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INTRODUCTION

Recently, Barro and Becker [1985] produced a model of economic growth where economic agents derive utility from their children's well–being and choose not only how much to save and consume during their life–cycle but also the number of children they would like to have. Thus parents have to make a choice between their consumption and their bequest of capital which provides for the well–being of their children. Also, in choosing the number of children, parents must consider the cost of raising them. The accumulation of capital therefore is closely related with the number of children which, just like consumption, provide utility to their parents through their own well–being. The additional element of choice, fertility, provides the simple intertemporal accumulation model with a richer structure and many of the standard comparative static results, such as the relation between consumption growth and the interest rate, have to be modified to account for the inclusion of fertility into the choice problem of the parents.

One of the recent interesting hypotheses of population growth is due to Easterlin (1973) (see also Becker (1981) chapter 7) who suggests the possibility of self–generating fluctuations in population growth. A large population will face stiffer economic competition, lower incomes, congestions and crowding if other means of production as well as the social infrastructure do not expand simultaneously. The result may be a decline in fertility as parents try to maintain an adequate standard of living for themselves. But why should capital and other means of production or the social infrastructure not expand with population size at a uniform rate? Are fluctuations a necessary or even possible outcome of this analysis? Using the Barro–Becker framework and relaxing some of their assumptions, we will attempt to answer this question. Our results show that under a broad class of preferences, fertility and per capita incomes not only move together but endogenously oscillate. (For another interesting analysis of these issues, see Kemp and Kondo (1985).)

In section 2 below we present the model and give our results in Theorem 1 and 2. In section 3 we consider some examples to illustrate our results. We look at a class of functions which reduce to the case considered by Barro and Becker for particular parameter configurations but which generally illustrates the oscillatory and monotonic patterns for population growth and capital accumulation.

2. The Model and Theorems

Barro and Becker consider a model where parents derive utility from their own consumption as well as from the utility of their children. Starting with a given stock of capital k_0, they maximize their utility, assuming that their children will do the same:

(1)
$$V(k_0) = \max_{c_0, n_0, k_1} \quad U(c_0) + \sum_{t=1}^{\infty} \prod_{w=0}^{t} \alpha(n_w) n_w U(c_t)$$

S.T.

$$(1-g)k_0 + F(k_0) = c_0 + n_0(k_1 + B_0)$$

where c_t is the consumption of generation t, n_t is one plus the endogenous population growth rate, B is the cost of raising children and is constant over time, k_t is the per capita stock, $F(k)$ is the production function, $U(c_0)$ is the utility derived from consumption and g is the depreciation rate. $\alpha(n)$ can be taken as a parameter of altruism towards children.[1,2] Barro and Becker assume $\alpha(n)\cdot n$ to be concave. In dynamic programming form we can write the problem as

(2) $$V(k_0) = \max_{c_0} [U(c_0) + a(n_0)V(k_1)]$$

S.T.

$$f(k_0) = c_0 + n_0(k_1 + B_0)$$

where $\alpha(n_0)\cdot n_0 = a(n_0)$ and $f(k) = F(K) + (1-g)k$. Here $a(n_0)$ is increasing and concave, which reflects that the utility of the parents is increasing at a diminishing rate with the number of children, for a given level of well−being $V(k_1)$ per child. We assume that there is a maximum sustainable level of capital stock $\bar{k}$ such that $f(k) < k$ for all $k > \bar{k}$. Substituting the budget constraint into the problem, we obtain[3]

(3) $$V(k_0) = \max_{k_1, n_0} [U(f(k_0) - n_0(k_1+B)) + a(n_0)V(k_1)]$$

The above problem, after choosing n_0 optimally as a function of (k_0,k_1) can also be written as

(4) $$V(k_0) = \max_{k_1} W(k_0, k_1, n(k_0, k_1))$$

Define $\bar{W}(k_0,k_1) \equiv W(k_0,k_1,n(k_0,k_1))$ and $\bar{W}_1 = \partial\bar{W}/\partial k_0$. Let k^* be a steady state satisfying $V(k^*) = \bar{W}(k^*,k^*) = \max_{k_1} \bar{W}(k^*,k_1)$. Let E be the set of steady states. We will use the following lemma to prove our main result.

LEMMA 1: (i) If $W_1(k_0,k_1)$ is strictly increasing in k_1, then an optimal path $\{k_t\}$ from any $k_0 > 0$, $k_0 \notin E$, is strictly monotone, i.e., $(\hat{k}_{t-1}-\hat{k}_t)(\hat{k}_t-\hat{k}_{t-1}) > 0$.

(ii) If $W_1(k_0,k_1)$ is independent of k_1, then the capital stock jumps to its steady state value in one period.e., $\hat{k}_2 = \hat{k}^*$.

(iii) If $W_1(k_0,k_1)$ is strictly decreasing in k_1, then an optimal path from any $k_0 > 0$, $k_0 \notin E$, fluctuates, i.e., $(\hat{k}_{t-1}-\hat{k}_t)(\hat{k}_t-\hat{k}_{t-1}) < 0$.

Note that at a steady state in Lemma 1 above we may have $k^* > 0$ or $k^* = 0$. Note also that we do not need to use the differentiability of the value function $V(k)$ to prove this lemma. This lemma may be rigorously proved in the manner given in Benhabib, Majumdar and Nishimura (1985). A short proof using the differentiability of $V(k_1)$ is given in the Appendix of the present paper. In the rest of the paper, we shall give the interpretation of the conditions imposed on $W_1(k_0,k_1)$. To do so, we assume the differentiability of $V(k_1)$.

We can now apply the above Lemma 1 to our problem given by (1). Let e be the elasticity of (a/a') with respect to n; that is,

(5) $$e \equiv (na'/a)\cdot d(a/a')/dn, \quad \text{where} \quad a' = da(n)/dn$$

THEOREM 1: If $e < 1$ (> 1), the capital stock oscillates (is monotonic). If $e = 1$ (the Barro–Becker case), the capital stock jumps to its steady state value in the first period.

Proof: The theorem follows from Lemma 1 if we can establish that the sign of W_{12} is the same as that of $e-1$. We set

$$W(k_0,k_1,n_0) = U(f(k_0) - n(k_1+B)) + a(n_0)V(k_1). \tag{6}$$

Maximizing $W(k_0,k_1,n_0)$ with respect to n and k_1 yields

$$W_n = -(k_1 + B)U'(c_0) + a'(n_0)V(k_1) = 0 \tag{7}$$

and

$$W_{k_1} = -nU'((c_0) + a(n_0)V'(k_1) = 0. \tag{8}$$

Using (7), we can obtain the optimal value of n_0 as $n(k_0,k_1)$ with the derivatives

$$\frac{dn}{dk_0} = \frac{(k_1+B)U''(c_0)f'(k_0)}{a''(n_0)V(k_1) + (k_1+B)^2 U''(c_0)} > 0 \tag{9}$$

$$\frac{dn}{dk_1} = \frac{-U'(c_0)-a'(n_0)V'(k_1) - (k_1+B)n_0U''(c_0)}{a''(n_0)V(k_1) + (k_1+B)^2 U''(c_0)} \tag{10}$$

Using (6) and (7), we can evaluate $\tilde{W}_{12}$ as follows:

(11) $$\overline{W}_{12} = W_{k_0k_1} + W_{k_0n_0}\left(\frac{\partial n_0}{\partial k_1}\right) + W_{n_0k_1}\left(\frac{\partial n_0}{\partial k_0}\right) + W_{n_0n_0}\left(\frac{\partial n_0}{\partial k_1}\right)\cdot\left(\frac{\partial n_0}{\partial k_0}\right)$$

where $W_{k_0k_1} = -n_0 f'(k_0)U''(c_0)$, $W_{k_0n_0} = -(k_1+B)^2\, U''(c_0)[f'(k_0)+1]$,

$W_{n_0n_0} = (k_1 + B)^2\, U''(c_0) + a''(n_0)V(k_1)$, $W_{n_0k_1} = -U'(c_0) + n_0(k_1+B)U''(c_0) + a'(n_0)V'(k_1)$.

Substituting into (11) and cancelling, we obtain

$$\overline{W}_{12} = \frac{f'(k_0)U''(c_0)}{a''(n_0)V(k_1)+(k_1+B)^2\, U''(c_0)}[(k_1+B)(a'(n_0)V'(k_1) - U'(c_0))-n_0a''(n_0)V(k_1)]$$

Solving for $V(k_1)$ and $V'(k_1)$ from (7) and (8) and substituting, we obtain

$$\overline{W}_{12} = \left[\frac{f'(k_0)U''(c_0)}{a''(n_0)V(k_1)+(k_1+B)^2\, V''(c_0)}\right]\left[(k_1+B)\left(\frac{n_0a'(n)U'(c)}{a(n_0)} -U'(c)\right) - \frac{n_0a''(n_0)(k_1+B)U'(c)}{a'(n)}\right]$$

(12) $$= \left[\frac{f'(k_0)U''(c_0)(k_1+B)U'}{a''(n_0)V(k_1)+(k_1+B)^2\, V''(c_0)}\right]\left[\frac{n_0a'(n_0)}{a(n_0)} - 1 - \frac{n_0a''(n_0)}{a'}\right]$$

The first square bracket on the right is positive by concavity. The second can be further simplified so that it equals

(13) $$\left[\left(\frac{n_0a'}{a}\right)\frac{(a'(n_0))^2 - a(n_0)a''(n_0)}{(a'(n_0))^2} - 1\right] = \left(\frac{n_0a'}{a}\right)\frac{d(a/a')}{dn} - 1 = e-1$$

Therefore the sign of W_{12} is the same as that of $e-1$. **Q.E.D.**

Theorem 1 gives conditions under which the capital stock is oscillatory or monotonic. We now turn to the analysis of how the fertility rate n changes with the capital stock. Theorem 2 below gives a result for the oscillatory case:

THEOREM 2: If $e = \left(\frac{n_0 a'}{a}\right)\frac{d(a/a')}{dn_0} < 1$, the fertility rate n oscillates in phase with the per capital stock k.

Proof: We have $dn_0/dk_0 = \partial n_0/\partial k_0 + (\partial n_0/\partial k_1)dk_1/dk_0$. From the proof of Lemma 1, we know that $e < 1$ implies $dk_1/dk_0 < 0$. Also from (9) in the proof of Theorem 1 we have $\partial n_0/\partial k_0 > 0$. From (8) and (10) in the proof of Theorem 1 we can compute how the optimal value of n_0 changes with k_1:

$$\frac{\partial n_0}{\partial k_1} = \frac{U'(c_0)\left(1-\frac{n_0 a'}{a}\right) - (k_1+B)n_0 U''(c_0)}{a''(n_0)V(k_1) + (k_1+B)^2\, U''(c_0)} \tag{14}$$

However, we also have

$$1 - \frac{n_0 a'}{a} \geq 1 - \frac{n_0[(a')^2 - aa'']}{aa'} \tag{15}$$

$$= 1 - e > 0.$$

Thus, under our concavity assumptions $\partial n_0/\partial k_1 < 0$ and $dn_0/dk_0 > 0$. Since under $e < 1$ the capital stock oscillates, so does n. **Q.E.D.**

Theorem 2 therefore lends support to the hypothesis that under some reasonable conditions on preferences, the fertility rate n_0 will tend to be high (low) when the per capita stock k_0 and per capita income $f(k_0)$ are high (low).

REMARK 1: It should be noted that in the oscillatory case the optimal trajectory can converge to the steady state or to a period–two cycle. These possibilities may be studied by the formal methods presented in Benhabib and Nishimura (1985). If $e-1$ changes sign, the dynamic behavior of trajectories can become more complicated and even chaotic.

3. Examples

In this section we give several examples which illustrate the monotonic and oscillatory cases discussed above. We also show that the special case considered by Barro and Becker, who use a constant relative risk aversion function for $a(n)$, corresponds to a parameter configuration on the borderline of the monotonic and oscillatory behavior in the class of Hyperbolic Absolute Risk Aversion (HARA) functions.

The general example which contains the monotonic, oscillatory as well as the Barro–Becker case is illustrated by a HARA function for $a(n)$ given by $\delta(n+Z)^A$. In this case, $e = n/(n+Z)$. Therefore if $Z > 0$ $(e < 1)$, the capital stock oscillates (Theorem 1) and n and k move in phase with each other (Theorem 2). Since k is bounded by $\bar{k}$ and $f(\bar{k}) \leq c + n(\bar{k}+B)$, we have $n \leq f(\bar{k})/(B+\bar{k}) = \bar{n}$.

Since $a = \delta(n+Z)^A$, we can choose δ (for any given A and Z) such that $a = \delta(\bar{n}+Z)^A < 1$. Therefore the utility sums in (1) will converge.

Note that the case $Z = 0$ corresponds exactly to the Barro–Becker case with $e = 1$. The monotonic case with $e > 1$ will correspond to $Z \leq 0$, with $Z \geq -n$ along the optimal path.

Figure 1 below illustrates the relation between the initial stock $\hat{k}_0$ and the optimal choice $\hat{k}_1$, where $\hat{k}_1 = h(\hat{k}_0)$ and $h(k)$ is the policy function for the monotonic case. Intersections of $h(\hat{k}_0)$ with the 45° line are steady states. Note that we may have multiple steady states, with stable ones alternating with unstable ones. Figure 2 illustrates the Barro–Becker case and Figure 3 the oscillating case which converges to the steady state. If the steady state becomes unstable, the trajectories may converge to a periodic cycle and can also become chaotic.

REMARK 2: Differentiating (2) along an optimal path, we have $V'(k_t) = V'(c_t)f'(k_t)$. Using (7), we obtain

$$n_t = \frac{a(n_t)V'(c_{t+1})}{V'(c_t)} f'(k_{t+1})$$

At a steady state, this becomes

$$a(n^*) = \frac{n^*}{f'(k^*)}$$

Thus if the steady state is efficient in the usual sense, that is, if $f'(k^*) > n^*$, then there is positive discounting of the future. On the other hand, at the steady state V

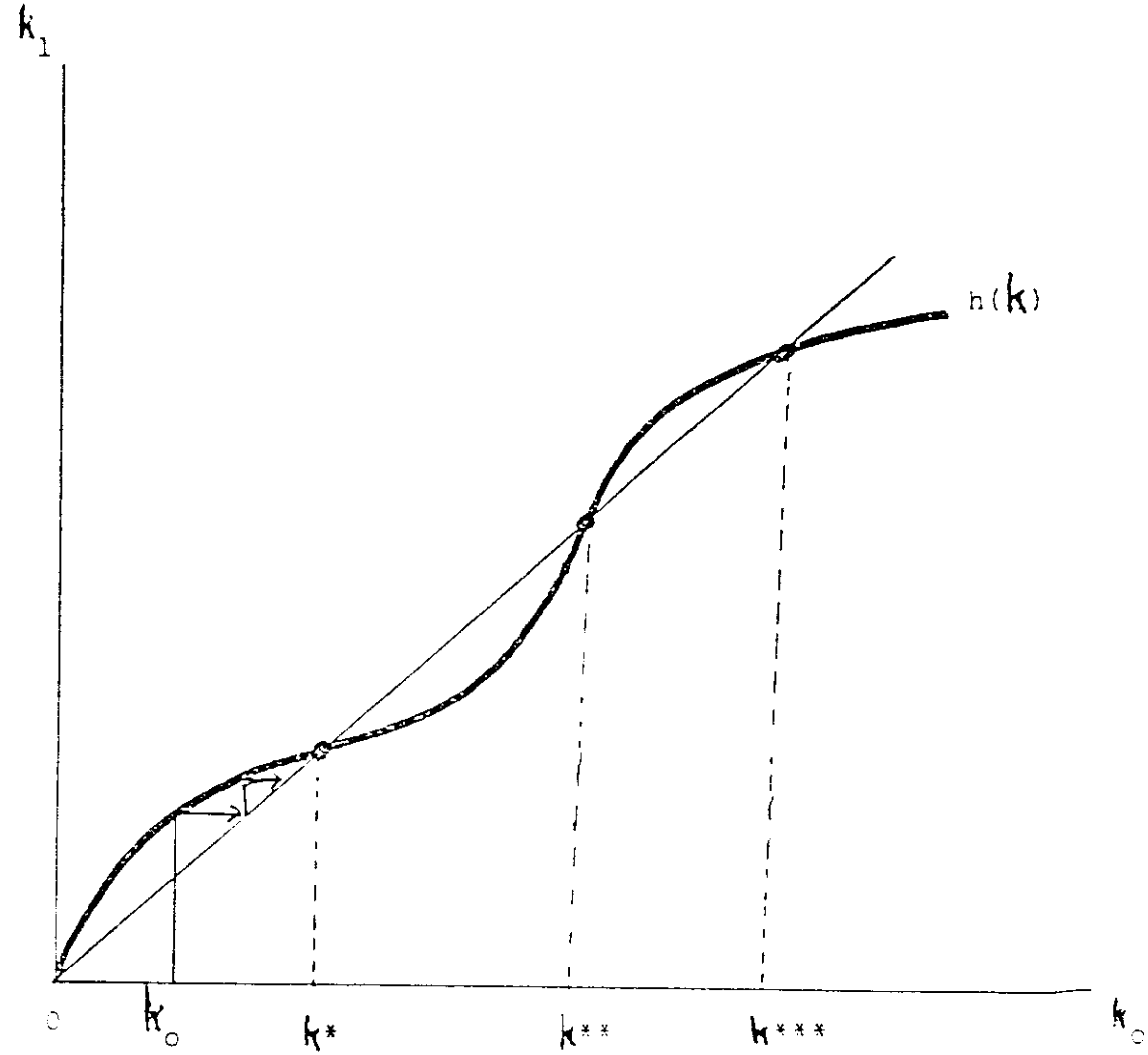

Figure 1

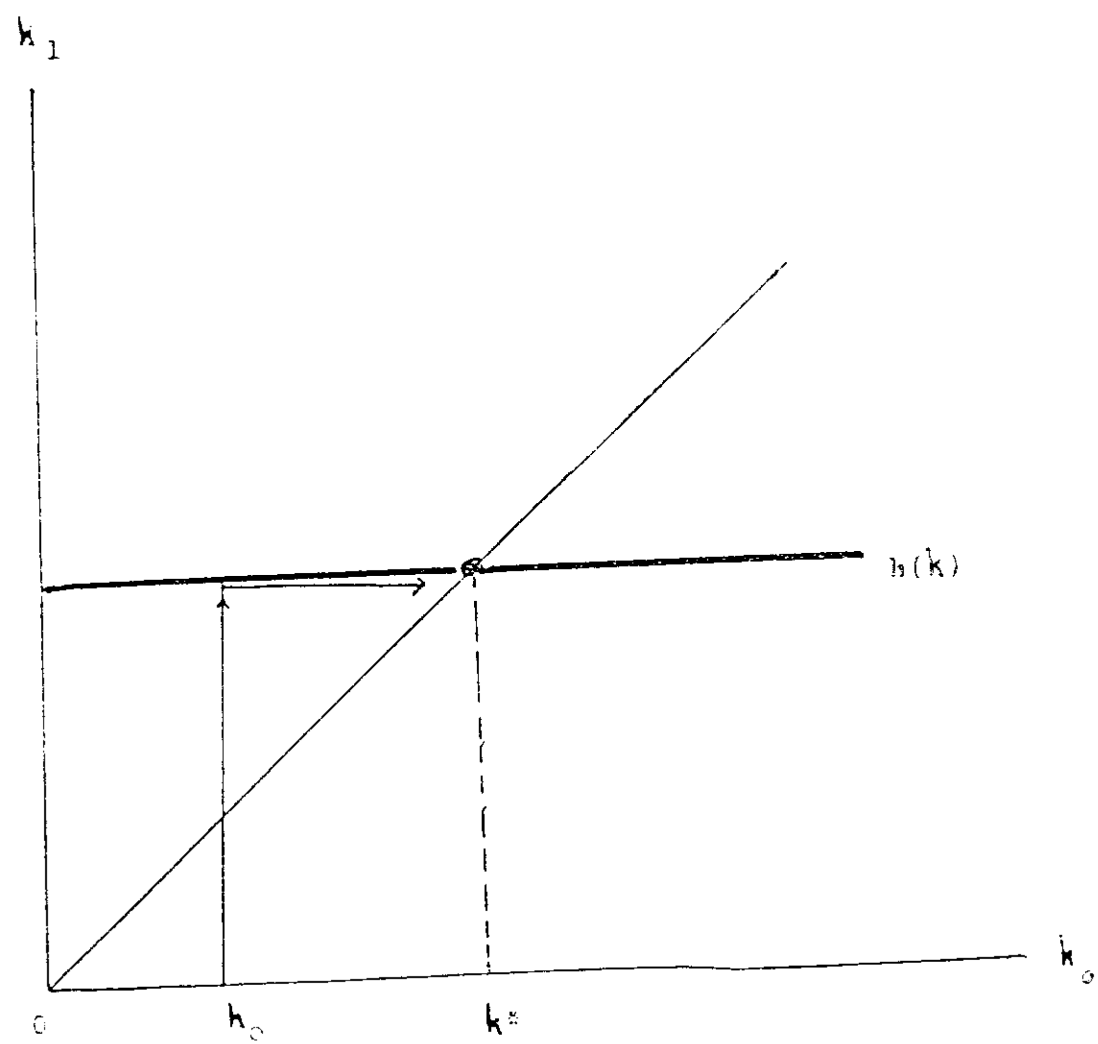

Figure 2

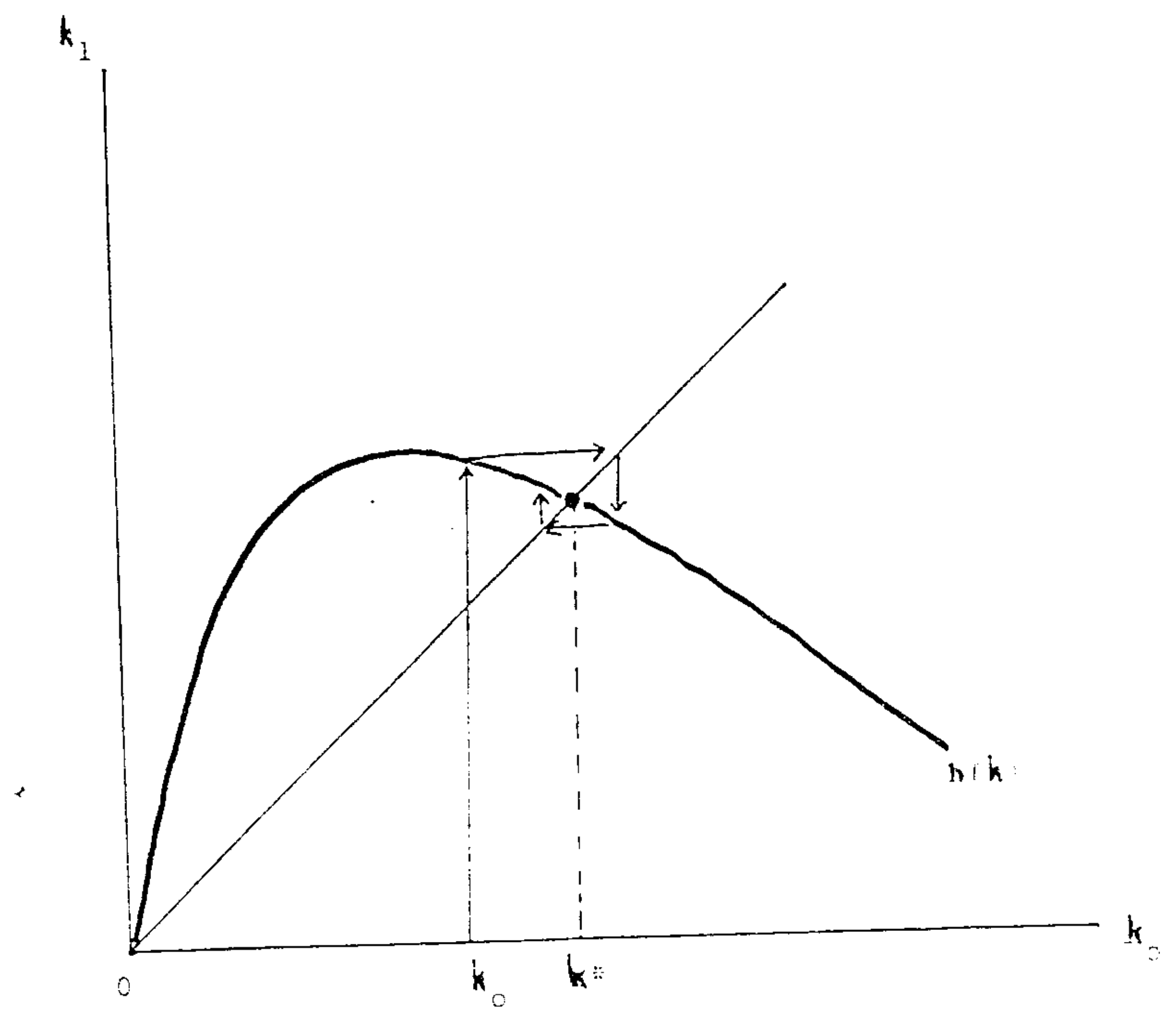

Figure 3

$= U/(1-a)$, so that $U/U' = (k+B)(1-a)/a'$. If U/U' is increasing in c and $(1-a) > 0$, then an increase in costs of child rearing will increase consumption provided we ignore the effect of B on the steady state values of n^*, k^* and $f'(k^*)$. This is discussed by Barro and Becker (1985). Also, changing the steady state interest rate via a perturbation of the production function will affect U/U' as well as the steady state consumption levels only via its impact on steady state values k^* and n^*.

The steady state value of n can be either bigger or smaller than one and will, among other things, depend on B, the cost of raising children. The following numerical example demonstrates this point. Let $U = 2c^{.5}$, $f(k) = k^{.333} + 0.75k$ and $a = 0.5(n+1)^{.667}$. For $B = 0.3385$, steady state values are $n^* = 0.9995$, $k^* = 0.5291$, $a = 0.7936$. For $B = 0.3383$, we have $n^* = 1.0003$, $k^* = 0.5280$, $a = 0.7938$. The effects of increasing δ on steady state values is ambiguous, since a higher marginal valuation of the future may lead to a higher steady state k (see equation (9)) and since $n = a \cdot f'(k)$, n may either increase or decrease.

APPENDIX

THEOREM (i): If $W_{12} > 0$, then an optimal path is strictly monotone.

Proof: Consider optimal paths $(\hat{k}_t)$, $(\hat{k}_t')$ from $\hat{k}_0$, $\hat{k}_0'$ respectively, where $\hat{k}_0' > \hat{k}_0$. Then

(16) $$W(\hat{k}_0,\hat{k}_1) \geq W(\hat{k}_0,\hat{k}_1')$$

(17) $$W(\hat{k}_0',\hat{k}_1') \geq W(\hat{k}_0',\hat{k}_1')$$

Hence

(18) $$W(\hat{k}_0',\hat{k}_1') - W(\hat{k}_0,\hat{k}_1') + W(\hat{k}_0,\hat{k}_1) - W(\hat{k}_0',\hat{k}_1) \geq 0$$

$$\int_{\hat{k}_0}^{\hat{k}_0'} [W_1(s,\hat{k}_1') - W_1(s,\hat{k}_1)]ds \geq 0$$

(19) $$\int_{\hat{k}_1}^{\hat{k}_1'} \int_{\hat{k}_0}^{\hat{k}_0'} W_{12}(s,t)ds\, dt \geq 0$$

Since $W_{12} > 0$ and $\hat{k}_0' > \hat{k}_0$, $\hat{k}_1', \geq \hat{k}_1$ must hold. We note that along the optimal paths,

(20) $$W_1(\hat{k}_0,\hat{k}_1) = W_{k_1} + W_n(\partial n/\partial k_0) = 0.$$

$W_{12} > 0$ implies

(21) $$0 = W_1(\hat{k}_0,\hat{k}_1) > W_1(\hat{k}_0',\hat{k}_1)$$

for $\hat{k}_0' > \hat{k}_0$. Hence $(\hat{k}_0',\hat{k}_1,....)$ cannot be an optimal path: $\hat{k}_1'$ must differ from $\hat{k}_1$. We have shown that $\hat{k}_0' > \hat{k}_0$ implies $\hat{k}_1' > \hat{k}_1$. This also means that $\hat{k}_0 \; (\gtrless) \; \hat{k}_1$ implies $\hat{k}_t \; (\gtrless) \; \hat{k}_{t+1}$. **Q.E.D.**

THEOREM (ii): If $W_{12} = 0$, every optimal path from any $\hat{k}_0 > 0$ jumps to a steady state in one step.

Proof: Let k^* be a steady state. It satisfies

(22) $$W_2(k^*,k^*) = 0.$$

Since $W_{12} = 0$, W_2 is independent of the value of $\hat{k}_0$. Hence $(\hat{k}_0,k^*)$ for any $\hat{k}_0 > 0$ satisfies

(23) $$W_2(\hat{k}_0,k^*) = 0.$$

Therefore $(\hat{k}_0,k^*,k^*,....)$ is an optimal path from any $\hat{k}_0 > 0$. **Q.E.D.**

THEOREM (iii): Over the domain where $W_{12}(k_0,k_1) < 0$ holds, optimal paths oscillate.

Proof: The inequality (19) and W_{12} are used to get $\hat{k}_0' > \hat{k}_0 \rightarrow \hat{k}_1' \leq \hat{k}_1$. $\hat{k}_1' = \hat{k}_1$ is excluded by the same argument as in the proof of (i). **Q.E.D.**

FOOTNOTES

1 We will assume that $\alpha(n) \cdot n$ is small enough, possibly bounded by 1 so that the sum converges.

2 In a competitive market and under constant returns, wages plus profit income will be a function of the given capital stock per worker and will equal $f(k)$. To save space we simply specify income to be equal to output.

3 Barro and Becker limit $a(n)$ to be of the constant relative risk aversion class. We will allow a broader class of functions $a(n)$.

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