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ABSTRACT

A misinterpretation of turnpike theorems suggests that persistent cyclic behavior in optimal growth models is only possible with large discount rates. This paper shows that for any positive discount rate there exists a large family of standard Cobb-Douglas technologies with three sectors which have optimal growth paths of persistent cycles.

Introduction

The possibility of persistent cycles in intertemporal equilibrium models, with or without infinitely lived agents, has already been established. Nevertheless, there is the misperception that such cycles are incompatible with long-lived agents and low discount rates.(1) This in part is due to a misinterpretation of turnpike theorems which, in optimal growth contexts, state that given a technology and the preferences of an infinitely lived representative agent, there exists a sufficiently low discount rate for which the equilibrium path converges to a steady state. The recent elegant results of Boldrin and Montrucchio [1986] in discrete time and Montrucchio [1988] in continuous time, on the other hand, show that any differentiable difference equation (differential equation, in continuous time) can be shown to correspond to the policy function of some well behaved concave dynamic program with sufficiently large discount rate. Paradoxically, these results may have also contributed to the misperception that a large amount of discounting is necessary to obtain cyclic trajectories. The early work of Cass and Shell [1976] and Brock and Scheinkman [1976] however make it clear that there is a trade-off between the level of discounting and the "curvature" of preferences plus technology in establishing turnpike theorems. The counterpart of the turnpike theorems can also be established by exploiting this trade-off: for any given positive discount rate there exists a "large" family of "nice" technologies that can generate persistent cycles as optimal growth paths. This paper, generalizing the method by which Benhabib and Nishimura [1979] constructed examples of persistent cycles, formally establishes this result in theorem 1 below. What is meant by "large" families is made precise. "Nice" technologies refer to classical Cobb-Douglas production functions with constant returns to scale and non-joint production. We should note that an

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(1) See for example Woodford, (1987).

issue not addressed in Boldrin-Montrucchio [1986] and Montrucchio [1988] is whether the concave programs that can generate cyclic or chaotic trajectories are compatible with constant returns to scale and non-joint production functions. In the last section we discuss how the results can be extended via perturbation to non-linear utility functions and non constant returns to scale. We also note that the results of this paper can easily be extended to discrete time models (like the one in Benhabib and Nishimura, (1985)) using similar methods.

1. Cycles in neoclassical growth models

We shall consider three Cobb-Douglas production functions:

$$y_i = b_i \prod_{j=0}^{2} k_{ji}^{\alpha_{ji}} , \qquad i = 0,1,2, \tag{1.}$$

$$\sum_{j=0}^{2} \alpha_{ji} = 1 \qquad \text{(constant returns)} , \tag{2.}$$

where y_0 is consumption good; y_1, y_2 are the capital goods; k_{0i} is the labor input to the ith good; k_{ji} is jth input to the ith good. We refer to this family of production functions as a three sector Cobb-Douglas technology. Maximizing y_0, subject to y_1, y_2 given, and to

$$\sum_{i=0}^{2} k_{0i} = 1 \ , \ \sum_{i=0}^{2} k_{ji} = k_j \ , \tag{3.}$$

yields $y_0 = \tau(y_1, y_2, k_1, k_2, 1)$. We rewrite this as $y_0 = c = T(y_1, y_2, k_1, k_2)$, where lower case letters indicates per capita values. T defines the efficiency frontier. We can now define an optimal growth problem:

$$\begin{gathered} Max \int_0^{\infty} T(y_1, y_2, k_1, k_2)\, e^{-(r-g)t} dt \ , \\ \text{subj. to} \quad \dot{k}_i = y_i - gk_i \ , \quad i = 1,2 ; \\ k_1(0), k_2(0) \text{ given} ; \end{gathered} \tag{4.}$$

the maximization is with respect to the paths y_1 and y_2, where $y_i(t) \geq 0$, $i = 1,2$, and

$T(y_1, y_2, k_1, k_2) \geq 0$; here $r-g$ is the discount rate and g is the depreciation rate plus the population growth rate; we assume $r-g>0$.

Simple assumptions assure that (k_1, k_2) will be bounded. We can now state the main Theorem:

Theorem 1.

For every positive discount rate $\delta^* = r - g > 0$ *and a representative agent with linear instantaneous utility function, there exists an eight parameter family of 3-sector Cobb-Douglas technologies (with non-joint production and Constant Returns to Scale (CRS)), such that for every open neighborhood* N *of* δ^**, and any choice of technology from the above family and some* $\delta \in N$ *the associated optimal growth problem has non-constant cycles in stocks, prices and outputs.*

The Cobb-Douglas economies which satisfy this condition are determined by a set of 8 parameters, which contains an open set.

Remark.

It is known that for economies with one capital good there can be no cycles. At least three capital goods are necessary in our construction. For economies with a larger number of capital goods the existence of periodic solutions is an obvious consequence of this result. In those cases the Cobb-Douglas technologies giving rise to cycles would constitute a family of more than eight parameters.

Proof.

The proof relies on the Hopf Bifurcation theorem. We briefly recall the main steps of the proof. For a statement and discussion of Hopf bifurcation, see Chow and Hale (1982).

From the Pontryagin Maximum Principle the necessary conditions for an optimum are:

5. $$\underset{y \in \mathbb{R}_+}{Max}\, H = T(y,k) + p(y-gk)$$

6. $$\dot{k}_i = y_i - gk_i$$

7. $$\dot{p}_i = -T_{k_i} + rg_i \,.$$

Let $T_{k_i} = w_i > 0$, $T_{y_i} = -p_i < 0$. Under mild conditions, this problem has a steady state (s.s.), denoted (k^*, y^*). For our Cobb-Douglas construction, a unique interior steady state will exist. In the neighborhood of the s.s., we will assume (it will be true for our technology) that $T_{y_i} < 0$ (T strictly concave in y, and also in k (not jointly)). The $\underset{y}{Max}\, H$ implies $q_i = -T_{y_i} = p_i$ since the maximum is interior. Furthermore, $-T_{y_i}(y,k) = p_i$ is invertible so that $y = y(k,p)$ (outputs are uniquely determined by factors and relative prices). Furthermore, $T_k = \tilde{w}(y(k,p),k) = w(k,p)$. (Since factor prices are determined by k,p). We therefore rewrite 6 and 7 above as

6′ $$\dot{k} = y(k,p) - gk$$

7′ $$\dot{p} = -w(k,p) + r$$

We now linearize the system given by 6′ and 7′ around the steady state. Let

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}$$

be the input coefficient matrix with a_{ij} the ith input to the jth good. a_{0j} is labor input to the jth good, a_{10} is the ith input to consumption good.

At the steady state,

8. $$[w_0, w]\left[\frac{a_{00}}{a_{\cdot 0}} \;\middle|\; \frac{a_{0\cdot}}{A}\right] = [1,p] \text{ (price = cost conditions)}$$

9. $$\left[\frac{a_{00}}{a_{\cdot 0}} \,\Big|\, \frac{a_{0\cdot}}{A}\right] \begin{bmatrix} c \\ y \end{bmatrix} = \begin{bmatrix} l \\ k \end{bmatrix} \text{ (full employment conditions) .}$$

Differentiating 9 and substituting for c from one of the equations in 9, we get

10. $$\left[\frac{\partial y}{\partial k}\right] = [A - (a_{\cdot 0} \cdot a_{0\cdot})/a_{00}]^{-1} .$$

Here we use the fact that input coefficients remain fixed, since p is fixed (Rycbzinski). Similarly, using Shepard's Lemma,

$$\left[\frac{\partial w}{\partial p}\right]' = [A - a_{\cdot 0} \cdot a_{0\cdot}/a_{00}]^{-1} = \left[\frac{\partial y}{\partial k}\right] .$$

where $'$ denotes transposition. Note that $\frac{\partial w}{\partial k} \equiv 0$.

Let $B = [A - a_{\cdot 0} \cdot a_{0\cdot}/a_{00}]$, thus linearizing around the steady state we derive:

11. $$J = \begin{bmatrix} B^{-1} - gI & [\partial y/\partial p] \\ 0 & -(B')^{-1} + rI \end{bmatrix}$$

where I is the identity matrix. Roots of J are roots of $B^{-1} - gI$, $-(B)')^{-1} + rI$.

To apply Hopf's Theorem, we have to show that for any $r > g$, we can construct a Cobb-Douglas technology with pure imaginary roots for $[-(B')^{-1} + rI]$. B^{-1} is derived from the input coefficient matrix implied by r via the non-substitution theorem. We first show that for any pair given by r and the steady state per unit input coefficient matrix, we can define a unique Cobb-Douglas technology. More precisely, once we define

12. $$\hat{A} \equiv \left[\frac{a_{00}}{a_{\cdot 0}} \,\Big|\, \frac{a_{0\cdot}}{A}\right]$$

we have:

Lemma 1.

For any given $r > 0$ *and steady state per unit input matrix* $\hat{A}$ *such that* $r\lambda_A < 1$, *there exists a unique set of s.s. prices* p_0, p, *and a unique Cobb-Douglas technology associated with it. Furthermore, given* $r > g$, *the steady state is unique.*

Proof of Lemma 1.

Since at the s.s. $w_i = rp_i$, $i = 1,2$, setting $w_0 = 1$ we have $p = a_0 + rp\,A$, and therefore $p = a_0[I - rA]^{-1}$, also $p_0 = a_{00} + rpa_{.0}$, (assuming of course conditions for $[I - rA]^{-1} \geq 0$, that is, dominant root of A, λ_A such that $r\,\lambda_A < 1$: this condition will have to be satisfied in our construction of the matrix A.). We have therefore that, for any pair (r,A), a vector p of prices and a vector w of factor costs are determined. We now proceed to determine the coefficients (b_i, α_{ij}) in the production functions:

$$y_i = b_i a_{0i}^{\alpha_{0i}}\, a_{1i}^{\alpha_{1i}}\, a_{2i}^{1-\alpha_{0i}-\alpha_{1i}}\ ,\ i = 0,1,2\ .$$

From cost minimization we derive:

$$\frac{w_0 a_{0i}}{w_1 a_{1i}} \equiv M_{1i} = \frac{\alpha_{0i}}{\alpha_{1i}}\ ;\ \frac{w_0 a_{0i}}{w_2 a_{2i}} \equiv M_{2i} = \frac{\alpha_{0i}}{1 - \alpha_{0i} - \alpha_{1i}}\ ;$$

and therefore, solving for α_{ij}:

$$\begin{aligned}
\alpha_{0i} &= M_{1i}M_{2i}(M_{1i} + M_{2i} + M_{1i}M_{2i})^{-1} \\
\alpha_{1i} &= M_{2i}(M_{1i} + M_{2i} + M_{1i}M_{2i})^{-1}\ , \\
1 - \alpha_{0i} - \alpha_{1i} &= M_{1i}(M_{1i} + M_{2i} + M_{1i}M_{2i})^{-1}\ .
\end{aligned}$$

Clearly α_{0i}, α_{1i}, $\alpha_{0i} + \alpha_{1i}$ are all in the interval (0,1). From the steady state condition $y = gk$ we now determine the vector (b_i), $i = 0,1,2$. This concludes the proof of lemma 1. ■

Lemma 1 shows that we can completely parametrize our economy by the triple $((r,\hat{A}),g)$, where $\hat{A}$ has to satisfy some restrictions (which will be discussed soon). We have now the last step: prove that we can determine a matrix $\hat{A}$ which also gives rise to a Hopf bifurcation. Denote

13. $$B \equiv [A - a_{\cdot 0} \cdot a_{0 \cdot} / a_{00}]$$

and define a subset of the B matrices, for any given $r > 0$, as

$$B_H(r) = \{B : \bar{b}_{11} + \bar{b}_{12} = 2r,\ (r - \bar{b}_{11})(r - \bar{b}_{22}) - \bar{b}_{12}\bar{b}_{21} > 0\},$$

where $\bar{b}_{ij}$ are the elements of the inverse of B. For any $B \in B_H(r)$, $-B^{-1} + rI$ has a pair of pure imaginary roots.

Notice that, because of equations 12 and 13, for every vector $(B,\ a_{\cdot 0},\ a_{0 \cdot},\ a_{00})$ there is a uniquely determined associated matrix $\hat{A}$ and vice versa.

Lemma 2.

For any $r > 0$ there exists a vector $(B,\ a_{\cdot 0},\ a_{0 \cdot},\ a_{00})$ such that (a) $B \in B_H(r)$, and (b) there exists a Cobb-Douglas technology for the associated $\hat{A}$ matrix. The set of parameters which satisfy (a) and (b) contains an open set in an 8-dimensional parameter space.

Proof of Lemma 2.

Recall first that for any $(B,\ a_{\cdot 0},\ a_{0 \cdot},\ a_{00})$ the matrix B^{-1} with elements $\bar{b}_{ij}$ and $\hat{A}$ are related by:

$$\hat{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & \dfrac{\bar{b}_{22}}{\mathrm{D}} + \dfrac{a_{10}a_{01}}{a_{00}} & -\dfrac{\bar{b}_{12}}{\mathrm{D}} + \dfrac{a_{10}a_{02}}{a_{00}} \\ a_{20} & \dfrac{\bar{b}_{21}}{\mathrm{D}} + \dfrac{a_{20}a_{01}}{a_{00}} & \dfrac{\bar{b}_{11}}{\mathrm{D}} + \dfrac{a_{20}a_{02}}{a_{00}} \end{bmatrix}$$

where $\mathrm{D} \equiv \bar{b}_{11}\bar{b}_{22} - \bar{b}_{12}\bar{b}_{21}$. In view of lemma 1, we now only need to show that we can choose the elements b_{ij} of B and a_{00}, $a_{0 \cdot}$, $a_{\cdot 0}$ so that the following conditions are satisfied:

(1) $\hat{A}$ is non negative;

(2) the row sums of $\hat{A}$ are less or equal to one;

(3) $\lambda_A r < 1$;

(4) $\bar{b}_{12} + \bar{b}_{22} = 2r, (\bar{b}_{11} - \bar{b}_{22})^2 - 4\bar{b}_{12}\bar{b}_{21} > 0.$

The following simple three steps procedure will prove our claim.

i) choose $\bar{b}_{11} \equiv b, \bar{b}_{22} = 2r - b$, with b any number in $[0,2r]$;

ii) choose $\bar{b}_{12}, \bar{b}_{21}$ such that $sign\ \bar{b}_{12} = -\ sign\ \bar{b}_{21}$; let $\bar{b}_{12} > 0$ (say) and $|\bar{b}_{21}| > \bar{b}_{12} \geq a$, a to be determined later;

Note that (4) is satisfied for any choice of $\bar{b}_{ij}$ described above; also note that $\mathrm{D} > a^2$, and that:

$$\frac{\bar{b}_{ij}}{\mathrm{D}} < \frac{4r^2}{a^2}, \quad |\frac{\bar{b}_{ij}}{\mathrm{D}}| < \frac{1}{a},$$

iii) now any choice of a_{10}, a_{02}, a_{00} such that $\frac{a_{10}a_{02}}{a_{00}} > \frac{1}{a}$ will insure that condition (1) is satisfied. Finally choose elements of $a_{\cdot 0}$, $a_{0\cdot}$ small so that conditions (2) and (3) are satisfied (for instance, $a_{10} = a_{20} = a_{01} = a_{02} = c$, for c small will be enough.).

With b as defined above, we have to determine an 8-dimensional vector of parameters: $(b, \bar{b}_{12}, \bar{b}_{21}, a_{00}, a_{10}, a_{20}, a_{01}, a_{02})$. For a suitably chosen pair of positive numbers (a,c), as in steps i-iii above, a subset of the admissible values for such parameters is given by *any* vector which satisfies the system of inequalities:

$$0 \leq b \leq 2r\,;\ 0 < a_{ij} < c,\ i = 0,1,2,\ j = 0,1,2;\ a_{10}a_{02} > a_{00}/a\ ;$$
$$|\bar{b}_{21}| > \bar{b}_{12} > a\ ;\ \bar{b}_{21} < 0\,.$$

This proves the second claim in lemma 2. ■

One last remark concludes now the proof of Theorem 1. The "positive speed" crossing condition is proved as in Benhabib-Nishimura (1979), (A4), p. 441.

2. Non linear utility

One last misconception might be left: that the Hopf bifurcation described above depends crucially on the linear utility or on the constant returns to scale condition. We now proceed to show that this is not the case.

In the optimal growth problem 4 above replace the instantaneous utility of the representative consumer with a neoclassical utility function U (U is concave and C^2 in the interior of the positive real line). The space of utility functions is endowed with the C^2 topology. The Maximum Principle now gives:

$$\dot{k} = y - g\,k \tag*{14.}$$

$$\dot{q} = -U'(T)T_k + rq \tag*{15.}$$

$$U'(T)T_y = q \tag*{16.}$$

where as above $c = T(y, k)$. Denoting:

$$p \equiv T_y(y, k) \tag*{17.}$$

we may rewrite 15 as $\dot{q} = -U'(T)(T_k + rp)$ and differentiating 16 we derive:

$$\dot{q} = U''(T)\dot{c}p + U'\dot{p} \tag*{18.}$$

Substituting for $\dot{c}$, and using 15 we derive:

$$U''(T_y y_k + T_k)(y - q\,k) + [U''T_y y_p + U']\dot{p} = U'(-w + rp)$$

Denoting $(T_y y_k + T_k)\ (y - g\,k) \equiv Q$, $U''T_y y_p + U' \equiv R$ we derive:

19. $$\dot{p} = -w + rp + \frac{U''}{U'}\left\{\frac{R}{1+(U''/U')R}\,[w-rp] - \frac{U'Q}{U'+U''R}\right\}.$$

$$\equiv -w + rp + \frac{U''}{U'}\,S(y,k)$$

From 17 we can write the optimal investment as a function of p and k, and we are left with the two equations:

14. $$\dot{k} = y(k,p) - g\,k$$

19. $$\dot{p} = -w(y(k,p),p) + rp + \frac{U''}{U'}\,S(y(k,p),k)$$

Note that 19 is a perturbation of 7′. When U''/U' is small, the perturbation is small. We now show that for a perturbation small enough the Hopf bifurcation is preserved. Denote $x_U \equiv (k_U, p_U)$ as the steady state corresponding to any U. Let $\epsilon \equiv -\frac{U''}{U'}$, evaluated at the steady state consumption, and denote by λ the vector of parameters used in Lemma 2 above. Linearization of the equations 14 and 19 gives a matrix

$$E(\lambda, x_U, \epsilon) = C(\lambda, x_U) - \epsilon D(\lambda, x_U)$$

where $C(\lambda, x_0)$ is the J matrix in 11. above; note that the coefficients of these matrices vary continuously in U.

In the previous section we have seen that as λ varies between two values λ_1 and λ_2 say, one of the eigenvalues of $C(\lambda, x_0)$ crosses the imaginary axis. Now Rouche's theorem yields the same conclusion for any neoclassical utility function, provided ϵ is small enough.

A similar argument shows that the Hopf bifurcation exists in the case of non-constant returns to scale (again, in the case of small deviations from the homogeneity of degree one) that preserves the concavity of the problem.

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