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ENDOGENOUS CYCLES WITH UNCERTAIN LIFESPANS IN CONTINUOUS TIME

by

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Introduction

This paper studies the dynamics of a modified version of the heterogeneous agent model that Blanchard (1985) built to study fiscal policy. Blanchard's assumption of exogenous labor income is here replaced by making endogenous both real wages and the supply of labor. Using constant elasticity of substitution utility and production functions, the resulting dynamic system is found to display endogenous real business cycles at a wide range of parameter values when both elasticities of substitution are less than unity. Cycles arise around efficient stationary states with positive real interest rates.

Relation to other cycle models. This model is one of the recent class of equilibrium business cycle models with full microfoundations. As such, unlike earlier ad hoc models of endogenous cycles (e.g., the multiplier/accelerator models of Samuelson <1939> and Hicks <1950>), it can describe the role of agents in the economy, and can in particular explain why rational, optimizing agents do not arbitrage away persistent cycles.

The cycle model developed here is endogenous and nonlinear, which distinguishes it from stable systems with stochastic shocks driving the cyclical dynamics. An early equilibrium model driven by stochastic shocks was developed by Lucas (1975). An inability of agents to distinguish between aggregate, unanticipated policy shocks and industry-specific relative shocks led to hedging through capital accumulation and thus, persistence of aggregate fluctuations. Serial correlation of the shocks was needed to yield cumulation of aggregate movements. Kydland and Prescott (1982) developed a real business cycle model driven by technology shocks. Multi-period lags in capital goods production propagated the shocks over time. Correspondence with actual business cycle data was improved if the shocks were autocorrelated and if current period utility was a function of a distributed lag of leisure (which allowed employment to vary more than productivity). Long and Plosser (1983) studied technology shocks in a multisector model with a single-period production lag. They found that independently distributed shocks are propagated over time if present and future consumption are normal goods, for then favorable shocks are partly spread into increased future production.

Endogenous business cycles have been identified in infinite horizon models. Benhabib and Nishimura (1979) showed that standard optimal growth models in continuous time with heterogeneous capital goods give rise to limit cycles for critical values of the subjective discount rate. For discrete time models Benhabib and Nishimura (1985) showed that cycles can occur even with only one capital good, as long as *inter alia* current and future capital stocks are substitutes in the current period objective function. In these technological cycles, the accumulation of one type of capital asset leads to a shift of the production possibility frontier in favor of other goods if they are more intensive in that type of capital asset, and a Rybczynski effect then leads to decumulation of that asset. Intertemporal arbitrage eliminates the resulting cycles if discounting is too weak.

Benhabib and Day (1982) found that periodic and chaotic trajectories could occur in a perfect foresight overlapping generations (OLG) model when young agents are net borrowers. Grandmont (1985) showed that endogenous cycles can arise (via flip bifurcations) in monetary OLG models where young agents are net savers and have rational forecast functions. Reichlin (1986) found that when a second asset is added to an OLG model, Hopf bifurcations can occur even if saving is a positive function of interest rates, as long as substitution between labor and capital is limited. In that case, a disturbance that increases saving above its steady state level increases both wages and, for a while, saving itself, until saving finally falls because of low interest rates.

Farmer (1985) showed that Hopf bifurcations may occur in an OLG model with a fixed real value of the government budget deficit (net of interest payments) as long as the government is a net holder of capital. Benhabib and Laroque (1986) showed that Hopf bifurcations can occur in OLG economies if the nominal quantity of outside money is negative (again, equivalent to government being a net purchaser of private capital). In other words, cycles occur in these cases when the private sector is a net debtor.

The model studied here is in a sense a hybrid case. As in monetary OLG models, the young have a motive for saving in that their endowment of labor declines over time. However, as in infinite horizon models, there are long-lived individuals who can foresee and attempt to arbitrage the profit opportunities implied by the cyclical path of the economy. As a contribution to the lines of inquiry cited above, this paper indicates that heterogeneity and turnover of the labor force is another posssible source of aggregate fluctuations.

The Blanchard model. Blanchard (1985) studied the role of fiscal policy when a representative agent has a finite but uncertain lifetime. Agents were heterogenous in that their asset holdings differed depending on their ages. However, each individual was assumed to face the same constant probability of death, irrespective of his age. This simplification made the model tractable by facilitating aggregation of individual behavior. An individual modified his subjective discount rate by adding to it the probability of dying, and the steady state interest rate would then lie between this modified and the unmodified discount rate. Shorter expected lifespans led to lower steady state capital. In an open economy version, steady state net foreign assets were positive only if the world interest rate was higher than the unmodified subjective discount rate. The finite horizons of agents implied an absence of Ricardian equivalence, so that deferring taxes by increasing government debt at a point in time generated initial wealth effects on consumption and a decumulation of real capital (or of foreign assets in the open economy version). Steady state consumption and capital (or net foreign assets) were then lower.

Blanchard also investigated several modifications. Using an isoelastic (instead of logarithmic) utility function, he found that a lower elasticity of substitution (higher relative risk aversion) implied a smoother consumption path for an individual, thus less saving early in life and lower steady state capital stocks. He also found that an exogenously declining labor income increased saving for retirement and thus implied higher steady state capital stocks, as well as the possibility of overaccumulation inefficiency. He allowed labor income to cycle exogenously in the open economy model, and found that finite lifespans implied cycles in aggregate consumption even if individuals held constant their own consumption paths. Fiscal policy could then smooth consumption across cohorts by smoothing after tax income.

Aggregate Human Capital

As in Blanchard (1985), individuals are assumed to be born into the labor force at rate p, which is also both the death rate and an individual's constant probability of dying. The population's size is normalized at unity, and its age distribution is stationary. In contrast to Blanchard, saving for retirement is here induced by allowing an individual's current period endowment of labor to decline at rate q. For convenience, agents are assumed to be born with a labor endowment of (p+q)/p. At time v, agents born at time s then have a labor endowment of:
$L(s,v) = [(p+q)/p]e^{q(s-v)}$, and the size of their cohort is the number born, p, times $e^{p(s-v)}$. Summing over all cohorts, the aggregate labor endowment at any time v is
$\int_{-\infty}^{v} L(s,v)pe^{p(s-v)}ds$, equalling unity.

An individual's human capital at time t is defined as the expected discounted present value of his labor endowment: $h(s,t) = E\int_t^\infty w_v L(s,v)e^{-\int_t^v r_\mu d\mu}\,dv$ (with w and r being the real wage and real interest rate). Expectations are assumed to be rational, so that an agent knows the future paths of factor prices and the probability of his being alive at time v-t in the future, which is $e^{-p(v-t)}$. Thus,

$$h(s,t) = \int_t^\infty w_v L(s,v)e^{-\int_t^v (r_\mu+p)d\mu}\,dv.$$

Aggregate human capital is then:

$$H_t = \int_{-\infty}^t h(s,t)pe^{p(s-t)}\,ds = \int_t^\infty w_v e^{-\int_t^v (r_\mu+q+p)d\mu}\,dv.$$ Therefore,

$$\dot{H}_t = -w_t + (r_t+q+p)H_t. \qquad (1)$$

H_t depends positively (negatively) on the sequence of future real wages (real interest rates). However, the time rate of change of H_t is larger for a lower (higher) current wage (interest rate). In other words, aggregate human capital will be higher when the economy is entering a high wage, low interest rate period, and it will be lower on leaving such a period.

Aggregate Capital Stock and Consumption

Individuals maximize expected utility over their lifetimes, and leave unintended bequests because they do not know when they will die. The bequests at any point in time are assumed to be distributed equally among the new born at that instant. This interpretation simplifies the derivation and yields the same aggregate results as Blanchard obtained using a Yaari insurance scheme.

As of time t, the problem of agents born at time s is (defining n(s,v) as labor supplied and β as the subjective discount rate):

$$\max E \int_t^\infty U(c(s,v),n(s,v))e^{-\beta(v-t)}dv = \max \int_t^\infty U(\cdot)\ e^{(\beta+p)(t-v)}dv.$$

The budget constraint is (with $k(s,v)$ equalling the agent's physical capital): $\dot{k}(s,v) = r_v k(s,v) + w_v n(s,v) - c(s,v)$. The transversality condition equates the expected discounted value of consumption of goods and leisure to human plus physical capital:

$$k(s,t) + h(s,t) = \int_t^\infty \{c(s,v) + w_v[L(s,v)-n(s,v)]\}e^{-\int_t^v (r_\mu+p)d\mu}\ dv.$$

Summing the budget constraint over all cohorts results in the definition of new aggregate variables (denoted by upper case letters):[1/] $\dot{K}_t = r_t K_t + w_t N_t - C_t$. (2)

Let X_v = the shadow value of capital at time v. Using a CES utility function of the following form:

$$U(c,n) = [c^a + (L-n)^a]/a,$$

then $E_1 = 1/(1-a)$ is the elasticity of substitution, and two of the Euler conditions are:

$$c(s,v)^{-1/E_1} = X_v \quad \text{and} \quad X_v w_v = [L(s,v)-n(s,v)]^{-1/E_1}.$$

Combining these yields: $L(s,v) - n(s,v) = c(s,v)w_v{}^{-E_1}$. (3a)

Aggregating (3a) gives: $1 - N_t = C_t w_t{}^{-E_1}$. (3b)

The third Euler condition is:

$\dot{X}_v = (\beta-r_v)X_v$, which can be solved as $X_v = X_t e^{\int_t^v (\beta-r_\mu)d\mu}$.

Substituting for X_v and X_t from the first Euler equation:

$$c(s,v) = c(s,t)e^{E_1\int_t^v (r_\mu-\beta)d\mu}.$$

Plugging this expression for $c(s,v)$ into the transversality condition, and using (3a) results in:

$k(s,t) + h(s,t) = c(s,t)Y_t$, (4)

where $Y_t = \int_t^\infty [1+w_v{}^{(1-E_1)}]e^{\int_t^v [(E_1-1)r_\mu-\beta E_1-p]d\mu}\ dv$. Thus, $1/Y_t$ is an individual's propensity to consume out of wealth.

Since Y_t does not depend on the time of birth s, equation (4) may be aggregated, again simply by definition:

$$C_t = (K_t + H_t)/Y_t. \tag{5}$$

Aggregate consumption thus depends only on aggregate wealth, and the economy-wide propensity to consume out of wealth $(1/Y_t)$. Thus,

$$\dot{C}_t = (\dot{K}_t+\dot{H}_t)/Y_t - (C_t/Y_t)\dot{Y}_t, \quad \text{where:}$$

$$\dot{Y}_t = -1 - w_t^{1-E_1} - Y_t[(E_1-1)r_t - E_1\beta - p].$$

Using (1), (2), (3b), and (5), a differential equation in aggregate consumption results:

$$\dot{C}_t = \frac{C_t}{K_t+H_t}[E_1(r_t-\beta)(K_t+H_t)-pK_t+qH_t]. \tag{6}$$

The Euler equations imply that individual consumption grows by a rate equal to $E_1(r-\beta)$. The growth rate of aggregate consumption differs because there is a constant influx of new labor force entrants with labor endowment and human capital 1+q/p times higher than the average.

The Dynamic System

Assuming constant returns to scale, the net production function, F(K,N), equals rK + wN, net of depreciation at rate δ. Dropping time subscripts, the differential equation system is (from (1), (2), and (6)):

$$\dot{K} = F(K,N) - C$$

$$\dot{C} = \frac{C}{K+H}(uK + H)[E_1(r-\beta)+q]$$

$$\dot{H} = (r+q+p)H - w$$

Here $u = [E_1(r-\beta)-p]/[E_1(r-\beta)+q]$ (7)
Using (3b) and $w = \partial F(K,N)/\partial N$, N can be expressed as a function of K and C (and then so can w and r).

Some work is needed to obtain stationary values in terms of exogenous parameters. Assuming positive consumption, the stationary values of H and K obey: $H = w/(r+q+p) = -Ku$. (8)
An economically relevant stationary state thus requires u, given by (7), to be negative. Using (7), K can then be expressed in terms of the endogenous w and r. The stationary value of consumption is given by:
$C = rK+wN$. Also, from (3b), $C = (1-N)w^{E_1}$. Thus,
$N = [1-rKw^{-E_1}]/[1+w^{(1-E_1)}]$. (9)

A CES production function is used, so that:
$F(K,N) = (a_1K^b+a_2N^b)^{1/b} - \delta K$. Profit maximization requires that the marginal product of capital equal the rental cost:
$r+\delta = a_1[a_1+a_2(N/K)^b]^{(1-b)/b}$. Substituting for N/K from (8) and (9):

$$[(r+\delta)/a_1]^{b/(1-b)} = a_1 + a_2\left[\frac{r + w^{E_1-1}(r+q+p)u}{-(w + w^{E_1})}\right]^b$$

Finally, w can be eliminated with the factor price frontier:
$w^b = a_2\{1 - [(r+\delta)^b/a_1]^{1/(b-1)}\}^{b-1}$. This results in a polynomial in only one endogenous variable, r, the order of which depends on the elasticities of substitution in consumption and production.

Computer-assisted iterative techniques were used to solve for stationary values of r given various parameter values. Finding stationary values of the other endogenous variables is then straightforward, and the local dynamics near stationary states can be assessed.

The eigenvalues of the Jacobian (J) of the dynamic system are those of the characteristic equation, $M^3 + z_1M^2 + z_2M + z_3 = 0$, where $z_1 = -(\text{trace } J)$, $z_3 = -|J|$, and z_2 is the sum of the second order principal minors of $|J|$. If the trace and determinant remain positive while the sign of the expression "$z_1z_2-z_3$" changes, then two roots of J pass through the imaginary axis, generically indicating the presence of a Hopf bifurcation (see Benhabib and Nishimura, 1979). Stability of the resulting cycles depends on higher order terms in the Taylor expansion of the dynamic system.

Although the complexity of this model does not permit derivation of necessary conditions for cycles in terms of exogenous parameters, examples of cycles could be found by trial-and-error search.[2] A sample of the results is given in Appendix 1, which indicate that changes in any of the eight exogenous variables can cause cycles. Cycles occur around positive stationary values of the interest rate, implying (given zero population growth) an absence of Phelps-Koopmans inefficiencies.

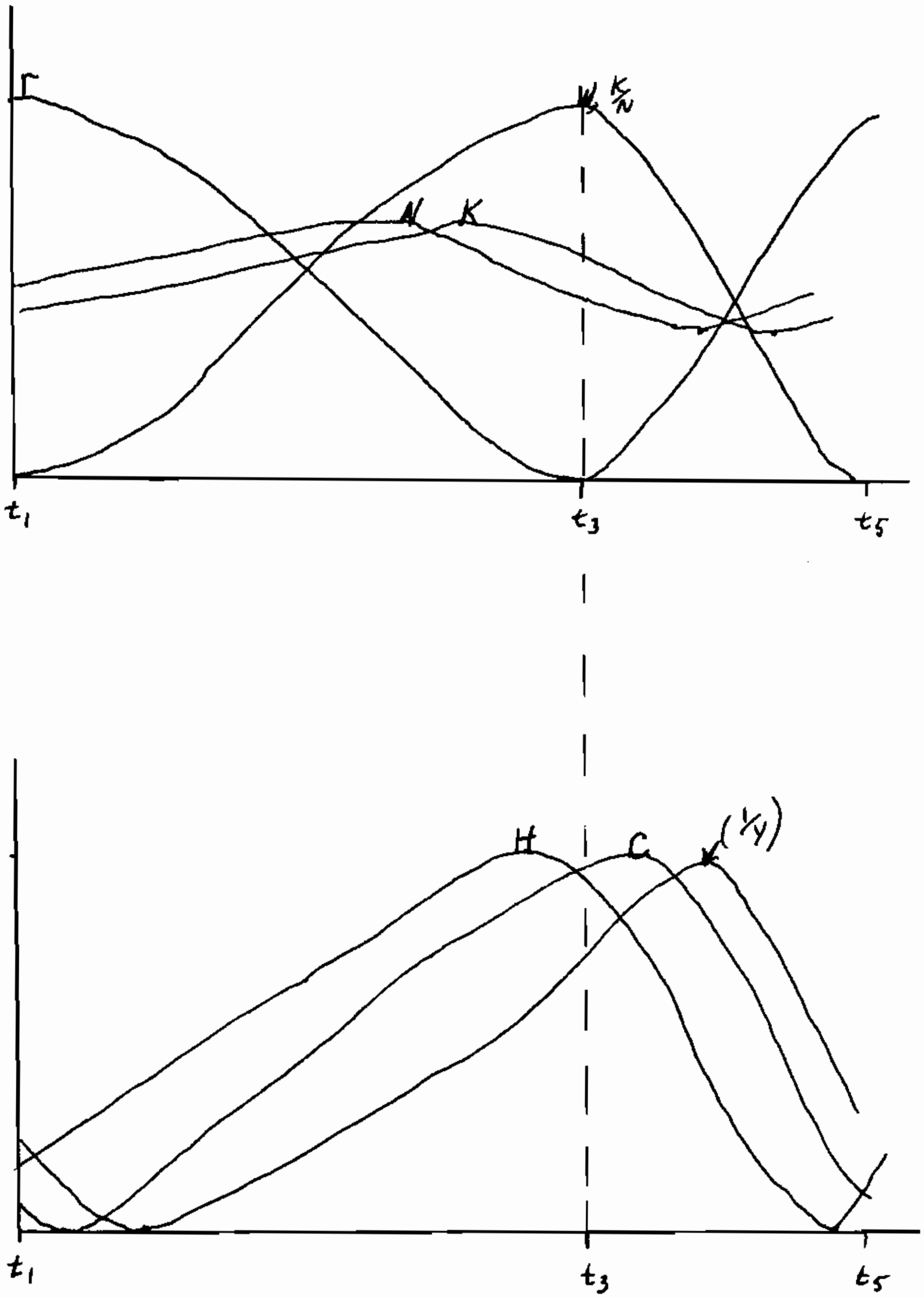
Interpretation

A crucial element of the account of cycles given below is the propensity to consume out of wealth ($1/Y_t$), the expression for which is given following equation (4). If the elasticity of substitution in utility is less than unity, this propensity depends negatively on the future path of wages, and positively on the future path of interest rates (the opposite of the effect of factor price paths on human capital). If the elasticity of substitution equals unity (Cobb-Douglas or log-linear utility), the propensity to consume is a constant, $(\beta+p)/2$, and a Hopf bifurcation cannot occur.[3] In addition, the following account implicitly relies on a sequence of turning points in aggregate variables which in Appendix 2 is shown to hold for the all the examples of cycles given in the text.

Suppose there is a slight increase in saving and investment from its steady state level. Given an elasticity of substitution in production less than unity, the resulting higher capital/labor ratio will cause a magnified increase in the wage/rental ratio (see Figure 1). Since the forthcoming factor price path is foreseen, human capital will rise (and the propensity to consume will fall) in anticipation of a high wage, low interest rate era.

As the effects on labor supply are crucial, equation (3b) is repeated below, using (5): $N = 1 - (K+H)/Yw^{E}1$.
There are three conflicting influences on labor supply:
(i) a positive response to a higher current period wage (*ceteris paribus*), which reflects current period substitution of consumption for leisure,
(ii) a tendency to choose more leisure because of higher wealth, and
(iii) intertemporal substitution effects which are reflected here in the propensity to consume out of wealth, $1/Y$.
In this continuous time model, a current period change in the wage/rental ratio does not effect intertemporal substitution, which depends rather on an integral of all future wages and interest rates. If the elasticity of substitution in utility is less than unity, the propensity to consume will fall initially in anticipation of even higher wages and lower interest rates

FIGURE 1*



* The implied sequence of turning points is shown to hold in Appendix 2. Given an elasticity of substitution in production of less than unity, the factor price ratio varies more than the capital/labor ratio. Other amplitudes and durations of phases of the cycle are suggestive only.

in the near future. In sum, the substitution effects will initially dominate wealth effects. As a result, labor income increases, allowing persistence of the higher level of saving and investment. The capital stock continues to rise.

Before wages reach their peak, the propensity to consume bottoms out and begins to increase in anticipation of reversed movements in factor prices. As a result, wealth and intertemporal substitution effects begin to dominate current period substitution effects on labor supply. Because human capital is also now near its peak, population turnover tends to strengthen the wealth effects and leads to further increases in consumption and leisure demands. This is because new labor market entrants have human capital which is $1+q/p$ times higher than the average. Those who enter the labor force late in an expansion tend to be lower suppliers of labor throughout their lifetimes than those who enter when the economy is in depression. Thus, even though an individual who lives through the cycle may continue to supply more labor relative to trend as wages continue to increase, turnover and heterogeneity of the labor force allow wealth effects to dominate when human capital is high, thus permitting aggregate labor supply to decline despite continuing increases in wages. The population turnover effects are crucial for the cycle. If $p = q = 0$, there is in effect a single infinitely-lived representative agent with a constant labor endowment, and a Hopf bifurcation cannot occur.[4]

As aggregate employment now begins to fall, the capital stock continues growing for a while, but at a slower rate. Increases in wealth and the propensity to consume imply rapidly increasing consumption. Gross saving eventually declines below the high level of depreciation and the capital stock and total wealth slowly begin to fall. Employment declines even faster than capital until finally a peak of the capital/labor and wage/rental ratios occur.

Despite falling wealth, employment continues to decline for a while because of current period and intertemporal substitution effects. The rate of growth in the propensity to consume slows as it approaches its peak (in anticipation of another reversal of factor price movements). Aggregate consumption thus reaches its peak and begins to fall. The downswing is now in full force.

The recovery begins after the propensity to consume peaks and begins to fall. Intertemporal substitution now strengthens growing aggregate wealth effects on labor supply. Eventually they begin to dominate current period substitution effects from the falling wage, and labor supply rises from its floor despite continuing declines in wages. As the propensity to consume and consumption are falling, saving begins to recover and the cycle repeats.

In summary, cycles occur here for the following reasons:

(i) A low elasticity of substitution in production implies that small changes in factor intensity are associated with magnified changes in factor prices, human capital and the propensity to consume out of wealth.

(ii) A low elasticity of substitution in utility implies that intertemporal substitution offsets wealth effects early on in an expansion or contraction phase, but stengthens wealth effects later on in each of those phases, thus reinforcing persistence but eventually helping control explosiveness of cyclical movements.

(iii) Population turnover and declining labor endowments prevent surviving individuals from arbitraging away the cycles in factor prices.

MATHEMATICAL NOTES

1/ Re: Equation (2) of text

To follow more closely the derivation used by Blanchard at this point, the aggregate capital stock is defined as:

$$K_t = \int_{-\infty}^{t} k(s,t)pe^{p(s-t)}ds$$

Differentiating this equation gives:

$$\dot{K}_t = k(t,t) - pK_t + \int_{-\infty}^{t} \dot{k}(s,t)pe^{p(s-t)}ds$$

However, k(t,t) is the bequest received at birth and (in the interpretation used in the text) this equals the fraction of the population dying times the aggregate capital stock (remembering that the total population size is normalized to be unity). Thus the first two terms on the right hand side above cancel each other. Substituting for $\dot{k}(s,t)$ from the budget constraint (here v=t) gives equation (2) of the text. In Blanchard's model, the right hand side of the budget constraint included the term, pk(s,v), as this was the Yaari insurance premium received by survivors. In Blanchard's version, the bequest received at birth, k(t,t), is zero, as the Yaari insurance agency inherits physical capital at death. Thus, the two approaches both yield the same aggregate equation (2).

2/ Re: Explicit expressions used for dynamic analysis

Following equation (7) in the text:

The CRS production function can be used to express N, w, and r as functions of K and C:

$N + Cw^{-E_1} = 1$ from (3b), and $w = F_2(K,N)$.

Using the implicit function theorem, N = N(K,C) where:

$N_2 = dN/dC = [E_1CF_{22}/F_2 - F_2^{E_1}]^{-1} < 0$, and

$N_1 = dN/dK = -E_1CF_{12}N_2/F_2 > 0$.

Then, of course $w = F_2(K,N(K,C))$ and $r = F_1(K,N(K,C))$.

Linearizing the dynamic system around stationary values, the Jacobian is:

$$J = \begin{bmatrix} r+wN_1 & wN_2-1 & 0 \\ J_4 & -wN_1 & J_6 \\ J_7 & J_8 & r+q+p \end{bmatrix} \qquad (10)$$

with $J_4 = [E_1(r-\beta)-p]C/(K+H) + CE_1(F_{11}+N_1F_{12})$

$J_6 = [E_1(r-\beta)+q]C/(K+H)$,

$J_7 = N_1(HF_{12}-F_{22}) + HF_{11}-F_{12}$, and

$J_8 = N_2(HF_{12}-F_{22})$.

The trace and the determinant of the Jacobian are:

$\text{tr } J = 2r + q + p$ and $|J| = A - B$, where

$A = J_6[(wN_2-1)J_7 - (r+wN_1)J_8] > 0$

and $B = (r+q+p)[(wN_2-1)J_4 + wN_1(r+wN_1)] > 0$

the signs of A and B holding for all the cases reported in the text. Denoting the characteristic equation of J by:

$M^3 + z_1M^2 + z_2M + z_3 = 0$, let $Z = z_1z_2-z_3$.

Following the algebra of Appendix 3, since $J_3=0$ here:

$$\begin{aligned} Z &= J_2[J_4(J_1+J_5)+J_6J_7] - (J_5+J_9)[(J_1+J_5)(J_1+J_9)-J_6J_8] \\ &= (wN_2-1)(rJ_4+J_6J_7) + (wN_1-r-q-p)[r(2r+wN_1+q+p)-J_6J_8] \\ &= A + Br/(r+q+p) + (\text{tr } J)[J_6J_8-r(r+q+p)]. \end{aligned}$$

Second derivatives of the CES production function used in calculating the values of $|J|$ and Z are:

$F_{11} = a_1a_2(b-1)N^bK^{-(1+b)}[a_1+a_2(N/K)^b]^{(1/b\ -2)}$

$$F_{22} = a_1a_2(b-1)K^bN^{-(1+b)}[a_1(K/N)^b+a_2]^{(1/b\ -2)}$$

$$F_{12} = -\ F_{11}K/N$$

The computer program used to solve for endogenous and dynamic values, given the exogenous parameters, is attached as Appendix 4.

3/ Re: Log-linear Utility

The following demonstrates that Hopf bifurcations cannot occur in this model for a general constant returns to scale technology if the elasticity of substitution between consumption and leisure is unity. From its definition following equation (4), $Y_t = 2/(\beta+p)$ when $E_1 = 1$. From (5), this implies that consumption is a constant fraction of total wealth. As a result, there are only two (rather than three) state variables in the resulting dynamic system. Using (1), (2) and (3b), the differential equation in aggregate consumption is then:

$$\dot{C}_t = (\dot{K}_t+\dot{H}_t)(\beta+p)/2 = C_t(r_t+q-\beta) - K_t(\beta+p)(q+p)/2,$$

after substituting for Y_t from above and for H_t from (5).

Dropping time subscripts, the new differential equation is:

$$\dot{K} = F(K,N) - C$$

$$\dot{C} = C(r+q-\beta) - K(\beta+p)(q+p)/2$$

The Jacobian of this system evaluated at a steady state is:

$$J = \begin{bmatrix} r+wN_1 & J_2 \\ J_3 & r+q-\beta-wN_1 \end{bmatrix}$$

where:

$J_2 = wN_2 - 1 < 0$, and

$J_3 = C(F_{11}+F_{12}N_1)-(\beta+p)(q+p)/2$

$= CN_2[(F_{11}F_{22}-F_{12}^2)C/w - wF_{11}] -(\beta+p)(q+p)/2$. Thus,

J_3 is also negative, given a concave production function.

For a Hopf bifurcation to occur in a continuous time system, parameter values must be such that a pair of complex conjugate roots cross the imaginary axis. With two state variables, this implies that the Jacobian must have a zero trace and a positive determinant at those parameter values. A zero trace requires that $r = (\beta-q)/2$, which implies that the diagonal terms of the Jacobian have opposite signs. Therefore, given concavity, $|J|$ is negative at such parameter values; the roots cannot be complex and a Hopf bifurcation does not occur.

4/ Re: No Population Turnover

Say $q = p = 0$. Then the differential equation in aggregate consumption reduces to:

$$\dot{C} = CE_1(r-\beta)$$

The stationary interest rate is then equal to the subjective rate of time preference, β. Therefore, the trace of the Jacobian (10) evaluated at a stationary state equals 2β. Then also J_6 above equals 0, and

$$J_4 = CE_1(F_{11}+N_1F_{12})$$

$$= CE_1[CE_1(F_{11}F_{22}-F_{12}^2)-F_{11}w^{1+E_1}]/(CE_1F_{22}-w^{1+E_1}),$$

which is unambiguously negative for a concave production function. Therefore:

$$|J| = -B = -\beta[(wN_2-1)J_4 + wN_1(\beta+wN_1)],$$

which is unambiguously negative.

A positive trace and negative determinant imply one negative and two positive real parts of the roots of the Jacobian. A Hopf bifurcation cannot occur.

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APPENDIX

EXOGENOUS PARAMETERS								STATIONARY VALUES						DYNAMIC VALUES		
q	p	ß	&	E1	E2	a1	a2	K	C	H	N	w	r	tr J	|J|*	Z*
.05	.03	.06	.1	.25	.234	80	340	.429	.058	.241	.87	.04	.05	.210	.200	-.001
.048	.03	.06	.1	.25	.234	80	340	.452	.059	.304	.87	.04	.05	.187	.182	.458
.05	.032	.06	.1	.25	.234	80	340	.489	.061	.416	.87	.05	.04	.156	.212	.902
.05	.03	.062	.1	.25	.234	80	340	.451	.059	.298	.87	.04	.06	.189	.204	.451
.05	.03	.06	.11	.25	.234	80	340	.478	.056	.401	.88	.05	.03	.148	.218	.853
.05	.03	.06	.1	.23	.234	80	340	.441	.059	.261	.87	.04	.06	.202	.292	.173
.05	.03	.06	.1	.25	.2342	80	340	.433	.058	.251	.87	.04	.06	.205	.207	.104
.05	.03	.06	.1	.25	.234	81	340	.438	.058	.258	.87	.04	.06	.202	.208	.164
.05	.03	.06	.1	.25	.234	80	341	.429	.058	.241	.87	.04	.07	.210	.206	.001
.04	.02	.05	.1	.30	.25	100	100	.496	.051	.161	.84	.02	.07	.195	.008	-.937
.04	.02	.05	.1	.30	.25	110	100	.790	.059	1.03	.87	.07	.01	.067	.036	.328
.04	.02	.07	.1	.07	.67	1.7	1	.464	.032	.299	.96	.02	.02	.100	.038	-.001
.04	.02	.07	.1	.1	.67	1.7	1	.592	.034	.462	.95	.03	.01	.074	.025	.071
.04	.02	.01	.1	.1	.2	1000	10	1.85	.127	.758	.84	.07	.04	.131	.302	-.049
.04	.02	.01	.1	.1	.2	1000	100	1.14	.077	.476	.90	.04	.03	.126	.281	.044
.04	.02	.10	.1	.01	.27	50	100	.501	.055	.259	.94	.03	.05	.168	.531	-.288
.04	.02	.10	.1	.01	.30	50	100	.457	.030	.242	.97	.02	.02	.105	.200	.094

NOTATION

1/q (1/p) = average expected working life (lifespan) of those already working, ß = subjective discount factor, E1 (E2) = elasticity of substitution between consumption and leisure (capital and labor). The production function:

$F(K,N) = (a_1K^b + a_2N^b)^{1/b} - \&K$, where E2 = 1/(1-b), & = depreciation rate, N = employment,

K = physical capital. C = consumption, H = human capital, w = real wage, r = real interest rate. $Z = z_1z_2-z_3$ in text, the statistic for Routh-Hurwitz theorem: tr J > 0, |J| > 0 and

Z >(<) 0 imply that the signs of the real parts of the roots of J are +-- (+++).

* times 1000.

Appendix 2

The Sequence of Turning Points

For the cases reported, the signs of the Jacobian are:

	∂K	∂C	∂H
$\partial \dot{K}$	+	-	0
$\partial \dot{C}$	-	-	+
$\partial \dot{H}$	-	-	+

It follows that cycle trajectories occur in the following way: In a boom in which K, C, and H are all increasing, K must change direction first. If instead, C changed, K would increase monotonically (and if H began to fall, C would also have to fall). With K falling, C cannot begin to fall until H does. Once all three state variables are falling, again K must change direction first, followed by H, then C.

For the CES production function used here, the slope of the factor price frontier is:

$$dw/d(r+\&) = - K/N = - \left[\frac{a_1 w}{a_2(r+\&)}\right]^{E_2}$$

Thus, wages and interest rates move inversely over a cycle, and capital intensity moves in the same direction as wages, and has the same turning points.

The turning points of aggregate variables are graphed in Figure 1 of the text. Aggregate human capital is just the discounted present value of future wages. Thus, human capital is highest when near-term interest rates are nearing their trough and near-term wages are nearing their peak, somewhere before ~~between points t_2 and~~ t_3 on the graph. Aggregate physical capital must peak while human capital is increasing, but before the peak of human capital. Employment peaks roughly when the ratio of consumption to wages is lowest. Therefore, if consumption peaked before t_3, the trough of employment would occur before the peak of consumption. In that case, capital intensity would peak before t_3, contradicting the requirements for a cycle. For a cycle to exist, therefore, aggregate consumption should peak <u>after</u> the peak of wages. Then the trough of employment would occur after the peak of consumption (thus after t_3). A peak of the capital/labor ratio exactly at t_3 is then possible.

<u>Individual behavior over the cycle</u>. The growth rate of individual consumption is $E_1(r-\beta)$. Individual consumption thus grows (declines) if the interest rate is larger (smaller) than the subjective discount rate. Consumption of an individual who survives over the cycle may be monotonic, but if β is within the range of interest rates over the cycle, an individual's consumption will peak <u>before</u> t_3, when r reaches its trough.

While individual labor supply trends downward due to the influence of a declining labor endowment, the resulting downward trend in consumption is somewhat offset by a downward trend in saving. Relative to its trend, individual labor supply reaches a trough shortly before the relative peak in individual consumption, as wages are then increasing.

Appendix 3

An Application of Routh Theorem with Three State Variables

$$J = \begin{bmatrix} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \\ J_7 & J_8 & J_9 \end{bmatrix}$$

$\text{tr } J = J_1 + J_5 + J_9$

$|J| = J_1(J_5J_9-J_6J_8)-J_2(J_4J_9-J_6J_7)+J_3(J_4J_8-J_5J_7)$

The characteristic equation of J is given by:

$M^3+z_1M^2+z_2M+z_3 = 0$, where

$z_1 = -\text{ tr } J$

$z_3 = -\ |J|$

$z_2 = J_5J_9-J_6J_8-J_2J_4-J_3J_7+J_1(J_5+J_9)$

Then let $Z = z_1z_2-z_3$.

$Z = J_2[J_4(J_1+J_5)+J_6J_7]+J_3[J_7(J_1+J_9)+J_4J_8] - (J_5+J_9)[(J_1+J_5)(J_1+J_9)-J_6J_8]$

By the Routh-Hurwitz theorem, for the eigenvalues of J to have all negative real parts: z_1, z_3, and Z must all be positive (see, e.g., G. Gandolfo, Economic Dynamics, North Holland). Let the characteristic equation of -J be given by $|-J-QI|=0$. Because it is a three-by-three matrix, $|-J-QI| = -|J+QI|$. Therefore $Q = -M$. The characteristic equation of -J is in full:

$Q^3+x_1Q^2+x_2Q+x_3 = 0$. It turns out that:

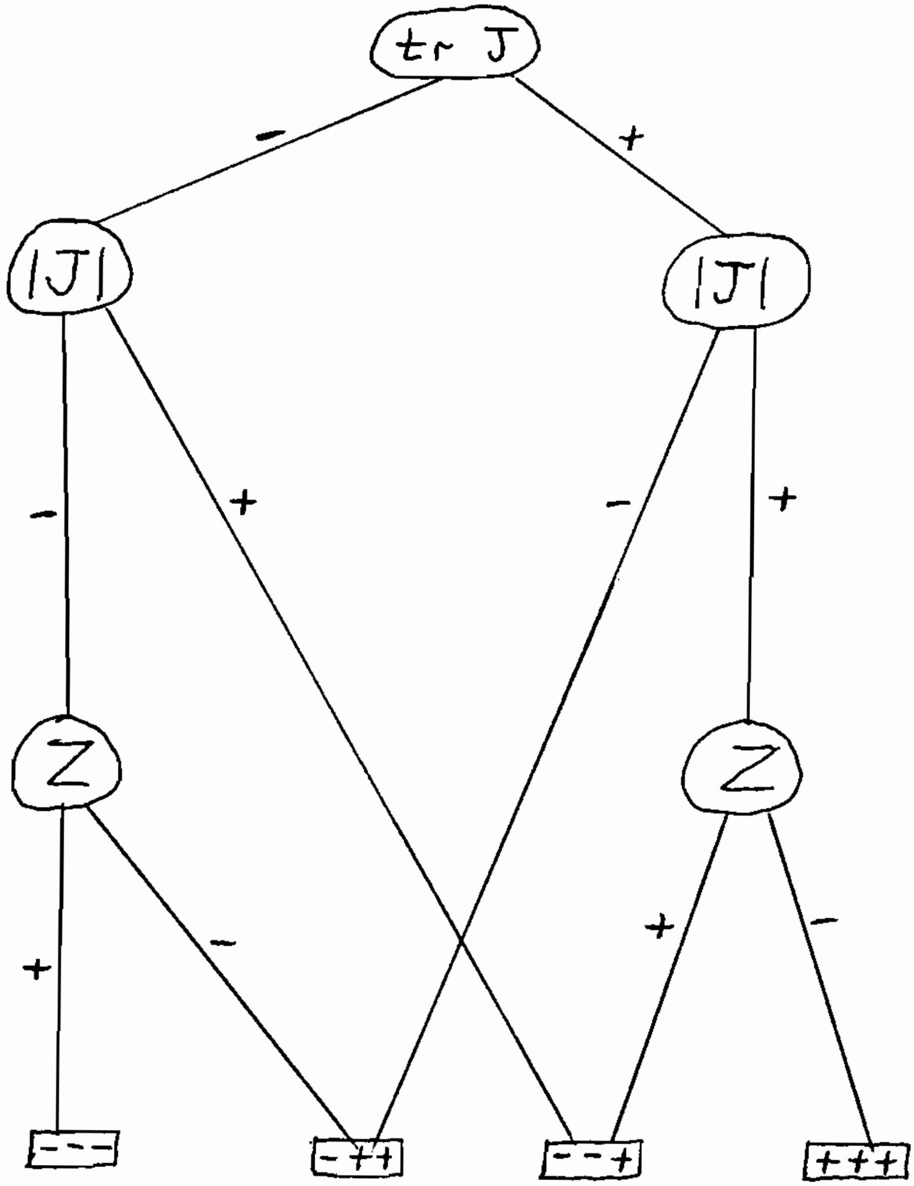
$x_1 = \text{tr } J = -\ z_1$

$x_3 = |J| = -\ z_3$

$x_2 = z_2$

Therefore the sign of $z_1z_2-z_3$ is opposite the sign of $x_1x_2-x_3$. Thus, for the eigenvalues of J to have all positive real parts, z_1, z_3 and $Z = z_1z_2-z_3$ must all be negative.

The results are charted below, where the branches of the tree diagram give the signs of the variables appearing at the nodes above them, and the signs of the roots of J are given in the boxes at the bottom:



Appendix 4

PROGRAM WRITTEN IN BASIC

```
1 REM "BR2"
2 F$= "#####.###":G$= "#####.##":H$= "###.###":E$="###.######"
5 POKE 16427,72    ' PAPER WIDTH
7 JL=0:ZL=0:RL=0:R=0:SS=0:DI=0:KL=0:CL=0:HL=0:LK=0:LC=0:LH=0
2000 REM ERROR MESSAGES
2030 PRINT "ERROR IN LINE";ERL
2040 PRINT "I5,I4,I3,I2,I1",I5;I4;I3;I2;I1
2045 PRINT "A1= ";A1
2050 RESUME 3010

10 DATA .05,.03, 80,340.7215,.06,.1,.25,-3.275
20 READ B,P,A1,A2,Q,D,U,E
40 GOTO 3007

100 REM SCREENWORK
110 GOSUB 2500:GOSUB 1700
120 PRINT "      B        P        Q         D       U        E         L"
:PRINT USING F$; B;P;Q;D;U;E;L
122 GNP = R*K+W*L + D*K:GI = GNP - C
125 PRINT  "      Z         J       R        W        K       A1      A2"
:PRINT USING F$; Z;J;R;W;K;A1;A2
127 PRINT "     GNP     GROSS I    C        N":PRINT USING F$; GNP,GI,C,L

130 T=0:INPUT "1 FOR LPRINT";T:IF T=1 THEN GOSUB 1000
150 INPUT "B P E";B,P,E:INPUT "Q D U";Q,D,U:INPUT "A1 A2";A1,A2
170 GOTO 110

1000 REM LPRINT RESULTS
1005 E2= 1/(1-E)
1010 LPRINT "EXOGENOUS PARAMETERS"
1020 LPRINT "      q       p     a1      a2       ";CHR$(254);"       &
    E1      E2"
1030 LPRINT USING G$; B;P;A1;A2;Q;D;U;E2
1040 LPRINT "STEADY STATE VALUES"
1050 LPRINT "      r        w        N        H        K        C"
1055 LPRINT USING F$; R;W;L;H;K;C
1070 LPRINT "CRITICAL VALUES FOR DYNAMICS"
1075 TJ= 2*R + B + P
1080 LPRINT "tr J =";:LPRINT USING H$; TJ,:LPRINT "    1000*/J/ = ";:LPRIN
T USING H$; J;:LPRINT "    1000*Z = ";:LPRINT USING E$; Z:LPRINT " "
1090 RETURN
```

```
1700 REM CALCULATE Z,J AND SS VALUES
1710 V= U*(R-Q)+B:ZZ= U*(R-Q)-P
1720 W= (A2[(1/E))*(1 - (A1[(1/(1-E)))*(R+D)[EE)[(1/EE)
1730 K= -V*W/(ZZ*(R+B+P))
1740 L= (W[U - R*K)/(W + W[U)
1745 IF L<0 THEN RETURN
1750 H= W/(R+B+P)
1770 C= R*K + W*L
1780 G= (L/K)[E
1790 G1= A1*A2*G*(E-1)*((A1+A2*G)[((1/E)-2))/K  ' G1= F11, WHERE F IS NET
PROD. FN
1800 G2= A1*A2*(E-1)*((A2+A1/G)[((1/E)-2))/(L*G) ' G2= F22
1810 G3= - K*G1/L   ' G3 = F12
1820 L2= 1/(U*C*G2/W -W[U)
1830 L1= -U*C*G3*L2/W
1835 J6= C*V/(K+H)
1840 J4= C*ZZ/(K+H) + U*C*(G1+L1*G3)
1850 J7= L1*(H*G3-G2) + H*G1-G3
1870 J8= L2*(H*G3-G2)
1890 J= -(R+W*L1)*(W*L1*(R+B+P)+J8*J6) + (1-W*L2)*(J4*(R+B+P)-J6*J7)
1900 Z= (W*L2-1)*(R*J4+J6*J7)+(W*L1-R-B-P)*(R*(2*R+W*L1+B+P)-J6*J8)
1910 J=1000*J:Z=1000*Z
1920 RETURN


2500 REM ITERATION TECHNIQUE FOR R
2503 EE= E/(E-1)
2505 X=1:UB= P/U +Q:IF UB>(A1[(1/E)-D) THEN UB= A1[(1/E)-D-.0000001
2510 LB= Q-B/U:IF LB<(-B-P) THEN LB=-B-P
2511 IF LB<(-D) THEN LB=-D
2512 HD=(UB-LB)/2:GOTO 2520
2515 PRINT "UB= ";UB,"LB= ";LB,"HD= ";HD
2520 F=.1:R= LB+HD:GOSUB 2700:TL=TT
2530 IF SGN(TT)<>SGN(TL) THEN F=.1*F
2535 TL=TT
2540 R= R -X*SGN(TT)*F*HD: GOSUB 2700
2550 IF F< .00001 THEN 2590
2555 IF R<LB OR R>UB THEN 2575
2570 GOTO 2530
2575 IF X=-1 THEN 2585
2580  X=-1:GOTO 2520
2585 PRINT "NO SOLUTION":R=0
2590 RETURN
2700 REM TRY R
2710 W= (A2[(1/E))*(1 - (A1[(1/(1-E)))*(R+D)[EE)[(1/EE)
2720 WS= W[(1-U) + 1:WN= W[(-U)
2730 G= (U*(R-Q)-P)*(R+B+P)/(W*(U*(R-Q)+B))
2740 TT= ((R+D)/A1)[(-EE) -A1 - A2*((G+R*WN)/(-WS))[E
2750 RETURN
```

```
3000 REM RUN DYNAMICS
3005 INPUT "B,P,A1,A2,Q,D,U,E";B,P,A1,A2,Q,D,U,E
3007 GOSUB 2500:GOSUB 1700:GOSUB 1000
3008 POKE 16916,0:CLS:POKE 16916,2:PRINT "K,C,H",K;C;H:PRINT "Z= ";Z
3010 INPUT "K,C,H"; K,C,H
3012 XX=0:INPUT "1 FOR EXIT";XX:IF XX=1 THEN 3005
3015 LPRINT "      K         C         H     EMPLOYM     REAL w     REAL r
   GNP"
3018 I=0
3020 GOSUB 3800
3025 PRINT "K,C,H",K,C,H
3030 G = A1*K[E + A2*L[E
3035 F = G[(1/E) - D*K
3040 DK = F - C
3050 R = A1*(A1 + A2*(L/K)[E)[((1-E)/E) - D
3070 V = U*(R-Q)+B:VV= V-B-P
3080 DC = (V*H + VV*K)*C/(K+H)
3085 GRI= DK + D*K
3090 DH = H*(R+B+P) - W
3095 GNP = R*K + W*L + D*K:IF I=15 THEN GOTO 3100
3096 IF I=0 THEN GOTO 3100
3097 IF SGN(DC)=SGN(LC) AND SGN(DK)=SGN(LK) AND SGN(DH)=SGN(LH) THEN 3105
3100 LPRINT USING E$; K;C;H;L;W;R;GNP
3102 KK=K+DK:CC=C+DC:HH=H+DH
3103 LPRINT USING E$; KK;CC;HH
3105 LC=DC:LK=DK:LH=DH:KL=K:CL=C:HL=H
3110 K=K+DK:H=H+DH:C=C+DC
3112 XX=0:IF I=5 THEN INPUT "1 TO EXIT";XX
3113 IF XX=1 THEN 3010
3115 I=I+1:IF I<15 THEN GOTO 3020
3120 GOTO 3010


3800 REM ITERATE FOR L
3805 X=1
3810 F=.01:L=.5:GOSUB 3970
3820 TL=TT
3830 L = L - X*SGN(TT)*F
3835 GOSUB 3970
3840 IF SGN(TT)<>SGN(TL) THEN F= .1*F
3845 IF L >=1 OR L<=0 THEN 3900
3850 IF F> .000001 THEN 3820
3879 PRINT "L= ";L,"TT= ";TT
3880 RETURN
3900 IF X>0 THEN 3930
3910 PRINT "NO SOLUTION FOR L"
3920 RETURN
3930 X=-1:GOTO 3810
3970 REM TEST L
3975 W = A2*(A2 + A1*(K/L)[E)[((1-E)/E)
3980 TT = L + C*W[(-U) -1
3985 RETURN
```