, \hat{k}_2) \tag{37}$$

If $\hat{c}_0 = 0$, (37) is maximized at $k_1 = \hat{k}_1 = f(\hat{k}_0)$. Consider

$$\begin{aligned} h'(k_1) = & -U_1(f(\hat{k}_0) - k_1, k_1) + U_2(f(\hat{k}_0) - k_1, k_1) \\ & + \delta U_1(f(k_1) - \hat{k}_2, \hat{k}_2) f'(k_1) \end{aligned} \tag{38}$$

$$\lim_{k_1\to\hat{k}_1} h'(k_1) = -\lim_{z_0\to 0} U_1(z_0, \hat{k}_1) + U_2(0, \hat{k}_1) + \delta U_1(\hat{c}_1, \hat{k}_2) f'(\hat{k}_1) = -\infty \tag{39}$$

This is a contradiction. Hence $\hat{k}_1 < f(\hat{k}_0)$ if $\hat{c}_0 > 0$. Next let $\hat{c}_0 = \hat{c}_1 = 0$. Then it is clear from assumptions (u.2) and (u.3) that there must exist some t such that $c_t > 0$ and $\hat{c}_t > 0$. We apply the same arguments given above and reach a contradiction. Hence $\hat{k}_0 > 0$ must imply $\hat{c}_t > 0$ for all $t \ge 0$. Q.E.D.

<u>Theorem 5.1</u>: Let (p.1), (p.2) (u.1)-(u.3) and (m) hold. Then any optimal path from $k_0 \in [0, \bar{k}]$ in problem (33) converges monotonically to a steady state.

This theorem may be proved exactly in the same way as Theorem 3.2. We omit the proof. A stronger convergence result may be proved by assuming the twice differentiability of utility function and replacing assumption (m) as follows:

(m′) $U_{12} - U_{11} > 0$ on $(0,\bar{k}) \times (0,\bar{k})$.

We do not, however, assume concavity. The stronger convergence result may be proved under much weaker assumptions than those corresponding to Theorem 4.1.

Theorem 5.2: Let (p.1), (p.2), (u.1)-(u.3) and (m′) hold. Then any optimal path ${}_0k$ from $k_0 \in [0,\bar{k}]$ in problem (34) converges to a steady state. Moreover, $\{k_t\}$ is a strictly monotone sequence for $k_0 \notin K^*$.

Proof: Since (m′) implies (m), Theorem 5.1 holds. Then we show that $k_0' \neq k_0$ implies $k_1' \neq k_1$ as was done in the proof of Theorem 4.1. Q.E.D.

So far we did not prove whether the origin is a locally stable steady state or not. To guarantee that it is a locally unstable steady state, we assume the following additional properties.

(p.3) $\lim_{k \to 0} f'(k) > \delta^{-1}$

(u.4) $U_{11}(c,k) < 0$, $U_{12}(c,k) \le 0$ for sufficiently small $c,k > 0$.

Theorem 5.3: Let (p.1)-(p.3), (u.1)-(u.4) and (m′) hold. Then any optimal path ${}_0k$ from $k_0 \in (0,\bar{k}]$ converges to a non-trivial steady state. Furthermore, $\{k_t\}$ is strictly monotone if k_0 is not a steady state.

Proof: We first prove that a non-trivial steady state is bounded away from 0. Consider the Euler equations satisfied at a steady state.

$$(\delta f'(k^*) - 1)U_1(c^*,k^*) + U_2(c^*,k^*) = 0. \tag{40}$$

Since $\lim_{k \to 0} \delta f'(k) > 1$, (40) cannot be satisfied for sufficiently small $k > 0$.

Suppose that an optimal path ${}_0k$ from $k_0 > 0$ converges to 0 so that $\{k_t\}$

is monotonically decreasing to 0. So $k_0 > k_1 > k_2 > 0$. Consider the Euler equation

$$(41) \qquad -U_1(c_0,k_1) + U_2(c_0,k_1) + \delta f'(k_1)U_1(c_1,k_2) = 0.$$

By rewriting equation (41), we obtain

$$(42) \qquad (\delta f'(k_1) - 1)U_1(c_1,k_2) + [U_1(c_1,k_2) - U_1(c_0,k_1)] + U_2(c_0,k_1) = 0.$$

The first and third terms are positive for k_0 sufficiently close to 0. Hence

$$(43) \qquad U_1(c_1,k_2) - U_1(c_0,k_1) < 0.$$

Since U_{12} and U_{11} are non-positive by assumption (u.4), it easily follows from (43) that $k_1 > k_2$ implies $c_0 < c_1$. Hence if k_0 is sufficiently close to 0, then along the path ${}_0k$, $k_0 > k_1 > k_2 > \ldots$, $c_0 < c_1 < c_2 < \ldots$ must hold. But the consumption path cannot increase monotonically as $c_t = f(k_t) - k_t$ converges to 0. Hence we have a contradiction. Therefore, for sufficiently small $k_0 > 0$, ${}_0k$ converges to a steady state $k^* > 0$. Q.E.D.

II.

In the second part of this section we consider a two-agent, pure exchange economy where the preferences of each agent are subject to assumptions in (u.1)-(u.5) and given by $u_t^i = V^i(c_t^i,u_{t+1}^i)$, $i = 1,2$, $t = 0,1,\ldots$. Pure exchange implies that $c_t^1 + c_t^2 = s > 0$ for all t. Since V^i's are monotonic in their arguments, we can invert V^2 to obtain

$c_t^2 = c(u_t^2,u_{t+1}^2)$ with $c_1 = \partial c(u_t^2,u_{t+1}^2)/\partial u^2 = 1/V_c^2$

and $c_2 = \partial c(u_t^2,u_{t+1}^2)/\partial u_{t+1}^2 = -V_u^2/V_c^2$. Substituting into the first agent's utility, we obtain $u_0^1 = V^1(s - c(u_0^2,u_1^2),u_1^1)$. We would like to maximize the intertemporal utility of one agent subject to a given utility of the other and give a global characterization of the time profile of utilities and consumptions for each agent. This problem has also been studied by Lucas and Stokey (1984) and we seek to generalize their results. Our problem can be set in a dynamic programming framework as follows:

$$u_0^1 = M(u_0^2) = \text{Max } V^1(s - c(u_0^2,u_1^2),M(u_1^2))\ ^4$$

where $0 \le c(u_0^2,u_1^2) \le s$. We say that u_{t+1}^2 is interior if $c(u_t^2,u_{t+1}^2) \in (0,s)$.

Theorem 5.4 (Monotonicity): If u_{t+1}^2 and u_{t+2}^2 are interior and $H(u_0^2,u_1^2) = -V_c^1(s - c(u_0^2,u_1^2),\ M(u_1^2))\cdot c_1(u_0^2,u_1^2)$ is strictly increasing in u_1^2, then along an optimal path $u_t^2 \neq u_{t+1}^2$ implies $(u_{t+1}^2 - u_t^2)(u_{t+2}^2 - u_{t+1}^2) > 0$.

Proof: Noting that $(-V_c^1c_1)$ increasing is the analogue of assumption M for Lemma 3.5, the proof is identical to that of the Lemma and Theorem 3.1. Q.E.D.

In Theorem 5.4 we do not need the twice differentiability of the utility possibility frontier M. In order to interpret the condition that H is strictly increasing, we further assume that V^1,V^2 satisfy (U.6), (U.7) and that $M(u_1^2)$ is differentiable.

Theorem 5.5: Along an interior optimal path, the utilites of agents are monotonic if u_{t+1}^i is a "normal good" in $V^i(c_t^i,u_{t+1}^i)$ for $i = 1,2$.

Proof: We have to show that the "normal good" hypothesis for future utility in the utility functions follows from $(- V_1c_1)$ increasing in u_1^2. We have

$\partial H/\partial u_1^2 = V_{cc}^1 \cdot (\partial c_0^2/\partial u_1^2)(\partial c_0^2/\partial u_1^2) - V_{cu}^1(\partial c_0^2/\partial u_0^2)(dM/du_1^2) - V_c^1(\partial^2 c_0^2/\partial u_0^2 \partial u_1^2)$.

First we evaluate $\partial^2 c_0^2/\partial u_0^2 \partial u_1^2$ using $\partial c_0^2/\partial u_1^2 = - V_u^2/V_c^2$ and $\partial c_0^2/\partial u_0^2 = 1/V_c^2$. A simple calculation yields $\partial^2 c_0^2/\partial u_0^2 \partial u_1^2 = 1/(V_c^2)^3[V_{cc}^2 V_u^2 - V_c^2 V_{cu}^2]$. To calculate dM/du_1^2, maximize $V^1(s - c(u_0^2, u_1^2), M(u_1^2))$ with respect to u_1^2 and solve the first order conditions to obtain $dM/du_1^2 = - V_u^2 V_c^1/V_c^2 V_u^1$. Substituting these terms into $\partial H/\partial V_1^2$ we obtain

$$\partial H/\partial u_1^2 = \{V_u^1 V_c^1[V_{cu}^2 V_c^2 - V_u^2 V_{cc}^2] + V_u^2 V_c^2[V_{cu}^1 V_c^1 - V_{cc}^1 V_u^1]\}/(V_c^2)^3 V_u^1.$$

The bracketed expressions in the numerator are positive if future utility is "normal" for V^1 and V^2. This completes the proof. Q.E.D.

Remark: Note also that if future utility is sufficiently inferior for at least one of the agents, $\partial H/\partial u_1^2$ may be negative and this will imply oscillations in utilities.

Footnotes

1. Multisector models with convex technology and additively separable preferences have been extensively studied. See McKenzie (1976), Brock and Scheinkman (1976), Cass and Shell (1976).

2. If g is the rate of depreciation plus populations growth, we can define $y_t = k_{t+1} - (1-g)k_t$. Then $c = \tau(k_{t+1} - (1-g)k_t, k_t) = T(k_{t+1}, k_t)$. For given k_t, dc/dy represents the slope of the production surface. For further analysis, see Benhabib and Nishimura (1984).

3. Strictly speaking, equations (10) holds only along an optimal path and substituting W′ into the expression for $\partial(V_1T_1)/\partial k_1$ is illegitimate. However, continuity considerations assure that the arguments that follow from this substitution will hold locally, which is all that is required to establish the global monotonicity or oscillation properties of the optimal path. A rigorous proof of the above arguments is given following Theorem (4.2) in section 4.

4. Note that $u_0^1 = M(u_0^2)$ is a utility possibility frontier.

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