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JOINT EXPLOITATION OF A PRODUCTIVE ASSET: A GAME-THEORETIC APPROACH

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ABSTRACT

It is generally believed that when two or more economic agents jointly exploit a common productive asset, there will be a tendency towards overuse or overconsumption, if there is no possibility of making binding commitments regarding the rates of use or consumption. Lancaster(1973) and Levhari and Mirman (1980) have studied specific examples of this phenomenon from a game-theoretic point of view, and in each case demonstrated the existence of a Pareto-inefficient Nash equilibrium of the corresponding dynamic game.

However, dynamic games often have multiple equilibria, and the question remains whether the games studied by these authors - and generalizations of those games - have efficient as well as inefficient equilibria. Indeed, the theory of repeated games suggests this possibility, although it must be emphasized that these games are not strictly repeated, since there is a state variable, namely the current stock of the productive asset, that changes through time in response to the players' actions.

In the present paper we explore the set of equilibria of a game-theoretic model of this type. We do find, in fact, that under certain circumstances there may be efficient as well as inefficient equilibria. Moreover, some of these equilibria have interesting features that are not present in repeated games.

In a repeated game, an important role is played by equilibria in which an inefficient equilibrium of the corresponding static game is repeated indefinitely. Under certain circumstances, especially if the players' discount rates are sufficiently low, an efficient outcome of the repeated game can be sustained as an equilibrium outcome by the players' threats to revert to the inefficient equilibrium in case of a deviation from the efficient path.

The equilibria of the dynamic games we study here that correspond to the repeated static equilibria are those in which each player uses a strategy in which his action at any date is independent of the current state of the state variable (the stock of the asset); we might call these "extreme equilibria." In these equilibria, the players run down the stock of the asset as fast as possible. By analogy with the terminology of repeated-game theory, we define a *trigger strategy equilibrium* to be a Nash equilibrium in which the players threaten to revert to an extreme equilibrium whenever a player is caught deviating from the target efficient path. The effectiveness of such threats depends, of course, on the "detection technology," i.e., on how much extra utility the deviating player can gain before his deviation is detected by the other players. In the model we study, efficient trigger-strategy equilibria may exist from some starting states but not others. More precisely, there is a stock level, say y', such that a trigger-strategy equilibrium exists from starting stocks greater than or equal to y', but not from those strictly less than y'. (This statement is meant to include the cases in which y' is zero or infinite.)

Under some circumstances, there may exist a new kind of equilibrium, which we call a *switching equilibrium*. We show that, in our model, whenever y' is positive (and finite), there is an open

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interval I with upper endpoint y' such that, from any starting stock in I there is an equilibrium of the dynamic game with the following structure: the players follow an inefficient but growing path until the stock reaches the level y', and then follow a trigger strategy (efficient) after that.

The use of a continuous-time model enables us to conveniently decouple the delay of information from the time interval between decisions. Although this leads to some conceptual and mathematical difficulties, we believe that it is an important contribution of our analysis.

In the continuous-time model that we study here, the stock at date t, $Y(t)$, evolves according to the differential equation,

$$Y'(t) = \eta[Y(t)] - c_1(t) - c_2(t) .$$

where (for the case of two players), $c_1(t)$ and $c_2(t)$ are the rates of consumption of the asset by players 1 and 2, respectively. The "production function," η, is assumed to be concave, and to take the value zero at both zero and some positive stock level. The strategy of each player determines his consumption rate at each time as a function of the previous history of the process, possibly with some delay. We assume that each player's utility for the game is equal to his total discounted consumption over the (infinite) duration of the game. This linearity of a player's utility in his consumption is the main special assumption of the model; however, we believe that many of the qualitative features of our results do not depend essentially on this linearity.

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The Joint Exploitation of a Productive Asset: A Game-Theoretic Approach

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1. Introduction

The phrase "tragedy of the common" evokes an image of an overgrazed pasture used in common by many husbandmen. By extension, it refers to a situation in which a producible asset is exploited jointly by several economic agents whose "noncooperative" behavior results in an overexploitation of the asset, i.e., an exploitation of the asset that is not jointly efficient (Pareto-optimal). Other than grazing, examples of this situation include fishing, forestry, and hunting. A novel example, and the one that first attracted our attention, has been studied by Lancaster (1973), who views the assets of a modern capitalist firm as being jointly exploited by the firm's owners and its unionized workers. For various reasons, the owners and workers cannot or do not bind themselves to long-term cooperative behavior, which leads to what Lancaster calls "the dynamic inefficiency of capitalism."

Following a direction suggested by the work of Lancaster and others,[1] we analyze a fairly general model of joint exploitation of producible assets as a dynamic, noncooperative game. Our goal is to understand the variety of "subgame-perfect" Nash equilibria of this game, and in particular to understand the conditions under which jointly efficient behavior can be sustained by such an equilibrium. Our program of research is suggested by the theory of repeated games, but the game we study is not in that category, because the stock of the asset will typically change in the course of the game, as a result of its innate productivity and of the actions of the players. As in the theory of repeated games, we do, indeed, find that under certain conditions efficient joint

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1 See the references cited below and in Section 10.

behavior can be sustained as a subgame-perfect equilibrium, and hence the tragedy of the common can be avoided! Also, because of the added complexity of the game, we find new and interesting types of equilibrium that do not appear in repeated games.

In an infinitely repeated game an important role is played by equilibria in which an inefficient noncooperative equilibrium of the corresponding static game is repeated indefinitely. It is well-known that under certain circumstances, especially if the players' discount rates are sufficiently low, an efficient outcome can be sustained as an equilibrium by the threats of players to revert to the inefficient noncooperative equilibrium path, once a "defection" from the optimal path is detected.

As noted above, presence of the producible asset leads to a dynamic game that is not strictly repeated. Specific examples of this kind have been studied by Lancaster (1973) and by Levhari and Mirman (1980), who demonstrated the existence of Pareto-inefficient Nash equilibria for these games. In the inefficient equilibrium obtained by Levhari and Mirman, each player uses a strategy such that his action at any date depends only on the current state (the stock of the asset). Such equilibria are sometimes called Markov-Nash equilibria (MNEs). By analogy to repeated games, we can explore the possibility of sustaining efficient equilibria by "trigger" strategies, that is, by the players' credible threats to revert to an MNE whenever any player deviates from the efficient path. We emphasize that dynamic games differ significantly from repeated games because the state variable, in our case the stock of the asset, changes through time in response to the actions of the players. Therefore, whether a trigger-strategy equilibrium exists, that is whether an efficient equilibrium can or cannot be enforced, depends not only on the discount rate and the "detection technology" (which determines how much extra utility the deviating player can gain before his deviation is detected by other players), but also on the current level of the state variable. In the model that we study, efficient trigger strategy equilibria may exist from some starting stocks but not from others. More precisely, there is a stock level, say y', such that a trigger-strategy equilibrium exists from starting stocks greater than or equal to y', but not from those strictly less than y'. (This statement is meant to include the cases in which y' is zero or infinite.)

Under some circumstances, there may exist a new kind of equilibrium, which we call a switching equilibrium. We show that, under certain conditions in our model, there is an open interval I with upper endpoint y' such that, from any starting stock in I there is an equilibrium of the dynamic game with the following structure: the players follow an inefficient path until the stock reaches the level y', and then follow an (efficient) trigger strategy after that.

An important feature of our analysis is an explicit modelling of delayed information. In our treatment of trigger-strategy and switching equilibria we assume that each player can observe the state of the system (the stock of the asset) with a fixed delay, i.e., at time t each player can observe the *history* of the state variable up through time $(t-\tau)$, where the delay τ is a fixed, positive parameter of the model. The larger the delay, the more a player can benefit from a "defection" from a prescribed path before his defection is detected and the other player can respond. In previous discrete-time models, this delay has been implicitly equated to the length of the period between decision times. The use of a continuous-time model makes it convenient for us to vary the delay, τ, as an independent parameter, and we consider this to be an important contribution of our analysis.

In the continuous time model that we study, the production (or gross output) of the stock of the asset depends at each point in time on its current level. We assume that the "production function" for the stock is concave. The rates of consumption of the players, which are constrained to be nonnegative and bounded, then determine the rate of growth of the stock. The game ends if and when the stock falls to zero. A player's total utility is equal to his discounted consumption over the duration of the game. This linearity of a player's utility is the main special assumption of the model; however, we believe that many of the qualitative features of our results may not depend essentially on this linearity.

At an efficient (Pareto-optimal) equilibrium the weighted sum of the players' total utility is maximized. Since the instantaneous utilities of the players are linear in consumption, this is equivalent to maximizing the discounted sum of the total consumption of the players. In Section 4 we show that the efficient consumption policy of the two players is to consume nothing until a certain level of the stock is reached. After that the total consumption of the players is equal to the output of the stock, so that the stock level is stationary. We call a consumption policy of this type a "frugal" policy. By contrast, if a player follows a "prodigal" consumption policy he always consumes at the upper boundary of his consumption set.

In Section 3 we characterize the solutions of a general class of optimal control problems that include the problem of efficient total consumption as well as the optimization problems that the players face in the dynamic game. The "frugal" and "prodigal" consumption policies are also precisely described in that section.

There are various types of Markov-Nash equilibrium: they can be "extreme", that is they can be equilibria for which at least one player always consumes on the boundary of his consumption set, or, in some special cases they can be "interior", that is they can be MNEs for which both agents simultaneously consume in the interior of their consumption sets over an interval of time. The analysis of MNEs is carried out under the assumption that information is *not* delayed. Section 5.1 discusses "extreme" MNEs and Section 5.2 discusses interior MNEs. Section 6 illustrates the preceding results with a specific example.

In Section 7 we discuss "trigger-strategy equilibria", with delayed information. We show that with trigger strategies it may be possible to sustain efficient equilibria with optimal consumption policies for certain intervals of initial stocks. Depending on the players' discount rate and on the amount of extra consumption that can be obtained by defecting (which in turn depends on the delay parameter), optimal consumption policies can be sustained only if the initial stock is above some critical value. It is also possible that optimal policies can be maintained for all initial stock levels or for no initial stock level.

In Section 8 we discuss "switching equilibria", in which the players employ generalized trigger strategies. We show that if initial stocks are in an interval I, whose upper endpoint is the critical value of stocks above which trigger strategies can sustain optimal policies, then along the initial segment of a switching equilibrium the players follow an inefficient but growing path. When the stock level reaches the upper endpoint of I, the players switch to optimal trigger strategies. (Switching equilibria exist only under certain conditions on the data of the model.)

In Section 9 we discuss some possible extensions of our model. We show how our results can be extended to allow maximal consumption rates to depend on the level of stocks. We do not however, investigate a closely related matter: the case in which effort or disutility is required to consume the asset, in particular where the effort or disutility required to consume a given quantity is negatively related to the stock of the asset.

Although we do not consider the general case in which the players' instantaneous utility functions are nonlinear, we do report briefly on the possibilities of trigger-strategy and switching equilibria for some generalizations of the discrete-time example studied by Levhari and Mirman (1980).

The analyses of the jointly efficient behavior and of the Markov-Nash Equilibria rely on the solution of a class of optimal control problems, which is described in Section 3. Unfortunately, we are not aware of any previous literature in which this class of problems is completely solved; our own mathematical arguments are largely relegated to the Appendix.

Some of the issues studied in this paper have also been raised in the earlier literature. In Section 10 we provide some brief, and probably incomplete, bibliographical notes.

2. The Model

Let $y(t)$ denote the stock of the asset at time t, and let $c_i(t)$ denote player i's rate of consumption ($i = 1, 2$). Player i's rate of consumption is constrained by

$$c_i(t) = 0 \text{ if } y(t) = 0,$$

$$0 \le c_i(t) \le \bar{c}_i, \quad i = 1, 2. \tag{2.1}$$

Let $n(y)$ be the gross rate of increase in the stock when the stock is y; then the law of motion of the stock is:

$$\dot{y}(t) = \begin{cases} n[y(t)] - c_1(t) - c_2(t), & y(t) > 0, \\ 0, & y(t) = 0 \end{cases} . \tag{2.2}$$

(we use the notation, $\dot{y}(t) = dy/dt$).

Assume that

(a) n is strictly concave and differentiable;

(b) $n(0) = n(y^0) = 0$, where $y^0 > 0$;

(c) $n(y) > 0$ for $0 < y < y^0$,

$n(y) < 0$ for $y > y^0$;

(d) $n'(\hat{y}) = \rho$ for some $\hat{y}$.

(Here $\rho > 0$ is each player i's rate of discount for future utility; see below.)

Given the time paths $c_1(t)$ and $c_2(t)$, and the initial stock $y_0 \equiv y(0)$, player i's total utility is

$$V_i(y) \equiv \rho \int_0^\infty c_i(t)\, e^{-\rho t}\, dt\,, \tag{2.3}$$

where it is understood that $y(t)$ follows the law of motion (2.2).

A strategy of player i determines his consumption at each time t as a function of t and of the histories of both players' consumption previous to time t. A player's strategy is *stationary* if for some function γ_i,

$$c_i(t) \equiv \gamma_i[y(t)]\,. \tag{2.4}$$

We should note at this point that the law of motion (2.2), and therefore the total utilities given by (2.3), may not be well-defined for all the strategy pairs chosen by the players. For example, if the two players adopt strategies such that $c_i(t) = 0$ for $y(t) < \hat{y}$ and $c_i(t) = \bar{c}_i$ for $y(t) \geq \hat{y}$, $i = 1, 2$, (2.2) will not be well-defined (when $\bar{c}_1 + \bar{c}_2$ is large enough). In order to avoid this difficulty and to restrict the strategies chosen by the players to those pairs for which (2.2) is well-defined, from now on we will assume that the total utility of each player will be negative for all the pairs of strategies for which (2.2) (and therefore (2.3)) is not well-defined. (For a rigorous approach that directly confronts problems of this type see Friedman [1971].)

3. A Basic Control Problem

In this section we shall analyze a class of *single-person* optimization problems without delay of information. This analysis will be used in subsequent sections to characterize the jointly efficient policies and the Markov-Nash Equilibria, for the model of the previous section.

As in the previous section, for $t \geq 0$, let $y(t)$ denote the stock of the asset at time t; the stock may be any real number. Let $c(t)$ denote the consumption of a single agent at time t; the net rate of increase of the stock is

$$\dot{y}(t) = m[y(t)] - c(t). \tag{3.1}$$

Here $m(\cdot)$ is a given function, which determines the gross rate of increase of the stock, i.e., when consumption is zero. We assume that:

1) m is differentiable and strictly concave;

2) $m(0) \leq 0$;

3) there is a (unique) strictly positive $\hat{y}$ such that $m'(\hat{y}) = \rho$.

4) m is eventually decreasing.

Let $\rho z(t)$ denote the agent's accumulated discounted consumption at time t, where

$$z(t) = \int_0^t e^{-\rho s} c(s) ds. \tag{3.2}$$

Note that

$$\dot{z}(t) = e^{-\rho t} c(t). \tag{3.3}$$

A consumption path is *admissible* if it is piecewise-continuous and satisfies

$$0 \leq c(t) \leq \bar{c}, \tag{3.4}$$

where $\bar{c} > 0$ is given. Let y_0 denote the initial stock, $y(0)$, and for any consumption path $c(\cdot)$ let T denote the first time if any that $y(t) = 0$. An admissible consumption path is *optimal* if it maximizes $z(T)$ in the set of all admissible consumption paths, subject to the previously stated constraints.

Notice that, in contrast to the model of the previous section, we do not require here that $c(t) = 0$ if $y(t) \leq 0$. However, this does not alter the value of any path, since no more utility is accumulated after time T, i.e., after the stock reaches zero.

Without loss of generality, we can confine our attention to stationary consumption paths (see Appendix A2, Lemma A2.5). Recall that a consumption path $c(\cdot)$ is stationary if there is a function, say γ, such that

$$c(t) = \gamma[y(t)]. \tag{3.5}$$

The function γ will be called a *policy*. Two special policies will turn out to be important, which we shall call "frugal" and "prodigal".

If $0 < m(\hat{y}) < \bar{c}$, then the *frugal policy* is defined by

$$\gamma(y) = \begin{Bmatrix} 0 \\ m(\hat{y}) \\ \bar{c} \end{Bmatrix} \text{ as } y \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \hat{y}. \tag{3.6a}$$

If $\bar{c} < m(\hat{y})$, let $\bar{y}$ be defined by $m(\bar{y}) = \bar{c}$; in this case the frugal policy is defined by

$$\gamma(y) = \begin{Bmatrix} 0 \\ \bar{c} \end{Bmatrix} \text{ as } y \begin{Bmatrix} < \\ \geq \end{Bmatrix} \bar{y}. \tag{3.6b}$$

The *prodigal policy* is defined by

$$\gamma(y) = \bar{c}, \quad \text{all } y. \tag{3.7}$$

We also define a policy to be *interior* if, for some open interval I,

$$0 < \gamma(y) < \bar{c}, \quad \text{for } y \text{ in } I.$$

As we shall see, only in special cases can an optimal policy be interior.

Proposition 1: Except for interior solutions that occur under certain special circumstances to be described below, an optimal policy must be either frugal or prodigal:

1) if $m(y) > 0$ for all $y > 0$ in some neighborhood of 0, then the optimal policy is frugal;

2) if $m(y) \leq 0$ for $0 \leq y \leq y_0$, then the optimal policy is prodigal;

3) if $m(0) < 0$ but $m(y_0) > 0$, then there is some number, say $y_c > 0$, such that the optimal policy is prodigal if $y_0 < y_c$ and is frugal if $y_0 > y_c$ (both are optimal if $y_0 = y_c$).

Proof: See Appendices 1 and 2.

As pointed out in Proposition 1, under some special circumstances an optimal policy can also be interior over some interval I of y. Suppose the net rate of increase of the stock, given by (3.1), takes the special form

$$\dot{y}(t) = \rho y(t) + K - c(t) \tag{3.8}$$

for $0 \leq y \leq z$, where K is a constant. Assume further that the value associated with a level of stock z at the upper boundary of I is given by $V(z)$ and the value associated with zero stocks is $V(0) = 0$. We shall consider two cases. First, suppose that the agent chooses a consumption path so that the stock $y(t)$, which starts at $y(0)$, reaches z at time S, before it reaches zero. Then the value of this consumption path is given by

$$\begin{aligned} v_1(y(0)) &= \rho \int_0^S e^{-\rho t} c(t)dt + e^{-\rho S} V(z) \\ &= \rho \int_0^S e^{-\rho t}[\rho y(t) + K - \dot{y}(t)]dt + e^{-\rho S} V(z). \end{aligned} \tag{3.9}$$

Integrating by parts, we obtain

$$v_1(y(0)) = \rho y(0) + K + e^{-\rho S}[V(z) - \rho z - K]. \tag{3.10}$$

Since S depends on the consumption path $c(\cdot)$ of the agent, he will choose $c(\cdot)$ to maximize S if $[V(z) - \rho z - K]$ is negative and minimize S if it is positive. We note, however, that if $K = V(z) - \rho z$ (or if S is infinite so that $y(t)$ never attains z), then $v_1(y(0))$ is independent of S. This implies that the agent is indifferent between consumption policies and he may choose any interior policy that takes $y(0)$ to z. We must, however, compare the value of such policies with the value of consumption policies that deplete the stock so that $y(t)$ reaches zero before it reaches z.

Consider a consumption policy $c(\cdot)$ for which $y(t)$ reaches zero at $t = T$. Since

$\rho(y(T)) = V(y(T)) = 0$ if $y(T) = 0$, we can derive the value of a consumption policy that depletes the stock at time T by using (3.10) and setting $z = 0$. We obtain

$$v_0(y(0)) = \rho y(0) + (1 - e^{-\rho T})K. \tag{3.11}$$

Note that $v_0(y(0))$ is maximized by minimizing T. This implies that the agent will choose to consume as fast as possible, at the rate $\bar{c}$, so that T is minimized. To establish whether the agent will prefer a consumption policy that reaches z or one that depletes the stock, we must compare $v_1(y(0))$ and $v_0(y(0))$. Consider, for the special case where $K = V(z) - \rho z$,

$$v_1(y(0)) - v_0(y(0)) = e^{-\rho T}K = e^{-\rho T}(V(z) - \rho z). \tag{3.12}$$

It is clear from (3.12) that if $K = V(z) - \rho z > 0$, the agent will prefer a consumption policy that allows the state to attain z. Since he is indifferent between consumption policies that attain z, all interior policies that allow the state to reach z are optimal. We will use this result in Section 5.2 below.

4. Optimal Policies

A pair of policies will be said to be *optimal* if it maximizes the total discounted consumption of the two players. Let

$$C(t) = c_1(t) + c_2(t),$$
$$\bar{C} = \bar{c}_1 + \bar{c}_2.$$

The total discounted consumption starting from an initial stock $y(0) = y_0$ is

$$V(y_0) = \rho \int_0^\infty C(t)e^{-\rho t}\,dt. \tag{4.1}$$

Our problem is to choose a piecewise continuous function $C(\cdot)$ to maximize (4.1) subject to (2.2) and

$$0 \le C(t) \le \bar{C}, \text{ all } t.$$

If in Proposition 1 of Section 3 one takes $m = n$, then, since $n(y) > 0$ for $0 < y < \bar{y}$, one obtains the following result from Part 1 of the Proposition.

Theorem 4.1. A pair of policies is optimal if and only if $C(\cdot)$ is frugal.

Let $\hat{V}(y)$ denote the maximum total discounted consumption, given that $y(0) = y$. For the remainder of this section we consider only the case in which

$$\bar{C} > \max_y n(y). \tag{4.2}$$

One can show that $\hat{V}'$ is concave.[2] In addition, we shall now show that

$$\hat{V}'(0+) = +\infty, \tag{4.3}$$

$$\hat{V}'(\hat{y}) = \rho. \tag{4.4}$$

Let $\hat{T}(y)$ be the first t such that $Y(t) = \hat{y}$, given that $y(0) = y$, and that an optimal policy is followed. Since the optimal policy is frugal,

$$\hat{V}(y) = \begin{cases} e^{-\rho\hat{T}(y)}\, n(\hat{y}), & y \le \hat{y}, \\ \bar{C}(1 - e^{-\rho\hat{T}(y)}) + e^{-\rho\hat{T}(y)}\, n(\hat{y}), & y \ge \hat{y}. \end{cases} \tag{4.5}$$

Hence, differentiating (4.5) with respect to y, one obtains

$$\hat{V}'(y) = \begin{cases} -\rho\hat{T}'(y) e^{-\rho\hat{T}(y)}\, n(\hat{y}), & y < \hat{y}, \\ \rho[\bar{C} - n(\hat{y})]\,\hat{T}'(y) n(\hat{y}), & y > \hat{y}. \end{cases} \tag{4.6}$$

For $y < \hat{y}$, $Y(t)$ *increases* to $\hat{y}$, and hence for $h > 0$, $\hat{T}(y)$ equals the time it takes to go from y to $(y+h)$ plus $\hat{T}(y+h)$; hence for small h, $\hat{T}(y) - \hat{T}(y+h)$ is approximately $h/n(y)$, and

$$\hat{T}'(y) = -\frac{1}{n(y)}, \quad y < \hat{y}. \tag{4.7}$$

(For a more formal treatment, see Lemma 1 of Appendix 4.) On the other hand, for $y > \hat{y}$, $Y(t)$ *decreases* to $\hat{y}$, and hence for $h > 0$, $\hat{T}(y+h)$ equals the time it takes to go from $(y+h)$ to y plus $\hat{T}(y)$; hence, for small h, $\hat{T}(y+h) - \hat{T}(y)$ is approximately $h/[\bar{C} - n(y)]$, and

2 For a discrete-time version of this statement, see (Radner, 1967).

$$T'(y) = \frac{1}{\bar{C} - n(y)}. \tag{4.8}$$

From (4.6)-(4.8) we have

$$\hat{V}'(y) = \begin{cases} \rho e^{-\rho \hat{T}(y)} \left[\dfrac{n(\hat{y})}{n(y)} \right], & y < \hat{y}, \\ \\ \rho e^{-\rho \hat{T}(y)} \dfrac{[\bar{C} - n(\hat{y})]}{[\bar{C} - n(y)]}, & y > \hat{y}. \end{cases} \tag{4.9}$$

In particular, since $\hat{T}(\hat{y}) = 0$ and $n(0) = 0$, (4.3) and (4.4) follow from (4.9).

5. Markov Nash Equilibria

A *Nash Equilibrium* is a pair of strategies, one for each player, such that neither player can increase his total discounted utility by unilaterally changing his strategy. A Nash Equilibrium is *Markov* if each player's strategy is stationary (in the sense of Section 2). We shall use the abbreviation MNE for "Markov Nash Equilibrium." In this section we shall characterize the set of all MNEs. (We shall consider only equilibria with "pure," i.e., nonrandomized strategies.)

If the two players use stationary strategies, say γ_1 and γ_2 respectively, then the discounted utility to player i from any time t on depends only on $y(t)$; denote this discounted utility by $V_i(y)$ if $y(t) = y$. (V_i also depends, of course, on γ_1 and γ_2.)

5.1 Extreme Nash Equilibria

Extreme Nash Equilibria are equilibria along which at least one agent consumes on the boundary of his consumption set at all times. Different types of Extreme Nash Equilibria can be classified according to the maximal output producible from the initial stock relative to the maximal consumption levels of the agents. The theorems below cover all the various cases.

Theorem 5.1. If $n(y_0) \le \bar{c}_i$, $i = 1, 2$, then the pair of prodigal strategies,

$$\left\{ c_1(t) = \gamma_1(y(t)) = \bar{c}_1, \quad c_2(t) = \gamma_2(y(t)) = \bar{c}_2 \right\},$$

constitute an Extreme Markov Nash Equilibrium.

Proof: We simply have to show that the best response to an agent consuming maximally is to also consume maximally. Setting $m(y_0) = n(y_0) - \bar{c}_i$, the proof follows directly from part 2 of Proposition 1 of Section 3.

Theorem 5.2. If $n(y_0) > \bar{c}_2$ and $n(y_0) < \bar{c}_1 + \bar{c}_2$, then either

a) There exists a unique y_c such that the pair of strategies for which agent 1 follows a frugal policy if $y_0 \geq y_c$ and a prodigal policy if $y_0 \leq y_c$, while agent 2 follows a prodigal policy for $y_0 > 0$, constitutes an Extreme Markov Nash Equilibrium or,

b) A pair of strategies where both agents follow prodigal policies for $y_0 > 0$ constitutes an Extreme Markov Nash Equilibrium.

Proof: First note that agent 2 is already consuming maximally and cannot increase his utility by an alternative policy. The optimality of the first agent's policy follows directly from part 3 of Proposition 1. Note that if $n(y_0) > \bar{c}_1$ as well, the Theorem will also hold with the numbering of the agents reversed.

Theorem 5.3. If $n(y_0) \geq \bar{c}_1 + \bar{c}_2$ then prodigal strategies for both agents constitute an equilibrium.

Proof: This is an immediate consequence of part 2 of Proposition 1.

Theorem 5.4. There are no Extreme Markov Nash Equilibria along which one agent consumes nothing over a positive time interval while the other consumes less than maximally.

Proof: See Appendix.

5.2 Interior Markov Nash Equilibria

A Markov Nash Equilibrium is interior over an interval $S = [t_0, t_1)$ if $0 < c_i(t) < \bar{c}_i$, $i = 1, 2$ for t in S. Suppose initial stocks $y(t_0)$ are given and that the players follows the consumption policies

$$\gamma_i(y) = n(y) - K_j - \rho y, \quad y(t_0) \leq z, \quad i, j = 1, 2 \quad \text{and} \quad i \neq j, \tag{5.2.1}$$

for z given and $K_i = V_i(z) - \rho z$, $i = 1, 2$. Assume further that

$$V_i(z) \geq \rho z\,, \quad i = 1, 2\,, \tag{5.2.2a}$$

$$2\rho y - n(y) + V_1(z) + V_2(z) - 2\rho z > 0 \quad \text{for} \quad y(t_0) < y < z\,, \tag{5.2.2b}$$

$$0 < n(y) - \rho y - V_i(z) + \rho z \leq \bar{c}_i\,, \quad i = 1, 2 \quad \text{for} \quad y(t_0) \leq y < z\,. \tag{5.2.2c}$$

Then the policies $[\gamma_1(y), \gamma_2(y)]$ given by (5.2.1) will constitute an interior Markov Nash Equilibrium on the interval $[t_0, t_1)$, where t_1 corresponds to the time at which the stock level z is reached.

To see that the pair given by (5.2.1) is an interior Markov Nash Equilibrium, first note that the players face the equation

$$\dot{y} = \rho y(t) + K_i - c_i(t)\,, \quad i = 1, 2$$

in choosing their consumption policies. From the discussion at the end of Section 3, consumption policies that allow the stock to attain z before reaching zero will be optimal (or best response) policies if the K_i's are chosen such that $K_i = V_i(z) - \rho z \geq 0$. This is assured by (5.2.2a). Equation (5.2.2b) assures that the stock will in fact grow from $y(t_0)$ to z under the policies $[\gamma_i(y), \gamma_2(y)]$, and (5.2.2c) assures that these policies are in fact interior.

6. An Example

Let μ, ν, and $\hat{y}$ be numbers such that

$$-\nu < 0 < \rho < \mu < \frac{\bar{C}}{\hat{y}}. \tag{6.1}$$

and define n by[3]

$$n(y) = \begin{cases} \mu y\,, & 0 \leq y \leq \hat{y}\,, \\ \mu\hat{y} - \nu(y - \hat{y})\,, & y \geq \hat{y} \end{cases} \tag{6.2}$$

3 Although in Section 3 we assumed m to be strictly concave, the analysis of necessary conditions can be carried out with concavity alone, so that the results of the previous sections are applicable to this example. Furthermore, the nondifferentiability of n at $\hat{y}$ presents no problems. Since any segment of an optimal path has to be optimal, if an optimal path satisfying the necessary conditions attains $\hat{y}$, the optimization problem can be spliced at $\hat{y}$. The necessary conditions are unaffected by this procedure and the results of the previous sections are applicable.

By Theorem 4.1, the optimal path is determined by

$$\hat{\gamma}(y) = \begin{cases} 0, & y < \hat{y}, \\ \mu\hat{y}, & y = \hat{y}, \\ \bar{C}, & y > \hat{y}. \end{cases} \tag{6.3}$$

Define $\zeta(y)$ by

$$\zeta(y) = \begin{cases} \mu y, & y < \hat{y}, \\ 0, & y = \hat{y}, \\ \mu\hat{y} - \nu(y-\hat{y}) - \bar{C}, & y > \hat{y}; \end{cases} \tag{6.4}$$

then the optimal path satisfies

$$\dot{y}(t) = \zeta(y), \quad y(0) = y. \tag{6.5}$$

Let $\hat{T}(y)$ be the first t such that $y(t) = \hat{y}$; then on $[0, \hat{T}(y)]$, (6.5) has the solution

$$y(t) = \begin{cases} ye^{\mu t}, & y < \hat{y}, \\ -\dfrac{\bar{C} - \nu\hat{y} - \mu\hat{y}}{\nu} + \left(y + \dfrac{\bar{C} - \nu\hat{y} - \mu\hat{y}}{\nu}\right)e^{-\nu t}, & y > \hat{y}. \end{cases} \tag{6.6}$$

Also, $y(t) = \hat{y}$ for all $t \geq \hat{T}(y)$.

One easily calculates:

$$\hat{T}(y) = \begin{cases} \left(\frac{1}{\mu}\right) \ln \left(\frac{\hat{y}}{y}\right), & y \le \hat{y}, \\ \left(\frac{1}{\nu}\right) \ln \left(1 + \frac{\nu(y-\hat{y})}{\bar{C} - \mu\hat{y}}\right) = \left(\frac{1}{\nu}\right) \ln(1+S), & y > \hat{y} \end{cases} \tag{6.7}$$

$$\hat{V}(y) = \begin{cases} \mu\hat{y}^{\left(1-\frac{\rho}{\mu}\right)} y^{\frac{\rho}{\mu}}, & y \le \hat{y}, \\ \left[1-(1+S)^{-\frac{\rho}{\nu}}\right]\bar{C} + (1+S)^{-\frac{\rho}{\nu}}\mu\hat{y}, & y > \hat{y}. \end{cases} \tag{6.8}$$

Notice that, since $\mu > \rho$, $\hat{V}$ is concave, and (4.3) and (4.4) are verified. We shall suppose that the players share total consumption equally, so that

$$\hat{V}_i(y) = \left(\frac{1}{2}\right) \hat{V}(y). \tag{6.9}$$

Consider now the limit of the optimal path as $\bar{C}$ increases without bound. We shall call this the case of *rapid consumption*. In this case, the second line of (6.8) reduces to

$$\hat{V}(y) = \rho(y - \hat{y}) + \mu\hat{y}, \quad y \ge \hat{y}. \tag{6.10}$$

[This is most easily checked by letting S tend to zero in (6.8), and writing

$$\bar{C} = \mu\hat{y} + \frac{\nu(y-\hat{y})}{S}; \tag{6.11}$$

cf. (6.7).]

For simplicity, we consider the extreme MNE only in the case of rapid consumption, with equal rates of consumption:

$$V_i^*(y) = \left(\frac{1}{2}\right) \rho y. \tag{6.12}$$

From (6.8)-(6.10),

$$\hat{V}(y) = \begin{cases} \mu\hat{y}^{\left(1-\frac{\rho}{\mu}\right)} y^{\frac{\rho}{\mu}}, & y \le \hat{y}, \\ \rho(y-\hat{y}) + \mu\hat{y}, & y \ge \hat{y}. \end{cases} \tag{6.13}$$

Comparing (6.13) with (6.12) we see that the loss in "welfare" in the extreme MNE compared to the optimal path, $\hat{V}(y) - 2V_i^*(y)$, rises monotonically from 0 at $y = 0$ to $(\mu - \rho)\hat{y}$ at $y = \hat{y}$, and remains at that level for all $y \ge \hat{y}$.

One can also easily calculate the path of an interior MNE (see Section 5.2). From (6.2) and (5.2.1) we have, if the interval I is below $\hat{y}$,

$$\begin{aligned} \dot{y}(t) &= \sigma y(t) + K, \\ \sigma &= 2\rho - \mu, \\ K &= K_1 + K_2. \end{aligned} \tag{6.14}$$

The solution of this differential equation is, with $Y(0) = y$,

$$y(t) = \left(y + \frac{K}{\sigma}\right)e^{\sigma t} - \frac{K}{\sigma}. \tag{6.15}$$

Case 1. $\sigma > 0$.

Let $z \le \hat{y}$ be the upper limit of the endpoint I, and let T be the first t such that $Y(t) = z$; then

$$\left(y + \frac{K}{\sigma}\right)e^{\sigma T} - \frac{K}{\sigma} = z,$$

or

$$T = \left(\frac{1}{\sigma}\right)\ln\left(\frac{\sigma z + K}{\sigma y + K}\right). \tag{6.16}$$

From (5.2.1)

$$\gamma_i(y) = (\mu - \rho)y - K_i, \quad i = 1, 2,$$

and so

$$V_i(y) = \int_0^T \rho e^{-\rho t}[(\mu - \rho)Y(t) - K_j]\,dt + e^{-\rho T}V_i(z). \tag{6.17}$$

One can verify directly from (6.15)-(6.17) that

$$V_i(y) = \rho y + K_i. \tag{6.18}$$

Case 2. $\sigma < 0$.

Suppose that we take $z = 0$; then from (6.18) and the fact that $V_i(0) = 0$ it follows that $K_1 = K_2 = 0$. Hence, from (6.15)

$$y(t) = ye^{\sigma t}, \tag{6.19}$$

so that $y(t)$ decreases monotonically towards the limit zero, as t increases without bound. Again, one can verify directly that $V_i(y) = \rho y$.

7. Trigger-Strategy Equilibria

Roughly speaking, a trigger-strategy equilibrium is a Nash equilibrium in which the two players "agree" on a "target path," and sustain the target path by threatening to switch to the extreme strategy (cf. Sec. 5.1) if they detect the other player in a departure from target path behavior. To make this precise, we must describe the circumstances under which one player's departure from target behavior can be detected by the other player. Such a departure will be called a *defection*.

For example, if a defection can be detected the instant it happens, the defector can gain no advantage and any target path that is superior for both players to the extreme MNE is sustainable. On the other hand, if there is sufficient delay between defection and its detection, then defection may be worthwhile. In this section we shall postulate just such a delay, and explore the existence of trigger-strategy equilibria that are Pareto superior to the extreme MNE, or even fully optimal.

We shall say that *information has a delay of* τ if at date t each player knows the history of the process only up to time $(t-\tau)$. The introduction of delay of information in our model raises a number of new problems. It is beyond the scope of this paper to deal with all of these problems in a thorough and general way. Fortunately, because of some special features of our model, it is not necessary to do so here.

First, we must specify how the game begins. We shall make the somewhat arbitrary assumption that each player does know the *initial* stock, $y(0)$, at time 0, but begins to obtain additional information about the process (other than his own consumption) only at time τ.

Second, we note that the stationary strategies defined by (2.4) are – in general – no longer feasible. However, the prodigal strategy – in which a player consumes at his maximal rate – remains feasible, because the player's consumption is independent of the state of process. Hence the pair of prodigal strategies is a Nash equilibrium, with positive delay of information, if the maximal rates of consumption are large enough (cf. Theorem 5.1). We shall continue to refer to this as the extreme MNE, and confine our discussion to the case in which the maximal rates of consumption are "large".

We consider now the situation in which the "target path" is the optimal one (Sec. 4). Let $\hat{y}(\cdot)$ denote the optimal path of the stock, and let $\hat{c}_i(\cdot)$ denote the corresponding optimal path of consumption for player i $(= 1, 2)$. The *trigger strategy* for player i is defined as follows: follow the consumption path $\hat{c}_i(\cdot)$ until the first time t (if any) at which player i detects that a defection has occurred, i.e., $y(t-\tau) \neq \hat{y}(t-\tau)$; thereafter consume as fast as possible.

The alert reader will have noticed that, if $y(\cdot)$ and $\hat{y}(\cdot)$ are differentiable, there can be no "first" time t at which $y(t-\tau) \neq \hat{y}(t-\tau)$. We shall make the following convention: each player "detects at time t that a defection has occurred at time $(t-\tau)$" if for some $\epsilon > 0$,

$$y(s) = \hat{y}(s) \quad \text{for} \quad t-\tau-\epsilon \leq s \leq t-\tau,$$

$$y(s) \neq \hat{y}(s) \quad \text{for} \quad t-\tau < s \leq t-\tau+\epsilon.$$

This convention can be justified by considering a sequence of discrete-time models approaching the continuous-time model. (For a direct and rigorous approach to such problems see Friedman (1971), Simon and Stinchcombe (1985) and Krasovskii and Subbotin (1988), Chapter 11.)

Against a player using the above trigger strategy, a player who defects at all should do so by consuming as fast as possible. Suppose he defects when the stock is y; we make the convention that he defects at time 0. Let $T(y)$ denote the time at which the stock is exhausted. The utility to player i of defecting when the stock is y is therefore

$$D_i(y) = \rho \int_0^{T(y)} \bar{c}_i e^{-\rho t}\, dt$$

$$= \bar{c}_i(1 - e^{-\rho T(y)})\,.$$

Recall that $\hat{V}_i(y)$ denotes the utility to i of following the optimal path from an initial stock y. The pair of trigger strategies will constitute a Nash equilibrium if and only if, for each i, and every y between $y(0)$ and $\hat{y}$, $D_i(y) \le \hat{V}_i(y)$.

We shall carry out a formal analysis only for the case in which the information delay, τ, is "small", and the maximum rates of consumption are "large". For convenience, suppose that the players have the same maximum rate of consumption, say $\bar{c}$. Then the functions D_i above will be the same, say D. We shall suppose that τ is "small", $\bar{c}$ is "large", and that $\delta = \bar{c}\tau$ is "moderate". Heuristically speaking, if one player defects when the stock is y, then if y is large enough the defector will in the short time interval τ be able to consume approximately δ, and thereafter the two players will equally share the remaining stock. If y is small enough, then in a time less than τ the defector will be able to consume approximately y before the stock is exhausted. In Appendix A4 we show that, for fixed $\delta > 0$,

$$\lim_{\substack{\tau \to 0 \\ \bar{c}\tau = \delta}} D(y) = \begin{cases} \rho\left[\delta + \left(\frac{1}{2}\right)(y - \delta)\right] = \dfrac{\rho(y+\delta)}{2}\,, & y \ge \delta\,, \\ \rho y\,, & 0 \le y \le \delta\,. \end{cases} \tag{7.1}$$

We shall call this the case of *rapid consumption.*

Suppose now that the target path is in fact the optimal path. Then defection is attractive at state y if $D(y) > \hat{V}_i(y)$. In the case of rapid consumption, if the players share equally,

$$\hat{V}_i(y) = \begin{cases} \left(\frac{1}{2}\right) e^{-\rho \hat{T}(y)}\, \pi(\hat{y})\,, & 0 < y \le \hat{y}\,, \\ \left(\frac{1}{2}\right)\left[\pi(\hat{y}) + \rho(y - \hat{y})\right]\,, & y \ge \hat{y}\,, \end{cases} \tag{7.2}$$

(this can be derived from (4.5), letting $\overline{C} = 2\overline{c}$ increase without limit.) Recall from (4.3) that

$$\hat{V}_i'(0) = +\infty ; \tag{7.3}$$

hence $D(y) < \hat{V}_i(y)$ for sufficiently small y.

On the other hand, if $\delta > \hat{y}$, then

$$D'(y) = \rho , \quad \hat{y} < y < \delta ,$$

whereas

$$\hat{V}_i'(y) = \frac{\rho}{2} , \quad \hat{y} < y .$$

Hence for sufficiently large δ and y, $D(y) > \hat{V}_i(y)$, so that defection at y will be attractive.

Finally, defection may be attractive at some $y < \hat{y}$ even if $D(y) < \hat{V}_i(y)$, provided that for some y',

$$D(y') > \hat{V}_i(y') , \quad y < y' \leq \hat{y} . \tag{7.4}$$

This is so because, given that defection will occur at y', the discounted utility of staying on the target path rather than defecting at y is less than $\hat{V}_i(y)$.

The example of Section 6 can be used to illustrate this phenomenon. (We continue to consider the case of rapid consumption.) Suppose that $\mu < 2\rho$, and define y_D by

$$\rho y_D = \hat{V}_i(y_D) . \tag{7.5}$$

From (6.13) one finds that

$$y_D = \left(\frac{\mu}{2\rho} \right)^{\frac{\mu}{\mu - \rho}} \hat{y} , \tag{7.6}$$

which is less than $\hat{y}$, since $\rho < \mu < 2\rho$. Take $\delta > y_D$ but close to y_D; then there will be a y_D' between δ and $\hat{y}$ such that

$$D(y) \begin{cases} > \hat{V}_i(y) , & \text{for } y_D < y < y_D' \\ \leq \hat{V}_i(y) , & \text{otherwise} . \end{cases} \tag{7.7}$$

Hence starting from any $y \geq y'_D$ the optimal path can be sustained by a trigger strategy equilibrium, but not from any state $y < y'_D$.

8. Switching Equilibria

In the last section we showed that, under certain circumstances, the optimal path could be sustained by a trigger-strategy equilibrium only if the starting stock were sufficiently large, i.e., at least as large as some critical value, say y'_D, with $0 < y'_D < \hat{y}$. In this section we show that, under the same circumstances, there can be an open interval I just below y'_D, and an equilibrium starting from any point in I, such that an inefficient path is followed until $y(t)$ reaches y'_D, and then the optimal path is followed thereafter. We shall call this a *switching equilibrium*.

The switching equilibria will consist of a pair of generalized trigger strategies. To define these strategies we first define a class of *switching paths*, as follows. Let c be a nonnegative number, with

$$n(y) - 2c > 0, \tag{8.1}$$

where $n(\cdot)$ is again the "production function" of (2.2), and $y = y(0)$ is the initial stock. We are interested in the case in which $0 < y < y'_D < \hat{y}$. Along the switching path, each player consumes at rate c until the stock $y(t)$ reaches y'_D; thereafter the players follow the optimal path. Formally, the switching path is defined by: for $i = 1, 2$,

$$\tilde{c}_i(t) = \begin{cases} c\,, & y \leq \tilde{y}(t) < y'_D\,, \\ 0\,, & y'_D \leq \tilde{y}(t) < \hat{y}\,, \\ \dfrac{n(\hat{y})}{2}\,, & \tilde{y}(t) = \hat{y}\,, \\ \bar{c}\,, & \tilde{y}(t) > \hat{y}\,, \end{cases} \tag{8.2a}$$

$$\tilde{y}'(t) = n[\tilde{y}(t)] - \tilde{c}_1(t) - \tilde{c}_2(t)\,, \tag{8.2b}$$

$$\tilde{y}(0) = y\,, \tag{8.2c}$$

where $\bar{c}$ is the maximal rate of consumption for each player. A *switching strategy* for player i is defined as follows: player i follows the consumption path (8.2a) until "the first time t (if any) such that $y(t-\tau) \neq \tilde{y}(t-\tau)$" (see Section 7 for a precise definition of this condition); thereafter player i

consumes at rate $\bar{c}$. Note that when $c=0$ a switching strategy is the trigger strategy of Section 7. A *switching equilibrium* is a Nash equilibrium in which each player uses a switching strategy (with the same c for both players) and the outcome is the switching path (8.2a, b, c), i.e., there is no defection.

The remainder of this section is devoted to deriving conditions on the data of the problem, and on c, for which a switching equilibrium exists. We shall see that under suitable conditions, a nondegenerate interval of such c-values exists. As in Section 7, we shall confine our attention to the case in which τ is small and $\bar{c}$ is large.

Let $V_c(y)$ denote the utility to each player if both players follow the switching path from an initial state y, and let $D_c(y)$ denote the corresponding utility to one player of defecting at time zero from the switching path to the maximal rate of consumption, $\bar{c}$, while the other player follows the switching strategy. Correspondingly, let $V_0(y)$ and $D_0(y)$ denote the utilities of following the optimal path and of defecting from the optimal path, respectively.

To simplify the notation, let $z = y_D'$, that is,

$$\left.\begin{aligned} D_0(y) &> V_0(y) \quad \text{for } y<z \text{ and close to } z, \\ D_0(z) &= V_0(z), \\ D_0(y) &< V_0(y) \quad \text{for } y>z. \end{aligned}\right\} \tag{8.3}$$

It follows from the definition of the switching strategies that

$$\left.\begin{aligned} D_c(y) &= D_0(y) \\ V_c(y) &= V_0(y) \end{aligned}\right\} \text{ for } y \geq z. \tag{8.4}$$

Hence, to show that it is not optimal to defect from the switching strategy at states $y<z$ but close to z, i.e., that $D_c(y)<V_c(y)$ at such states, it suffices to show that

$$D_c'(z-) > V_c'(z-), \tag{8.5}$$

where the "minus signs" in (8.5) indicate derivatives from the left.

For the remainder of this section we confine our attention to starting states $y<z$, but close to z. Let $\bar{T}$ denote the time to reach z *along the switching path*. Fix τ; for y sufficiently close to z, we

have $\bar{T} < \tau$. This implies that if one player defects, the nondefector will switch his consumption rate from c to 0 (at time $\bar{T}$) before he detects the defection (at time τ).

Let $T(y)$ denote the time at which the state reaches zero following a defection at time 0 from an initial state y. As before

$$D_c(y) = \bar{c} \int_0^{T(y)} \rho e^{-\rho t} dt$$

$$= \bar{c}(1 - e^{-\rho T(y)}).$$

Hence

$$D_c'(y) = \rho \bar{c} e^{-\rho T(y)} T'(y). \tag{8.6}$$

To evaluate $T'(y)$, consider a "small" *increase* in y, say dy, resulting in corresponding changes $d\bar{T}$, $dy(\bar{T})$, $dy(\tau)$, and dT. (Here $dy(\bar{T})$ denotes the change in the state at the original time $\bar{T}$.) By Lemma 1 of Appendix 4, $d\bar{T}$ is approximately

$$\frac{-dy}{n(y) - 2c}. \tag{8.7}$$

Over the interval $(\bar{T} - |d\bar{T}|, \bar{T})$ the rate of change of the state, $y'(t)$, is now $(n[y(t)] - \bar{c})$ rather than $(n[y(t)] - \bar{c} - c)$, since the nondefector has switched at time $\bar{T} - |d\bar{T}|$ from consuming at rate c to consuming at rate 0. Hence $dy(\bar{T})$ is approximately

$$c\,|d\bar{T}|. \tag{8.8}$$

By Lemma 2 of Appendix 4, $dy(\tau)$ is approximately

$$\left(\frac{n[y(\tau)] - \bar{c}}{n[y(\bar{T})] - \bar{c}}\right) dy(\bar{T}), \tag{8.9}$$

and by Lemma 1 of Appendix 4, dT is approximately

$$\frac{-dy(\tau)}{n[y(\tau)] - 2\bar{c}}. \tag{8.10}$$

Combining (8.7)-(8.10), and letting dy tend to zero, we have

$$T'(y) = \left(\frac{c}{n(y) - 2c}\right)\left(\frac{n[y(\tau)] - \bar{c}}{n[y(\bar{T})] - \bar{c}}\right)\left(\frac{-1}{n[y(\tau)] - 2\bar{c}}\right). \tag{8.11}$$

Observe that

$$\lim_{\substack{y \to z \\ y < z}} y(\bar{T}) = z\,;$$

also $y(\tau)$ approaches some limit as y approaches z from below. Hence, from (8.11)

$$T'(z-) = \left(\frac{c}{n(z) - 2c}\right)\left(\frac{n[y(\tau)] - \bar{c}}{n(z) - \bar{c}}\right)\left(\frac{-1}{n[y(\tau)] - 2\bar{c}}\right), \tag{8.12}$$

where, in (8.12), $y(\tau)$ corresponds to the initial condition, $y(0) = z$.

We now examine the behavior of $T'(z-)$ as τ tends to zero and

$$\bar{c} = \frac{\delta}{\tau}. \tag{8.13}$$

From (8.12) and (8.13)

$$\frac{T'(z-)}{\tau} = \left(\frac{c}{n(z) - 2c}\right)\left(\frac{\delta - \tau n[y(\tau)]}{\delta - \tau n(z)}\right)\left(\frac{1}{2\delta - \tau n[y(\tau)]}\right). \tag{8.14}$$

Since $y(\tau)$ is between 0 and z, it follows from (8.14) that

$$\lim_{\tau \to 0} \frac{T'(z-)}{\tau} = \frac{c}{2\delta[n(z) - 2c]}. \tag{8.15}$$

Also, as in Section 7,

$$\lim_{\tau \to 0} T(z) = 0,$$

so that, by (8.6) and (8.15),

$$\lim_{\tau \to 0} D_c'(z-) = \frac{\rho c}{2[n(z) - 2c]}. \tag{8.16}$$

We now turn our attention to $V_c'(z-)$. Recall that $\bar{T}(y)$ is the time to reach z along the switching path, given that $y(0) = y$ (we show explicitly the dependence of $\bar{T}$ on y). Since each player consumes at rate c until time $\bar{T}$, and then follows the optimal path,

$$V_c(y) = c \int_0^{\bar{T}(y)} \rho e^{-\rho t}\, dt + e^{-\rho \bar{T}(y)} V_0(z)$$

$$= c(1 - e^{-\rho \bar{T}(y)}) + e^{-\rho \bar{T}(y)} V_0(z)$$

$$V_c'(y) = \rho[c - V_0(z)] e^{-\rho \bar{T}(y)} \bar{T}'(y). \tag{8.17}$$

By Lemma 1 of Appendix 4,

$$\bar{T}'(y) = \frac{-1}{n(y) - 2c}. \tag{8.18}$$

Also, $\bar{T}(y)$ tends to 0 as y tends to z from below. Hence

$$V_c'(z-) = \frac{\rho[V_0(z) - c]}{n(z) - 2c}. \tag{8.19}$$

Comparing (8.16) and (8.19), we see that, if τ is sufficiently small, it is not optimal to defect from the switching strategy at states $y < z$ but close to z if

$$\frac{c}{2} \geq V_0(z) - c,$$

or equivalently

$$c \geq \left(\frac{2}{3}\right) V_0(z); \tag{8.20}$$

cf. (8.5). Recall that we also require (8.1), i.e., that the switching path actually grows; thus we need

$$c < \frac{n(y)}{2}. \tag{8.21}$$

For y sufficiently close to z (and $< z$), there is an open interval of c-values satisfying (8.20)-(8.21) provided that

$$\frac{2}{3} V_0(z) < \frac{n(z)}{2}. \tag{8.22}$$

By (8.3) and (7.1), if τ is small then $V_0(z)$ is approximately $\rho(z+\delta)/2$, so if

$$\rho(z+\delta) < \left(\frac{3}{2}\right) n(z)\,, \tag{8.23}$$

then (8.22) is satisfied for sufficiently small τ.

For the example of Section 6, (8.23) reduces to

$$\rho(z+\delta) < \frac{3\mu z}{2}\,, \tag{8.24}$$

where z is the solution of

$$\left(\frac{1}{2}\right)\mu\hat{y}^{\left(1-\frac{\rho}{\mu}\right)} z^{\frac{\rho}{\mu}} = \frac{\rho(z+\delta)}{2}\,. \tag{8.25}$$

Let ξ be the solution of

$$\left(\frac{1}{2}\right)\mu\hat{y}^{\left(1-\frac{\rho}{\mu}\right)} \xi^{\frac{\rho}{\mu}} = \rho\xi\,, \tag{8.26}$$

or

$$\left(\frac{\xi}{\hat{y}}\right) = \left(\frac{\mu}{2\rho}\right)^{\frac{1}{1-\frac{\rho}{\mu}}}\,. \tag{8.27}$$

(Recall that $\rho < \mu$.) Then

$$\xi < \hat{y} \quad \text{if and only if} \quad \rho > \frac{\mu}{2}\,. \tag{8.28}$$

From (8.25) and (8.26), z will be close to ξ if δ is close to ξ, in which case (8.24) is approximately equivalent to

$$\rho < \left(\frac{3}{4}\right)\mu\,. \tag{8.29}$$

In other words, putting (8.28) and (8.29) together, in the example of Section 6, a switching equilibrium will exist for $y < z$ and y close to z if τ is sufficiently small, δ is close to ξ, and

$$\frac{1}{2} < \frac{\rho}{\mu} < \frac{3}{4}\,. \tag{8.30}$$

In this case it will suffice to take

$$\left(\frac{2}{3}\right) V_0(z) \le c < \frac{\mu y}{2}, \tag{8.31}$$

where $V_0(z)$ is given by the left-hand side of (8.25).

It is clear that this is a "robust" example.

9. Extensions

In this section we briefly discuss some of the possible extensions and generalizations of the model presented in the previous sections.

In the model presented in Section 2 we assumed that each player's consumption was bounded by a fixed maximal consumption level (see (2.1)). We can replace this assumption by

$$0 \le c(t) \le \bar{C}[y(t)], \tag{9.1}$$

where $\bar{C}(\cdot)$ is a continuous, strictly increasing function, with $\bar{C}(0) = 0$. Such a formulation would be appropriate if the realized rate of consumption at time t were a function of both "effort", say $e(t)$, and the current stock, $y(t)$:

$$c(t) = F[e(t), y(t)], \tag{9.2}$$

where F is continuous, increasing in both arguments, and

$$F(e, 0) = 0, \quad \text{all } e, \tag{9.3}$$

$$0 \le e(t) \le \bar{e}. \tag{9.4}$$

This expresses the idea that the productivity of effort depends on the current stock (condition (9.2)), that effort is totally unproductive when the stock is zero (condition (9.3)), and that there is a limit, $\bar{e}$, to the effort (rate) that can be expended (condition (9.4)). Condition (9.1) follows from (9.2)-(9.4) by taking

$$\bar{C}(y) = F(y, \bar{e}). \tag{9.5}$$

If we assume that effort has no disutility, then the model analyzed in the paper is a limiting case of the above model, with

$$\bar{C}(y) = \begin{cases} \bar{c}, & y > 0, \\ 0, & y = 0. \end{cases} \tag{9.6}$$

Notice, however, that in this limiting case, the function $\bar{C}(\cdot)$ is discontinuous at 0, and constant thereafter.

The analysis of the paper is applicable to this alternative model, with only minor modifications. In fact, if the function $\bar{C}(\cdot)$ satisfies certain conditions (described below), then the analysis will be somewhat simplified because the stock will never reach zero in finite time (provided the initial stock is positive). In particular, this will lead to a simpler proof of the existence of an optimal policy (cf. Appendix A1).

To investigate the conditions under which the stock will never fall to zero in finite time (i.e., for every policy and for every t, $y(t) > 0$, provided $y(0) > 0$), we first note that, if consumption is always at the maximum rate, then the differential equation for the stock is

$$\begin{aligned} \dot{y}(t) &= n[y(t)] - \bar{C}[y(t)] \\ &= -f[y(t)], \end{aligned} \tag{9.7}$$

where

$$f(y) = \bar{C}(y) - n(y), \tag{9.8}$$

$\bar{C}(y)$ is the upper limit on the *total* consumption of the two players, and $n(\cdot)$ is the "net increase function" described in Section 2. Note that $f(0) = 0$, and f is continuous at 0.

Suppose that $y(\cdot) > 0$, and let

$$T = \inf\{t : y(t) > 0\}.$$

From the differential equation (9.7) we see that T can be finite only if, for some $a > 0$, $f(y) > 0$ for $0 < y < a$, and $y(0) < a$. We shall therefore confine our attention to this case; then $y(t)$ is decreasing and

$$\lim_{t \to \infty} y(t) = 0.$$

We shall prove:

Proposition 9.1:

$$T = \int_0^{y(0)} \frac{dx}{f(x)}\,; \tag{9.9}$$

hence T is finite if and only if the integral in (9.9) is finite.

As an implication of the Proposition we have:

Corollary. T is infinite if there are numbers $\epsilon > 0$ and $\alpha < \infty$ such that, for $0 < y < \epsilon$,

$$f(y) \leq \alpha y\,. \tag{9.10}$$

On the other hand, T is finite if there are strictly positive numbers ϵ, α, and β such that, for $0 < y < \epsilon$,

$$f(y) > \alpha y^{\beta}\,. \tag{9.11}$$

To prove (9.11), from (9.7) we have

$$\begin{aligned} \frac{dt}{dy} &= -\frac{1}{f(y)} \\ t &= -\int_{y(0)}^{y(t)} \frac{dx}{f(x)} \\ &= \int_{y(t)}^{y(0)} \frac{dx}{f(x)}\,. \end{aligned}$$

Hence $y(t) = 0$ at some finite t if and only if the integral in (9.9) is finite, in which case (9.9) is valid. (It is also valid, of course, when $T = \infty$.)

Here is an example for which (9.10) is not satisfied and yet T is infinite:

$$f(y) = y(1 - \ln y)\,. \tag{9.12}$$

To see that the integral in (9.9) is infinite, make the change of variable $z = 1 - \ln x$; one then verifies that

$$\int_y^{y(0)} \frac{dx}{f(x)} = \ln\left[\frac{1 - \ln y}{1 - \ln y(0)}\right],$$

which tends to infinity as y tends to zero.

Another possible extension is to allow nonlinear instantaneous utility functions for the players. We should note that in the discrete-time example of Levhari and Mirman (1980), the players have logarithmic instantaneous utility functions and the production technology is "Cobb-Douglas". For that particular example, we can show that the existence of trigger-strategy equilibria is independent of the initial level of the stock of the asset. If the utility of defection dominates (resp. is dominated) by the utility of cooperation for some value of the stock then it does so for all values of the stock. Therefore switching equilibria are ruled out. The same is true for instantaneous utility functions of the constant relative risk aversion type, coupled with linear production technologies.

If we allow instantaneous utility functions of the HARA (hyperbolic absolute risk aversion) type, which generalize the constant relative risk aversion utilities, switching equilibria become possible in the Levhari-Mirman model (discrete time). Below we report an example.

Let the instantaneous utility functions of the agents be given by $U(c) = (c+n)^{\alpha}M/\alpha$ and let technology be linear so that the stock k grows according to

$$k(t+1) = a[k(t) - c_1(t) - c_2(t)].$$

M, α, n and a are constants such that $M > 0$, $\alpha \leq 1$ and $a > 1$. The players maximize the discounted sum of their utilities, given by $\sum_{t=0}^{\infty} \beta^t U_i(c_i(t))$, $0 < \beta < 1$. The value of cooperation from an initial stock k, assuming the players derive equal benefits from cooperation, is given by

$$V^c(k) = (M/\alpha)2^{-\alpha}(1-x)^{\alpha-1}[k + (a/(a-1))n]^{\alpha},$$

where $x = \beta^{(1/1-\alpha)}a^{(\alpha/1-\alpha)}$. Players may also choose to defect from cooperation. If they do, they face a trigger strategy where the opponent tries to eat all the remaining stock in the subsequent period. Of course, it will be optimal for the original defector to try to do the same in that period and we assume that the players then equally share the remaining stock. From then on, both players receive the utility of zero consumption, since there can be no production without stocks. If a player chooses to defect from cooperation, he must choose the optimal amount to consume in the first period in which he defects. Since marginal utility is diminishing, in general he will choose to leave some stocks for the next period, even though the output next period will be shared by the two players.

If it is optimal to defect from cooperation for stocks below a level k_0 but not for stocks above k_0, and if the initial stock k is below k_0, we can sometimes construct a switching equilibrium. Both players will find it in their interest to switch to cooperation if the stock exceeds k_0, under the threat of the trigger strategies described above. For k slightly below k_0, it is possible to construct symmetric Markov Nash Equilibria, where the consumption policy of each player is a best response to that of the other player, and along which the stock grows and eventually exceeds k_0. Since players anticipate switching to cooperation when k exceeds k_0, the equilibrium policies reflect this anticipation. Therefore, such interior Markov Nash Equilibrium policies are different from Markov Nash Equilibrium policies where players do not expect to switch to cooperation once k_0 is overtaken. In particular, along Markov Nash Equilibria where a switch takes place, there will be less consumption and greater saving compared to Markov Nash Equilibria where there is no switching to cooperation.

We consider an example where $n=1$, $M=1$, $\alpha=0.1$, $\beta=0.3$ and $a=6$. There are no nonnegative cooperative steady states for this example and consumptions as well as the stock grow indefinitely. Value functions are nevertheless well-defined because the utility function is sufficiently concave and there is sufficient discounting of the future. The value of defection dominates the value of cooperation for stock levels k below (approximately) 1.6. However, a symmetric and growing interior MNE can be constructed for stocks below 1.6. For $k = 1.5$, there is an MNE which grows so that a level of $k \approx 1.7898$ is reached in one step. At that point, there is a switch to cooperation and the stock continues to grow. It is also possible to compute equilibria where k_0 is exceeded after multiple steps.

Finally, it would be interesting to completely characterize the set of equilibria, and to study how this set depends on the initial state, that is the initial stock of the asset. Of particular interest may be general results on equilibria – like the switching equilibria – along which threat strategies are not implemented but are used to sustain second best strategies.

10. Brief Bibliographical Notes

The possibility of using trigger-strategies to enforce cooperative outcomes in repeated games has been widely discussed in the literature. Several authors have also discussed the possibility of using credible threats to enforce cooperation in dynamic games. Tolwinski (1982) provides an excellent discussion and suggests the use of MNEs as credible threats to sustain efficient cooperative equilibria in a discrete-time model. He provides an example with a linear production function and quadratic utilities. He also discusses strategies that pose more severe threats (minimax strategies) to enforce cooperation although he notes that such strategies are not perfect or credible in his example and that the players would have to have some reason to believe the threats would be carried out. (See also Haurie and Pohjola (1987), Haurie and Tolwinski (1984).)

Tolwinski, Haurie and Leitmann (1986), using the ideas of A. Friedman (1971), discuss a class of (non-zero sum) differential games in continuous time that allow for history-dependent (although not subgame-perfect) trigger strategies to enforce cooperative outcomes. Hämäläinen, Haurie and Kaitala (1985) study a fishery-management game in which they introduce a delay parameter that allows the defecting player to go undetected for a fixed time period. They then analyze the enforceability of cooperation with trigger strategies as the delay parameter is varied. They note the possibility that the enforceability of cooperation may depend on the state variable and that it may break down along the implied cooperative trajectory.

Cave (1987) discusses the possibility of enforcing cooperation with MNE threats for the example provided by Levhari and Mirman (1980). (See also Reynolds (1988).) As noted in the previous section, in the Levhari-Mirman example the possibility of enforcing a cooperative solution depends on the discount rate and the parameters of their utility and production functions, but is independent of the initial level of the stock. A similar situation occurs in an example of durable-goods duopoly studied by Ausubel and Deneckere (1987), who also characterized the set of all subgame-perfect equilibria.

Oudiz and Sachs (1986) also discuss the possibility of using credible threat strategies for a dynamic game in the context of a macroeconomic model of international policy coordination. They note the possibility of enforcing any dynamic outcome that dominates the total utility achievable by deviating and then reverting to an MNE.

Benhabib and Ferri (1987) provide an example of a "switching" equilibrium in a dynamic game between a union and a firm that produces with a linear technology and is subject to adjustment costs in changing its labor force.

Survey papers by Kaitala (1986) and Pohjola (1986) review the applications of the dynamic game literature to the management of fisheries and to macroeconomics, respectively. They provide a useful bibliography of the literature that is related to some of the issues discussed in this paper.

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APPENDIX

A1. Existence of an Optimal Control for the Model of Section 3

In the control problem of Section 3, we may take $\dot{y}(\cdot)$ as the control, rather than the consumption path $c(\cdot)$. Then

$$\hat{V} = \sup_{\dot{y}} \int_0^T \rho e^{-\rho t} \left\{ m[y(t)] - \dot{y}(t) \right\} dt , \tag{A1.1}$$

subject to the constraints:

$$m[y(t)] - \bar{c} \le \dot{y}(t) \le m[y(t)] , \quad \text{all } t ;$$

$$y(0) = y_0 ;$$

$$\dot{y} \text{ measurable} ;$$

$$T = \inf \{t : y(t) = 0\} .$$

T may be infinite, and will be called the *stopping time*. In the above, the initial state, $y_0 > 0$, is given, as well as the maximum rate of consumption, $\bar{c} > 0$, and the function $m(\cdot)$. The function $m(\cdot)$ is concave and continuously differentiable, $m(0) \le 0$, $0 < m'(0) < \infty$, and $m'(y) < 0$ for some $y > 0$.

Since consumption is bounded above by $\bar{c}$, we know that $\hat{V}$ is finite. Where there is no danger of confusion, we shall use the symbols $\dot{y}$ and y to denote the respective *paths*, $\dot{y}(\cdot)$ and $y(\cdot)$. Accordingly, let $V(\dot{y})$ denote the integral on the right-hand-side of (A1.1), i.e., the total discounted consumption corresponding to the control $\dot{y}$. Let $(\dot{y}_n)$ be a sequence of controls such that

$$\lim_{n \to \infty} V(\dot{y}_n) = \hat{V} , \tag{A1.2}$$

and let T_n be the stopping time corresponding to $\dot{y}_n$.

We shall show that a subsequence of the sequence of controls $(\dot{y}_n)$ converges (in a suitable sense) to an optimal control. The main difficulty is that, even if a sequence of controls $(\dot{y}_m)$ does converge, the corresponding sequence of stopping times (T_m) need not.†

Let L be the space of all measurable (real-valued) functions f on $[0, \infty)$ such that

$$\int_0^\infty \rho e^{-\rho t} [f(t)]^2 \, dt < \infty ,$$

and for f and g in L, write

$$\langle f, g \rangle = \int_0^\infty \rho e^{-\rho t} f(t) g(t) \, dt . \tag{A1.3}$$

Endow L with the metric corresponding to the "inner product" (A1.3), i.e., the *distance* between f and g is

$$\langle f - g, f - g \rangle^{1/2} . \tag{A1.4}$$

(With this metric, L is a Hilbert space, sometimes denoted $L^2(\mu)$, where μ is the measure with the density function $\rho e^{-\rho t}$.)

One says that the sequence (f_n) in L converges *weakly* to f in L if, for every g in L,

$$\lim_{n \to \infty} \langle f_n, g \rangle = \langle f, g \rangle ; \tag{A1.5}$$

in this case we denote the convergence by the symbols

$$f_n \xrightarrow[w]{} f . \tag{A1.6}$$

We shall use the following well-known proposition (see, e.g., Dunford and Schwartz, 1958, Theorem II. 3.28, p. 68).

Proposition. If a sequence (f_n) in L is uniformly bounded, i.e., if there is a K such that, for all n and t, $|f_n(t)| \leq K$, then there is a subsequence (f_m) of (f_n) that converges weakly to some f in L.

Returning to our sequence $(\dot{y}_n)$, define:

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† Except for dealing with this difficulty, our proof follows that of Ekeland and Turnbull (1983), Theorem 2, p. 86.

$$\dot{x}_n(t) = \begin{cases} \dot{y}_n(t), & 0 \le t < T_n, \\ 0, & t \ge T_n; \end{cases}$$

$$x_n(t) = \begin{cases} y_n(t), & 0 \le t \le T_n, \\ 0, & t \ge T_n. \end{cases}$$

From the constraints on $\dot{y}_n$, and the assumptions about the function m, it follows that the sequence $(\dot{x}_n)$ is uniformly bounded. Hence, by the foregoing Proposition, there is a function $\dot{x}$ in L and a subsequence of the sequence $(\dot{x}_n)$ that converges to $\dot{x}$. Without loss of generality, we shall suppose that the original sequence converges weakly to $\dot{x}$, i.e.,

$$\dot{x}_n \xrightarrow[w]{} \dot{x}. \tag{A1.7}$$

We shall show that $\dot{x}$ is an optimal control.

For every t and s, define

$$g_t(s) = \begin{cases} \left(\frac{1}{\rho}\right) e^{\rho s}, & s \le t, \\ 0, & s > t; \end{cases}$$

then

$$\begin{aligned} x_n(t) &= y_0 + \int_0^t \dot{x}_n(s)\, ds \\ &= y_0 + \langle \dot{x}_n, g_t \rangle . \end{aligned} \tag{A1.8}$$

Note that g_t is in L; hence, by (A1.7) and (A1.8), for every t,

$$\lim_{n\to\infty} x_n(t) = y_0 + \langle \dot{x}, g_t \rangle$$

$$= y_0 + \int_0^t \dot{x}(s)\, ds$$

$$= x(t). \tag{A1.9}$$

Note that (A1.9) states that x_n converges to x *pointwise*. Let

$$T = \inf\{t : x(t) = 0\}.$$

In order to show that $\dot{x}$ is optimal, we must show that

$$V(\dot{x}) = \int_0^T \rho e^{-\rho t} \left\{ m[x(t)] - \dot{x}(t) \right\} dt = \hat{V}. \tag{A1.10}$$

Write

$$c_n(t) = m[x_n(t)] - \dot{x}_n(t),$$
$$c(t) = m[x(t)] - \dot{x}(t); \tag{A1.11}$$

from (A1.2) and (A1.10), we want to show that

$$\lim_{n\to\infty} \int_0^{T_n} \rho e^{-\rho t} c_n(t)\, dt = \int_0^T \rho e^{-\rho t} c(t) dt, \tag{A1.12}$$

or at least that (A1.12) holds for some infinite subsequence of (c_n).

Observe that

$$\int_0^{T_n} \rho e^{-\rho t} c_n(t)\, dt = A_n - B_n, \tag{A1.13}$$

where

$$A_n = \int_0^T \rho e^{-\rho t} c_n(t)\, dt, \tag{A1.14}$$

$$B_n = \int_{T_n}^T \rho e^{-\rho t} c_n(t)\, dt. \tag{A1.15}$$

We consider first the sequence (A_n). Since the sequence $m[x_n]$ is uniformly bounded on $[0, T]$, it follows from (A1.9), the continuity of m, and Lebesgue's Dominated Convergence Theorem that

$$\lim_{n \to \infty} \int_0^T \rho e^{-\rho t} m[x_n(t)]\, dt = \int_0^T \rho e^{-\rho t} m[x(t)]\, dt . \qquad \text{(A1.16)}$$

If we define

$$h(t) = \begin{cases} 1, & t < T, \\ 0, & t \geq T, \end{cases}$$

then, since $(\dot{x}_n)$ converges weakly to $\dot{x}$,

$$\begin{aligned} \lim_{n \to \infty} \int_0^T \rho e^{-\rho t} \dot{x}_n(t)\, dt &= \lim_{n \to \infty} \langle \dot{x}_n, h \rangle \\ &= \langle \dot{x}, h \rangle \\ &= \int_0^T \rho e^{-\rho t} \dot{x}(t)\, dt . \end{aligned}$$

This last, with (A1.16) and (A1.11), implies

$$\lim_{n \to \infty} A_n = \int_0^T \rho e^{-\rho t} c(t)\, dt . \qquad \text{(A1.17)}$$

To study the behavior of (B_n), we first need:

Lemma 1.

$$\lim_{n \to \infty} \inf\ T_n \geq T .$$

Proof. First, if $x_n(t) = 0$ then $x_n(t') = 0$ for all $t' \geq t$. Hence, if $x_n(t) > 0$, then $x_n(t') > 0$ for all $t' < t$. Hence, if $x_n(t) > 0$ then $T_n > t$. Second, for every $t < T$, $x(t) > 0$; hence, by (A1.9), there exists N_t such that $n \geq N_t$ implies $x_n(t) > 0$, which (by the first observation) implies that $T_n > t$. Thus, for every $t < T$, $T_n \leq t$ only finitely often, which completes the proof of the lemma.

We now distinguish two cases.

Case 1. *$T_n < T$ infinitely often.*

In this case, it follows from Lemma 1 that there is a subsequence, say (T_k), of (T_n) such that $T_k < T$ and

$$\lim_k T_k = T .$$

Since the functions c_n are uniformly bounded on $[T_k, T]$, it follows that the sequence (B_k) converges to zero. Hence, in this case, $\dot{x}$ is optimal.

Case 2. *$T_n < T$ at most finitely often.*

In this case, there is a finite N such that $n \geq N$ implies $T_n \geq T$, and

$$-B_n = e^{-\rho T} \int_0^{T_n - T} \rho e^{-\rho t} c_n(T+t)\, dt . \tag{A1.18}$$

In (A1.1), note that $\hat{V}$ depends on the initial state, y_0; we indicate this dependence by the notation $\hat{V}(y_0)$. From (A1.1) and (A1.18) it follows that

$$|B_n| \leq e^{-\rho T} \hat{V}[x_n(T)] , \quad n \geq N . \tag{A1.19}$$

By (A1.9) and the fact that $x(T) = 0$

$$\lim_{n \to \infty} x_n(T) = 0 .$$

Note that $\hat{V} \geq 0$ and $\hat{V}(0) = 0$. Hence the proof for Case 2 will be complete if we can prove the following:

Lemma 2. **$\hat{V}$ is continuous at zero.**

Proof. Consider two problems, with different functions m, say m_1 and m_2, such that for every state y, $m_1(y) \geq m_2(y)$. It follows that the two corresponding functions $\hat{V}$ satisfy the same inequality, i.e., $\hat{V}_1(y) \geq \hat{V}_2(y)$ for all y. We shall prove the lemma by showing that it is true if we replace the given function m by another one that is everywhere as large but has a particularly simple form. Consider a function M of the form

$$M(y) = \begin{cases} \mu y\,, & 0 \le y \le \bar{y}\,, \\ \mu\bar{y} - \nu(y-\bar{y})\,, & y \ge \bar{y}\,, \end{cases} \tag{A1.20}$$

for some positive constants μ, ν, and $\bar{y}$. (Cf. the example of Section 6.)

For any function m satisfying our assumptions, there is a function of the form (A1.20) that majorizes m (see figure). Without loss of generality we take $\mu > \rho$.

The proof of the lemma is a bit lengthy, so we sketch it here. First, every consumption path is bounded between 0 and $\bar{c}$. Hence, for every $\epsilon > 0$ there exists a τ_ϵ such that, for all consumption paths $c(\cdot)$,

$$\int_{\tau_\epsilon}^{\infty} \rho e^{-\rho t} c(t)\,dt \le \epsilon\,. \tag{A1.21}$$

Second, we shall show that, for the function M, and for every $\tau > 0$,

$$\lim_{y_0 \to 0} \sup_{\dot{y}} \int_0^{\tau} \rho e^{-\rho t} c(t)\,dt = 0\,, \tag{A1.22}$$

where the sup in (A1.22) is subject to the constraints following (A1.1). The establishment of (A1.22) will be the core of the proof of the lemma. Combining (A1.21) and (A1.22), we shall then have the conclusion: for every $\epsilon > 0$,

$$\lim_{y \to 0} \hat{V}(y) \le \epsilon\,, \tag{A1.23}$$

which will complete the proof of the lemma.

It remains to establish (A1.22). Since any measurable control can be approximated by a simple control (i.e., a control that is constant on each of a finite set of intervals), it is sufficient to establish (A1.22) for simple controls. We do this by induction. Furthermore, *in this argument we take the consumption path $c(\cdot)$ to be the control,* rather than $\dot{y}(\cdot)$.

Referring to (A1.20), the specification of M, if y_0 is sufficiently small, then for all controls $c(\cdot)$,

$$y(t) < \bar{y}\,, \quad 0 \le t \le \tau\,. \tag{A1.24}$$

In what follows, we shall suppose this to be the case. The path $y(\cdot)$ then satisfies the differential equation

$$\dot{y}(t) = \mu y(t) - c(t). \tag{A1.25}$$

Without loss of generality, suppose that

$$\mu > \rho, \tag{A1.26a}$$

$$y_0 < \frac{\bar{c}}{\mu}. \tag{A1.26b}$$

As a first step in the argument, suppose that the consumption path is constrained to be constant, say equal to c, on the time interval $[0, \tau]$. The solution to the differential equation (A1.25) is then

$$y(t) = \left(y_0 - \frac{c}{\mu}\right) e^{\mu t} + \frac{c}{\mu}, \tag{A1.27}$$

and the total discounted utility on $[0, \tau]$ is

$$v = c(1 - e^{-\rho T}), \tag{A1.28}$$

where

$$T = \min\left[\tau, \inf\{t : y(t) = 0\}\right]. \tag{A1.29}$$

Define

$$V_1(y_0) = \max\{v : 0 \leq c \leq \bar{c}\}. \tag{A1.30}$$

We shall show that the optimal value of c is $\bar{c}$, and that

$$V_1(y_0) = \min \begin{cases} \bar{c}\left[1 - \left(1 - \dfrac{\mu y_0}{\bar{c}}\right)^{\frac{\rho}{\mu}}\right] = W(y_0), \\ \bar{c}(1 - e^{-\rho\tau}). \end{cases} \tag{A1.31}$$

The first line of (A1.31) will be applicable if y_0 is small enough so that, with $c = \bar{c}$, $y(t)$ reaches zero before time τ. To be precise, this will happen if $y_0 < \bar{y}$, where $\bar{y}$ is the solution of

$$\left(\bar{y} - \frac{\bar{c}}{\mu}\right)e^{\mu\tau} + \frac{\bar{c}}{\mu} = 0.$$

(Cf. equation (A1.27).)

To prove (A1.31), we see from (A1.27) and (A1.29) that T is nonincreasing in c; for sufficiently small c, $T = \tau$, whereas for larger c, T is strictly decreasing in c. It is therefore useful to distinguish two cases:

Case 1. $T = \tau$.

In this region, v is linear and strictly increasing in c.

Case 2. $T < \tau$.

In this region, T is determined by

$$\left(y_0 - \frac{c}{\mu}\right)e^{\mu T} + \frac{c}{\mu} = 0; \tag{A1.32}$$

cf. (A1.27). By a straightforward calculation one gets

$$v = c\left[1 - \left(1 - \frac{\mu y_0}{c}\right)^{\frac{\rho}{\mu}}\right], \tag{A1.33}$$

$$\frac{dv}{dc} = 1 - \left(1 - \frac{\mu y_0}{c}\right)^{\frac{\rho}{\mu}} - \frac{\rho y_0}{c}\left(1 - \frac{\mu y_0}{c}\right)^{\frac{\rho}{\mu} - 1}. \tag{A1.34}$$

Another straightforward calculation shows that $dv/dc > 0$. Hence, combining the two cases, we see that $dv/dc > 0$ on $[0, \bar{c}]$, so that $\bar{c}$ is the optimal value of c. The formula (A1.31) then follows from (A1.33), provided the solution of (A1.32) is no greater than τ; in any case, the right-hand-side of (A1.31) is an upper bound on $V_1(y_0)$.

Now consider a finite number of points t_k ($k = 1, \ldots, K$), with

$$0 = t_K < t_{K-1} < \cdots < t_1 < \tau \tag{A1.34}$$

and suppose that

$$c(t) = \begin{cases} c_k, & \text{for} \quad t_k \le t < t_{k-1}, \ k \ge 2, \\ c_1, & \text{for} \quad t_1 \le t \le \tau. \end{cases} \tag{A1.35}$$

(Note that the points t_k *increase* as k decreases.) Given the points (t_k), let $V_K(y_0)$ denote the maximum total discounted consumption on the interval $[0, \tau]$ that is attainable with consumption paths of the form (A1.35). We shall show that there is a positive constant R such that

$$V_K(y_0) \le y_0 R e^{(\mu - \rho)(t_1 - t_K)}; \tag{A1.36}$$

furthermore, R is independent of the number and position of the points (t_k).

We first calculate R. From the first line of (A1.31),

$$W'(y_0) = \rho \left(\frac{\bar{c}}{\bar{c} - \mu y_0} \right)^{1 - \frac{\rho}{\mu}},$$

$$W'(0) = \rho. \tag{A1.37}$$

It follows from these properties of W' and from (A1.31) that, for any upper bound on y_0, say $\bar{y}_0$, there is an $R > \rho$ such that

$$V_1(y_0) \le R y_0. \tag{A1.38}$$

Furthermore, by making the upper bound $\bar{y}_0$ sufficiently small, one can take

$$\rho < R < \mu.$$

We shall also take $\bar{y}_0$ sufficiently small so that (A1.24) and (A1.26b) are satisfied.

We have shown in (A1.38) that (A1.36) holds for $K = 1$. Suppose now that (A1.36) holds for K; we shall show that it holds for $(K + 1)$. Using a standard "dynamic programming" argument, for given points (t_k),

$$V_{K+1}(y_0) = \max_c \left\{ c(1 - e^{-\rho T}) + e^{-\rho T} V_K[y(T)] \right\}, \tag{A1.39}$$

where

$$T = \min \left[t_K, \ \inf\{t : y(t) = 0\} \right].$$

By (A1.36), the expression in curly brackets in (A1.39) does not exceed

$$v = c(1 - e^{-\rho T}) + e^{-\rho T} y(T) R e^{(\mu - \rho)(t_1 - t_K)} . \tag{A1.40}$$

From (A1.27),

$$y(T) = \left(y_0 - \frac{c}{\mu} \right) e^{\mu T} + \frac{c}{\mu} ;$$

substituting this in (A1.40) and rearranging terms we see that

$$v = c f(T) + R S e^{(\mu - \rho) T} y_0 , \tag{A1.41}$$

where

$$f(T) = 1 - e^{-\rho T} - \left(\frac{RS}{\mu} \right) e^{-\rho T} (e^{\mu T} - 1) ,$$

$$S = e^{\mu(t_1 - t_K)} .$$

We shall now show that v is decreasing in c for small c, and increasing for large c, but that v is maximized at $c = 0$.

Case 1. $T = t_K$.

In this case, from (A1.41), v is linear in c, and the coefficient of c is $f(t_K)$. We shall show that $f(t_K) < 0$, and hence that $dv/dc < 0$. Note that $f(0) = 0$, and

$$f'(x) = \rho e^{-\rho x} + \left(\frac{\rho RS}{\mu} \right) e^{-\rho x} (e^{\mu x} - 1)$$

$$- RS e^{-\rho x} e^{\mu x} .$$

Since $\rho < R < \mu$,

$$\begin{aligned} f'(x) &< \rho e^{-\rho x} + \rho S e^{-\rho x} (e^{\mu x} - 1) - \rho S e^{-\rho x} e^{\mu x} \\ &= \rho e^{-\rho x} (1 - S) \\ &< 0 , \end{aligned}$$

the last inequality following from $S > 1$. Hence $f(x) < 0$ for $x > 0$.

Case 2. $T < t_K$.

In this case, $y(T) = 0$, and so from (A1.40)

$$v = c(1 - e^{-\rho T}).$$

Comparing this with (A1.28) we see that we have already analyzed this expression in Case 2 of the study of $V_1(y_0)$, where we showed that $dv/dc > 0$.

Hence the maximum of v is attained either at $c = 0$ or at $c = \bar{c}$. If $c = 0$, then $T = t_K$ and, from (A1.41),

$$\begin{aligned} v &= RSe^{-(\mu - \rho)t_K} y_0 \\ &= y_0 Re^{(\mu - \rho)t_1} \end{aligned} \tag{A1.42}$$

which is the right-hand-side of (A1.36). (Recall that $S = \exp[\mu(t_1 - t_K)]$.) If $c = \bar{c}$ and $T = t_K$, then by Case 1, $dv/dc < 0$, so v is not maximized at $\bar{c}$. If $c = \bar{c}$ and $T < t_K$, then

$$v = V_1(y_0) \leq R y_0,$$

which is less than (A1.42). This establishes that (A1.36) holds for $(K + 1)$, and completes the proof of Lemma 2, and of the existence of an optimal control.

With an extension of the preceding argument one can easily go on to derive the exact form of the optimal control in the example of Section 6. We omit the details.

A2. Proof of Proposition of Section 3

To prove the Proposition, we shall first slightly transform the problem into an equivalent one, and then apply Pontryagin's Maximum Principle to characterize the optimal paths for various configurations of m and the initial conditions.

We first note that, since $m(y) \leq 0$ for $y \leq 0$, once $y(t)$ reaches 0 it can never again become strictly positive. Indeed, $y(t)$ will become and stay strictly negative as soon after time T as $c(t)$ is strictly positive on some open interval.

In summary, for an optimal path either $y(t)$ has a limit ($T = \infty$), or T is finite; in the latter case, positive utility (consumption) could not be accumulated after time T without causing $y(t)$ to become and stay strictly negative.

It follows that we can replace our optimization problem by the following equivalent problem, which is in a form more convenient for the application of Pontryagin's Maximum Principle:

Choose $T \geq 0$ (possibly infinite) and a piecewise-continuous function $c(\cdot)$ to maximize $z(T)$ subject to:

$$\dot{z}(t) = e^{-\rho t} c(t), \tag{A2.1a}$$

$$\dot{y}(t) = m[y(t)] - c(t), \tag{A2.1b}$$

$$0 \leq c(t) \leq \bar{c}, \tag{A2.1c}$$

$$z(0) = 0, \quad y(0) = y_0, \tag{A2.1d}$$

$$y(T) \geq 0 \text{ if } T < \infty, \quad \liminf_{T \to \infty} y(t) \geq 0 \text{ if } T = \infty. \tag{A2.1e}$$

In the above formulation, $y_0 > 0$ is given.

Second, we note that the optimal consumption policy may be taken to be stationary as shown in Lemma A2.5 below, so that the differential equation governing $y(\cdot)$ is autonomous,

$$\dot{y}(t) = m[y(t)] - \gamma[y(t)],$$

i.e., the time derivative of the stock depends only on the current stock. (Note that this is true even if m and γ are discontinuous.) Hence *for an optimal path, $y(\cdot)$ is monotone* (for $y(t) \geq 0$). From this it follows that either (1) $T < \infty$ or (2) $T = \infty$ and $\lim_{t \to \infty} y(t) = y(\infty)$ exists.

Because of (A2.1a), we are dealing with a so-called non-autonomous system, and hence we shall use Theorem 4, p. 59, of (Pontryagin et al., 1964 hereafter referred to as PBGM).

To state the necessary conditions, we need some additional notation. Let ϕ be a number, ψ a function of time, and H a function of two variables, consumption and time, defined by

$$H(c, t) = \phi e^{-\rho t} c + \psi(t)[m(y(t)) - c]. \tag{A2.2}$$

The function H is called the *Hamiltonian*. By Theorem 4, p. 59, of PBGM, if $\hat{c}(\cdot)$ is an optimal consumption path, subject to (A2.1a-f), then:

1) for every t, $0 \leq t \leq T$, $\hat{c}(t)$ maximizes $H(c, t)$ subject to $0 \leq c \leq \bar{c}$; (A2.3a)

2) $\phi \geq 0$; (A2.3b)

3) $\dot{\psi}(t) = -\psi(t)\, m'[y(t)]$ (A2.3c)

4) $\max_{T} H(c, T) = 0.$ (A2.3d)

We shall call (A2.3a)-(A2.3d) the conditions NC. In case $T = \infty$, we have to interpret the so-called "transversality" condition (A2.3d) as

$$\lim_{t \to \infty} H[\hat{c}(t), t] = 0. \qquad \text{(A2.3d')}$$

One can show† that $\phi > 0$; hence we can take $\phi = 1$. Define $\lambda(t) = e^{\rho t}\psi(t)$; then H can be written

$$H(c, t) = e^{-\rho t}[(1 - \lambda(t))c + \lambda(t)m(y(t))], \qquad \text{(A2.4)}$$

and (A2.3c) becomes

$$\dot{\lambda}(t) = \lambda(t)\,[\rho - m'(y(t))]. \qquad \text{(A2.5)}$$

Hence (A2.3a) implies:

Lemma A2.1.

$$\hat{c}(t) = \begin{Bmatrix} \bar{c} \\ 0 \end{Bmatrix} \text{ as } \lambda(t) \begin{Bmatrix} < \\ > \end{Bmatrix} 1. \qquad \text{(A2.6)}$$

† If $\phi = 0$, necessary conditions (see PBGM) require $\psi(t) \neq 0$ for all t. Consider first the case where $m(y_0 < \bar{c}$. If $\psi(t) < 0$ (A2.3a) implies $c(t) = \bar{c}$. Then (A3.d) implies that $\psi(T)[n(y(T)) - \bar{c}] = 0$ which is not possible since T must be finite, $y(T) = 0$ and $m(0) \leq 0$. Therefore $\psi(t) > 0$. But then (A2.3a) implies $c(t) = 0$ for all t, $m(y_0) \leq \bar{c}$, $c(t) = \bar{c}$ is feasible and clearly optimal without recourse to any of the technical arguments above. (See Cases 1.2, parts II and III and 1.3 below.)

To complete the proof of the Proposition, we consider in turn each of the three cases listed there.

Case 1. *$m(y) > 0$ for all $y > 0$ in some neighborhood of 0.*

In this case, $m(y) > 0$ for $0 < y \leq \hat{y}$. We must consider three subcases, according as $\bar{c} > \hat{y}$ or $\bar{c} \leq \hat{y}$.

Case 1.1. *$\bar{c} > m(y)$ for all y.*

This case will be treated in some detail. The remaining cases can then be disposed of briefly.

Suppose first that $y_0 < \hat{y}$. We shall show that the optimal path y will increase to $\hat{y}$.

Lemma A2.2. If $0 < y_0 < \hat{y}$, then $\lambda(0) > 1$.

Proof: If $\lambda(0) = 1$ then, by (A2.5), $\dot{\lambda}(0) < 0$ and $\lambda(t) < 1$ for all sufficiently small positive t; hence if $\lambda(0) \leq 1$ we may, without loss of generality suppose that $\lambda(0) < 1$. In this case, $c(0) = \bar{c}$ (by Lemma A2.1), $\dot{y}(0) < 0$ by (A2.1b)), and $y(t)$ would start out decreasing. Either $\lambda(0) \leq 0$ and $\dot{\lambda}(0) \geq 0$, or $\lambda(0) \geq 0$ and $\dot{\lambda}(0) \leq 0$; in either case $\lambda(t)$ would remain less than 1 until $y(t)$ reaches 0, at some finite time T. But then $\lambda(T) < 1$, $c(T) = \bar{c}$, $m[y(T)] = m(0) = 0$, so that $H[c(T), T] > 0$, contradicting the "transversality" condition, (A2.3d).

Having proved Lemma A2.2, we know that $\lambda(0) > 1$. Hence $c(0) = 0$, $\dot{y}(0) > 0$, and y is increasing; also $\dot{\lambda}(0) < 0$ and λ is decreasing. If, for some t_1, $\lambda(t_1) = 1$ and $y(t_1) < \hat{y}$, then $\dot{\lambda}(t_1) < 0$, and immediately thereafter $\lambda(t)$ would be < 1, $c(t)$ would equal $\bar{c}$, and $\dot{y}(t)$ would be < 0, which would contradict the *monotonicity* of the optimal y. *Hence $y(t)$ reaches $\hat{y}$ in finite time t_1, and*

$$\lim_{t \to t_1} \lambda(t) \geq 1 . \tag{A2.7}$$

Note that $\lambda(t_1) = 1$ can be achieved by integrating (A2.5) and choosing an appropriate value of the constant of integration. This last and the continuity of λ together imply that $\lambda(t_1) = 1$. Also, $y(t_1) = \hat{y}$ implies that $\dot{\lambda}(t_1) = 0$, and the monotonicity of y implies that $\dot{y}(t_1) \leq 0$. Indeed, $\dot{y}(t_1)$

must be zero, otherwise $y(t)$ would fall below $\hat{y}$ for $t > t_1$; but since y is autonomous, it would then have to increase again (by reference to the case in which $y_0 < \hat{y}$). Hence

$$y(t) = \hat{y}, \quad \lambda(t) = 1, \quad \text{for } t \geq t_1. \tag{A2.8}$$

This, in turn, implies that

$$c(t) = m(\hat{y}), \quad \text{for } t \geq t_1. \tag{A2.9}$$

Now consider the case in which $y_0 > \hat{y}$. We shall show that the optimal path y will decrease to $\hat{y}$.

Lemma A2.3. If $y_0 > \hat{y}$, then $\lambda(0) < 1$.

Proof: As in the proof of Lemma A2.2, if $\lambda(0)$ were ≥ 1, then we could without loss of generality take $\lambda(0) > 1$. This would imply

$$c(0) = 0, \quad \dot{y}(0) > 0, \quad \dot{\lambda}(0) > 0,$$

so that y and λ would increase, λ would remain greater than 1, and $c(t)$ would be 0 for all t. This is clearly non-optimal.

Hence

$$\lambda(0) < 0, \quad c(0) = \bar{c}, \quad \dot{y}(0) < 0, \quad \dot{\lambda}(0) > 0.$$

By an argument symmetric to that used for the case in which $y_0 < \hat{y}$, one can then show that $y(t)$ decreases to $\hat{y}$ at some finite time t_1, and that (A2.8) and (A2.9) hold for $t \geq t_1$.

Finally, if $y_0 = \hat{y}$, the stationarity of the optimal policy implies that (A2.8) and (A2.9) hold for all t (take $t_1 = 0$).

Notice that, for all initial conditions, $T = \infty$ and there is a $t_1 < \infty$ such that (A2.1) and (A2.2) hold. Hence, by (3.4), the transversality condition (A2.3d') is satisfied.

Notice, too, that we did not explicitly use the constraint (A2.1e) on the final stock, i.e., $y(T) = y_1$. In fact, we showed that the necessary conditions for an optimal policy could be satisfied only for $y_1 = \hat{y}$. In other words, no optimal policy exists for the problem (A2.1a-f) if $y_1 \neq \hat{y}$.

If for some y, $m(y) > \bar{c}$ (i.e., we are not in Case 1.1), then let $\bar{y}$ and $\bar{y}'$ be the two roots of $m(y) = \bar{c}$, with $\bar{y} \leq \bar{y}'$ (in the "boundary case", $\bar{y} = \bar{y}'$).

Case 1.2. $\hat{y} < \bar{y}$.

The analysis for $y_0 < \hat{y}$ is the same as that in Case 1.1. If $y_0 > \hat{y}$, then $\lambda(0) > 1$ as in Case 1.1, $\lambda(t)$ is decreasing, and

$$\lim_{t \to \infty} \lambda(t) = 1$$

(the limit may be reached in finite time). One easily verifies the following results:

I) If $\hat{y} \leq y_0 < \bar{y}$, then the situation is as in Case 1.1.

II) If $y_0 = \bar{y}$, then $y(t) = \bar{y}$ and $c(t) = \bar{c}$ for all t.

III) If $\bar{y} < y_0$, then $c(t) = \bar{c}$ for all t, and $y(t)$ converges to $\bar{y}'$ (possibly in finite time).

Case 1.3. $\bar{y} \leq \hat{y}$.

If $y_0 < \bar{y}$, then the analysis proceeds as in Case 1.1, except that one substitutes $\bar{y}$ for $\hat{y}$. If $y_0 \geq \bar{y}$, then the situation is as in Case 1.2.

Case 2. $m(y_0) \leq 0$.

This case requires only an elementary argument. Since $y(t)$ is monotone non-increasing, nothing can be gained by postponing consumption. Hence the optimal policy is to consume at the maximal rate until the stock falls to zero. Note that along the optimal path $m(y)$ must remain non-positive since y is non-increasing. Indeed, with discounting, postponing any consumption at all is strictly inferior.

Note that, since T is finite and $y(T) = 0$, the transversality condition (A2.3d) requires that (cf. (A2.4))

$$1 - \lambda(T)\bar{c} + \lambda(T)m(0) = 0$$

or equivalently,

$$\lambda(T) = \frac{\bar{c}}{\bar{c} - m(0)}. \tag{A2.10}$$

This determines the constant of integration for (A2.5). One easily verifies that $\lambda(0) < 1$ and that $\lambda(t)$ decreases monotonically to $\lambda(T)$.

Case 3. $m(0) < 0$ *but* $m(y_0) > 0$.

In this case it is easy to verify that both the frugal and prodigal policies can satisfy the necessary conditions for optimality. Let $\bar{y}$ and $\bar{y}'$ be the two roots of $m(y) = \bar{c}$, with $\bar{y} \leq \bar{y}'$, as in Case 1.2. First we note that if $\bar{y}$ exists and $y_0 \geq \bar{y}$, the prodigal policy $\bar{c}$ can be maintained indefinitely and is the optimal policy. We therefore concentrate on cases where $y_0 \leq \bar{y}$ so that $m(y_0) < \bar{c}$. Let $\check{y}$ be the smallest root of $m(y) = 0$, which must be smaller than $\bar{y}$. If $y_0 \leq \check{y}$, Case 2 is applicable so we shall confine our attention to the case where $y_0 \in (\check{y}, \bar{y})$. We shall also limit ourselves to the case where $\hat{y} \leq \bar{y}$ since the case where $\hat{y} > \bar{y}$ can be treated with minor modifications, as was done for the Case 1.2.

The prodigal policy will satisfy the necessary conditions for optimality in exactly the same way as in Case 2, with $\lambda(0) < 1$, $\lambda(t)$ decreasing, $\lambda(T) = \dfrac{\bar{c}}{\bar{c} - m(0)}$ and $y(T) = 0$. The frugal policy will also satisfy the necessary conditions in exactly the same way as described in Case 1.1. Since there are two paths satisfying the necessary conditions, we must compare their values to determine the best one.

For $y_0 < \hat{y}$, the total utility generated under the prodigal policy is $\bar{c}(1 - e^{-\rho T})$ and total utility generated under the frugal policy is $e^{-\rho t_1} m(\hat{y})$. (As before T is the smallest t such that $y(t) = 0$ under the prodigal policy and t_1 is the first t such that $y(t) = \hat{y}$ under the frugal policy.) The difference, which depends on y_0, is given by

$$D(y_0) = \bar{c}(1 - e^{-\rho T}) - e^{-\rho t_1} m(\hat{y}).$$

Note that $\lim_{y_0 \to \check{y}} t_1 = \infty$ so that $D(y_0) > 0$ for y_0 sufficiently close to $\check{y}$.

If $y_0 \geq \hat{y}$, the frugal and prodigal policies are identical until $\hat{y}$ is reached and the sign of the difference in total utilities is independent of y_0. The difference, in this case, is given by

$$D(y_0) = e^{-\rho t_1}[\bar{c}(1 - e^{-\rho(T-t_1)}) - m(\hat{y})].$$

where $(T - t_1)$ is the time it takes to decumulate $\hat{y}$ to zero under the prodigal policy. For $y_0 = \hat{y}$, t_1 is zero and $D(\hat{y}) < 0$ if $\bar{c}(1 - e^{-\rho T}) - m(\hat{y}) < 0$.

Although the sign of $D(y)$ can be positive or negative, we can establish that there exists at most one critical point y_c such that $D(y) > 0$ for $y \leq y_c$ and $D(y) \leq 0$ for $y \geq y_c$.

Lemma A2.4. $D(y)$ can cross zero at most once in the interval $[\tilde{y}, \bar{y})$.

Proof: Suppose to the contrary that there exists y_c and y_c' such that $D(y_c) = D(y_c') = 0$ and $D(y) < 0$ for $y \in [x, y_c'] = I$, where $y_c \leq x < y_c'$. First consider the case where $y_c' \neq \hat{y}$. Since the only optimal policies are the frugal and prodigal policies, $D(y) < 0$ for y in I implies that the frugal policy dominates the prodigal policy over the interval I, and that the prodigal policy is optimal in a right-neighborhood of y_c'. It follows that y_c' is a stationary point of an optimal path. This contradicts the necessary conditions that rule out stationary points other than $\hat{y}$. Now consider the case $y_c' = \hat{y}$ so that $D(\hat{y}) = 0$. Then it follows that $D(y) = 0$ for $y > \hat{y}$ because the frugal and prodigal policies are identical for y greater than $\hat{y}$. Therefore $D(y)$ cannot cross zero for values of y greater than $\hat{y}$. This completes the proof of Lemma A2.4.

Note that if y_c is a root of $D(y) = 0$ in the interval $[\tilde{y}, \hat{y}]$, it must be unique by Lemma A2.4. Therefore a prodigal policy is optimal for $y \leq y_c$ and a frugal policy is optimal for $y \geq y_c$, as claimed in part (3) of Proposition 1.

Lemma A2.5. The optimal consumption path may be taken to be stationary.

Proof: Recall that the form of the transversality condition (A2.3d') depended on the optimal path $y(\cdot)$ having a limit as t tends to infinity, when T is infinite. This last property followed from the monotonicity of $y(\cdot)$, which in turn followed from the stationarity of the optimal consumption policy. If the optimal consumption policy were not (necessarily) stationary, then the transversality condition (A2.3d) would take the more general form:

$$\lambda(T)y(T) = 0, \quad \text{if } T < \infty$$

$$\lim_{t \to \infty} e^{-\rho t} \lambda(t) y(t) = 0, \quad \text{if } T = \infty \tag{A2.11}$$

(see Kamien and Schwartz, 1981, pp. 147-148). By (A2.3a), $c(0)$ depends on $\lambda(0)$ such that $c(0) = 0$ if $\lambda(0) > 1$, $c(0) = \bar{c}$ if $\lambda(0) < 1$ and $c(0) \in [0, \bar{c}]$ if $\lambda(0) = 1$. Using arguments identical to those in Lemma A2.2 and A2.3, it is easily shown that $\lambda(0) = 1$ if and only if $y(0) = \hat{y}$. (Otherwise we either contradict (A2.11) or have to set $c(t) = 0$ for $t > 0$, which is clearly non-optimal.) Furthermore the same arguments require that if $y(0) = \hat{y}$, then $c(0) = m(\hat{y})$; it follows that $\lambda(0) = 1$ requires $c(0) = m(\hat{y})$. Therefore, since $\lambda(0) < 1$ for $y(0) > \hat{y}$ by Lemma A2.3 and $\lambda(0) > 1$ for $0 < y(0) < \hat{y}$ by Lemma A2.2, $y(0)$ determines $c(0)$.

Under the above policy rules for consumption, (A2.1b) and (A2.5) form an autonomous pair of differential equations; $(\lambda(t), y(t))$ is completely determined by $(\lambda(0), y(0))$. We must show therefore that $\lambda(0)$ is determined by $y(0)$ alone. Given $y(0)$, $\lambda(0)$ is determined by the transversality condition, (A2.11), and by $y(0)$.

A3. Proof of Theorem 5.4

Without loss of generality, assume $c_2(t) = 0$, $c_1(t) \geq 0$ in some interval $[t_0, t_1)$. The policy $c_1(t)$, if it is to be optimal over $[t_0, t_1)$, must maximize

$$V(y(t_0)) = \rho \int_{t_0}^{t_1} c_1(t) e^{-\rho t} dt + e^{-\rho t_1} V(y(t_1)), \tag{A3.1}$$

where $\dot{y} = m(y) - c_1(t)$. Here, $V(y(t_1))$ is the value function for player 1 from stocks starting at $y(t_1)$.

First we note that under stationary strategies where $c_i(t) = \gamma_i(y(t))$, $i = 1, 2$, $y(t)$ cannot remain fixed over $[t_0, t_1)$, since that would confine player 2 to consume nothing at all forever, and this cannot be a stationary equilibrium policy. We also note that the necessary conditions (2.5) and (2.6) also apply to the problem (A3.1). From (A2.5), since $y(t)$ is not constant, we observe that

$\dot{\lambda} \neq 0$, $\lambda \neq 1$ except at some isolated values of t. Therefore, from (A2.6), for almost all t, $c_1(t) = \bar{c}$ or $c_1(t) = 0$. Suppose now that $c_1(t) \neq \bar{c}$ over some subinterval $[t_0, t_1')$ so that both players are not consuming (except possibly at isolated values of t) and suppose that one or both of the players expect to switch to maximal consumption at some t_1'. Then irrespective of whether $y(t)$ grows or declines after t_1', both players have an incentive to switch to a positive level of consumption prior to but sufficiently close to t_1'. The reason is that if $y(t)$ can grow after a player switches to maximal consumption, he could have done better by switching earlier. If $y(t)$ declines or remains constant, one of the players could do better by switching to a consumption level that maintains $y(t)$ just below $y(t_1')$. Therefore the pair $c_1(t) = 0$, $c_2(t) = 0$ is not an equilibrium over an interval of time of positive length. This concludes the proof.

A4. The Utility of Defection

We first prove (7.1). Recall that the utility to the defector of defecting at time 0, with $y(0) = y$, is

$$D(y) = \bar{c}(1 - e^{-\rho T(y)}), \tag{A4.1}$$

where $T(y)$ is the first time that the stock reaches 0. Recall also that

$$\bar{c}\tau = \delta, \tag{A4.2}$$

where τ is small and $\bar{c}$ is large. From (A4.1),

$$D'(y) = \rho\bar{c}e^{-\rho T(y)}T'(y). \tag{A4.3}$$

We are interested in the case in which $y < \hat{y}$.

Two lemmas will be useful:

Lemma 1. Suppose that $0 \leq a < b$, $a < y < b$, $a \leq z \leq b$, $y \neq z$, and $x(\cdot)$ is the solution of the differential equation,

$$x'(t) = f[x(t)],$$

$$x(0) = y,$$

where f is continuous and nonzero on $[a, b]$. Suppose further that $x(t)$ reaches z in finite time, so that for $T < \infty$, T is the first t such that $x(t) = z$; then

$$T = \int_y^z \frac{dx}{f(x)},$$

$$\frac{dT}{dy} = -\frac{1}{f(y)}.$$

Proof of Lemma 1. Provisionally fix y. Since f is continuous and nonzero, $x(\cdot)$ is strictly monotone, and hence has an inverse, say $g(\cdot)$; i.e., $t = g(x)$ along the path defined by the solution of the differential equation. Also, $g(y) = 0$ and $g(z) = T$; hence

$$T = g(z) - g(y) = \int_y^z g'(x)dx.$$

But along the path,

$$g'(x) = \frac{1}{x'(t)} = \frac{1}{f(x)},$$

so that

$$T = \int_y^z \frac{dx}{f(x)}.$$

Differentiate this with respect to y to complete the proof of the lemma.

Lemma 2. Under the hypothesis of Lemma 1, let s be a fixed time such that

$$a < x(s) < b;$$

then

$$\frac{dx(s)}{dy} = \frac{f[x(s)]}{f(y)}.$$

Proof of Lemma 2. Let $x(t)$ be the solution of the differential equation corresponding to $y(0) = y + h$, and let k_h be the time it takes to go from y to $y + h$, i.e., $x_0(k_h) = y + h$. First,

$$\lim_{h \to 0} k_h = 0;$$

hence

$$\lim_{h \to 0} \frac{x_0(s + k_h) - x_0(s)}{k_h} = x_0'(s) = f[x_0(s)] .$$

Second,

$$\lim_{h \to 0} \frac{k_h}{h} = \lim_{h \to 0} \frac{k_h}{x_0(k_h) - x_0(0)} = \frac{1}{f(y)} .$$

But

$$\frac{x_h(s) - x_0(s)}{h} = \frac{x_0(s + k_h) - x_0(s)}{k_h} \cdot \frac{k_h}{x_0(k_h) - x_0(0)} ,$$

so that the conclusion of the lemma follows immediately from the above two limits.

We turn now to analysis of (A4.3). We shall see that $T'(y)$ and $D'(y)$ have a discontinuity at that value of y, say y_τ, for which $T(y) = \tau$. Therefore we shall distinguish two cases, according to which y is less than or greater than y_τ.

Case 1. $y < y_\tau$, $T < \tau$.

Recall that for $t < \tau$, only the defector is consuming at rate $\bar{c}$, and the nondefector is not consuming at all. Hence, by Lemma 1,

$$T'(y) = \frac{-1}{n(y) - \bar{c}} = \frac{1}{\bar{c} - n(y)}$$

$$D'(y) = \frac{\rho \bar{c} e^{-\rho T(y)}}{\bar{c} - n(y)} , \quad y < y_\tau . \qquad \text{(A4.4)}$$

Case 2. $y > y_\tau$, $T > \tau$.

Recall that for $t > \tau$, each player is consuming at rate $\bar{c}$. Let $S = T - \tau$, then

$$\frac{dT}{dy} = \frac{dS}{dy}$$

$$= \frac{dy(\tau)}{dy} \cdot \frac{dS}{dy(\tau)} .$$

By Lemma 1,

$$\frac{dS}{dy(\tau)} = \frac{-1}{n[y(\tau)] - 2\bar{c}} = \frac{1}{2\bar{c} - n[y(\tau)]}.$$

By Lemma 2,

$$\frac{dy(\tau)}{dy} = \frac{n[y(\tau) - \bar{c}]}{n(y) - \bar{c}}.$$

Hence

$$T'(y) = \frac{\bar{c} - n[y(\tau)]}{(2\bar{c} - n[y(\tau)])(\bar{c} - n[y])}$$

$$D'(y) = \frac{\rho\,\bar{c}\,e^{-\rho T(y)}(\bar{c} - n[y(\tau)])}{(2\,\bar{c} - n[y(\tau)])(\bar{c} - n(y))}, \quad y > y_\tau. \qquad \text{(A4.5)}$$

By (A4.2), as τ goes to zero both $n(y) - \bar{c}$ and $n(y) - 2\bar{c}$ go to minus infinity, and hence T goes to zero. We can also show that y_τ goes to δ, as follows. Write

$$y(\tau) - y(0) = \int_0^\tau y'(t)dt$$

$$= \int_0^\tau (n[y(t)] - \bar{c})dt,$$

so that

$$y(0) - y(\tau) = \int_0^\tau \left(\frac{\delta}{\tau} - n[y(t)]\right)dt$$

$$= \delta - \int_0^\tau n[y(t)]dt.$$

Hence,

$$\lim_{\tau \to 0} [y(0) - y(\tau)] = \delta.$$

If we take $y(0) = y_\tau$, $y(\tau) = 0$, then we get

$$\lim_{\tau \to 0} y_\tau = \delta. \qquad \text{(A4.6)}$$

Putting together (A4.4)-(A4.6), we get

$$\lim_{\tau \to 0} D'(y) = \begin{cases} \rho, & 0 < y < \delta, \\ \frac{\rho}{2}, & y > \delta. \end{cases} \tag{A4.7}$$

This last, together with $D(0) = 0$, implies

$$\lim_{\tau \to 0} D(y) = \begin{cases} \rho y, & 0 \le y \le \delta, \\ \frac{\rho(\delta + y)}{2}, & y \ge \delta, \end{cases} \tag{A4.8}$$

which is (7.1).