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BY

Steven J. Brams

D. Marc Kilgour

and

Morton D. Davis

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NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

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Steven J. Brams
Department of Politics
New York University
New York, NY 10003
USA

D. Marc Kilgour
Department of Mathematics
Wilfrid Laurier University
Waterloo, Ontario N2L 3C5
CANADA

Morton D. Davis
Department of Mathematics
City College of New York
New York, NY 10003
USA

ABSTRACT

Each of two players independently draws a number at random from [0, 1]. After observing its number, each player can then choose to offer or not offer to exchange its number for the other player's number. A player's payoff is the number it holds after the players have made their choices and a possible exchange has occurred. Four different games, which assume different conditions for an exchange, are analyzed:

1. AND. Both players must offer to exchange for a trade to occur.
2. OR. One player's offer is sufficient for a trade to occur.
3. MIX. Both players must offer for a trade to occur with certainty; if only one player offers, a trade occurs with probability p.
4. PLUS. Same as AND, except the payoff to the loser if a trade occurs is a number between the lower and the higher numbers drawn.

Games (1) - (3) are constant-sum, whereas game (4) is nonconstant-sum. The Nash equilibria in each of these games are:

1. Neither player ever offers, no matter now low its number is.
2. Both players always offer, no matter how high their numbers are.
3. The players select some common threshold--below which they always offer--that is monotonically increasing in p.
4. Neither player ever offers, no matter how low its number is.

The equilibrium in (4) highlights a possible conflict between the criterion of stability and the criterion of maximization of expected value, which calls for the players to offer under certain conditions. On the other hand, if PLUS is altered so that the players choose each other's thresholds, both will always offer. The unraveling downward in AND and PLUS, upward in OR and the alteration of PLUS, and the intermediate results in MIX are discussed.

UNRAVELING IN EXCHANGE GAMES

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1. Introduction

Traditionally, "unraveling" has meant "to disengage or separate the threads" (Webster's Ninth New Collegiate). More recently, it has come to mean a coming apart of a situation--related to its traditional meaning of disentangling--but without the connotation of something's being cleared up in the process. Quite the contrary: unraveling now connotes breakdown or disintegration, with the presumption that matters have overstepped their bounds, gotten out of hand.

This is the meaning that we intend in the title of this paper. Our analysis focuses on the strategic forces that cause unraveling to occur. We shall illustrate these forces with several exchange games that lead to different equilibrium outcomes. Some involve a kind of spiraling down of a process, whereby an exchange never occurs. Others lead to an upward spiraling, whereby an exchange always occurs. And there are also games with intermediate outcomes, in which there are limits on unraveling.

The foregoing games are all constant-sum: if an exchange occurs, what one player gains the other player loses. We also analyze a nonconstant-sum game, in which the loser's losses are reduced by up to the amount that would give it the same payoff as the winner if an exchange occurs. We show that the greater incentive that the players have to exchange in this game may be inconsistent with its equilibrium outcome.

To illustrate what we mean by unraveling, consider the following game, first proposed by Littlewood and reported in Bollobás (1986, p. 26):

> There is an indefinite supply of cards marked 1 and 2 on opposite sides, and of cards marked 2 and 3, 3 and 4, and so on. A card is drawn at random by a referee and held between the players A, B so that each sees one side only. Either player may veto the round, but if it is played the player seeing the high number wins.

What winning means is unspecified, but assume it results in a positive payoff, whereas losing results in a negative payoff and not playing results in a zero payoff. The surprising result in this game is that

> every round is vetoed. If A sees a 1 the other side is 2 and he must veto. If he sees a 2 the other side is 1 or 3; if 1 then B must veto; if he does not then A must. And so on by induction.

We would put the argument somewhat differently. A will certainly veto 1, so B's choice--on seeing 2 on a 1-2 card--has no effect. Therefore, B loses nothing by vetoing 2 if the card is 1-2. On the other hand, if the card is 2-3, B does have something to lose if he sees 2 and does not veto it, so his dominant strategy is to veto 2 whenever he sees it. Given that 2 is vetoed, a player who sees 3 has a dominant strategy to veto it whenever he sees it, and so on. Consequently, there is never any winner: the vetoes cause the process to unravel.

It is the inferences that the players make from information they know they both possess--their common knowledge--that is the key to the unraveling in Littlewood's game. Similarly in our exchange games, common knowledge has a cascading effect on players' strategy choices, which has been noted by other analysts in economic and related contexts (Milgrom and Stokey, 1982; Ribeiro da Costa Werlang, 1989).

2. Two Exchange Games: AND and OR

Consider the following situation. Player P draws a number x from some interval [a, b], whose numbers are distributed according to a known distribution. P can then choose to exchange that number for another number y drawn independently from some other interval, whose numbers are distributed according to a known distribution. It is easy to show that P, whom we assume to be the sole decision maker, maximizes the expected value of the number it holds by exchanging x iff $x < E(y)$, where $E(y)$ is the expected value of the number drawn from the second distribution.

For simplicity, assume that the distributions from which x and y are drawn are the same. If x and y are uniform over [0, 1], for example, P will exchange iff $x < 1/2$. This policy of trading will net P the following expected value:

$$\int_0^{1/2} E(y)dx + \int_{1/2}^{1} xdx = \frac{1}{4} + \frac{3}{8} = \frac{5}{8} \qquad (1)$$

since $E(y) = 1/2$.

Next assume that the decision to exchange is not made by a single player P, who first draws x and then decides whether or not to exchange it for y. Instead assume that two players, A and B, independently draw x (A) and y (B) from some known distribution. An exchange occurs iff both players agree to trade. Thus, the situation is no longer a one-person game against nature, in which P alone determines the outcome, but a two-person game in which both players must offer to exchange in order for a trade to occur.

Let a <u>threshold</u> for player A be a number a such that A is willing to trade iff A's draw x satisfies $x < a$. Similarly, denote player B's threshold by b, so that B is willing to trade iff $y < b$. (We investigate the possibility of nonthreshold strategies in section 3.) In any particular play of this game, A's value is

$$V_A(a,b) = \begin{cases} y \text{ if } x < a \text{ and } y < b & \text{(both players offer to exchange)} \\ x \text{ otherwise} & \text{(at most one player offers to exchange),} \end{cases}$$

and B's value, $V_B(b,a)$, is analogous.

Assume that x and y are uniformly distributed over [0, 1] and drawn independently. A's expected value is

$$EV_A = EV_A(a,b) = \int_0^1 \int_0^1 x\,dy\,dx + \int_0^a \int_0^b (y-x)\,dy\,dx \qquad (2)$$

$$= 1/2 + ab^2/2 - a^2b/2.$$

The first term on the right-hand side of (2) is the expected value of A's initial draw, x. If x and y are both less than the thresholds, a and b, respectively, there is an exchange, and A receives an additional amount, y-x (second term), which may be negative.

Fix b. Because

$$\frac{\partial EV_A}{\partial a} = b^2/2 - ab,$$

it follows that if $b > 0$,

$$\frac{\partial EV_A}{\partial a} \begin{matrix} > \\ = \\ < \end{matrix} 0 \text{ according as } a \begin{matrix} < \\ = \\ > \end{matrix} b/2.$$

Consequently, A maximizes EV_A by choosing $a = b/2$. If $b = 0$, EV_A does not depend on a.

Clearly, any equilibrium must have either $b = 0$ or $a = b/2$. Analogous consideration of EV_B indicates that, at any equilibrium, either $a = 0$ or $b = a/2$. The intersection of these conditions shows that the only possible equilibrium is $a = b = 0$, which makes trading a zero-probability event. (Hence, however low a number each player draws, the players will never offer to exchange.) At this <u>never-trade equilibrium</u>, $EV_A(0,0) = EV_B(0,0) = 1/2$.

An intuitive way of thinking about this never-trade result is to imagine that one player (say, B) sets an initial threshold, b_1. Then if A should learn about b_1, A's best response is to set a threshold of $a_1 = b_1/2$. But B's best response to a_1, should B learn about a_1, is to set a new threshold $b_2 = a_1/2 = b_1/4$. Thereby the best responses of each player to the other player's previous threshold unravel down to 0 (in the limit). Put another way, both players cannot optimally respond to the other player's threshold except at $a = b = 0$, making this the unique Nash equilibrium of the exchange game, from which neither player would have an incentive to depart unilaterally.

We call this exchange game AND, because <u>both</u> players must agree to a trade before it is consummated. We next define a new exchange game, called OR, in which a trade is consummated if <u>either</u> player--or both--

offers to exchange its number for the other player's number. Using the same thresholds as previously, A's value in any particular play of OR is

$$W_A(a,b) = \begin{cases} y \text{ if } x < a \text{ or } y < b & \text{(either player offers to exchange)} \\ x \text{ otherwise} & \text{(neither player offers to exchange),} \end{cases}$$

and B's value, $W_B(b,a)$, is analogous.

As before, assume that x and y are uniformly distributed over [0, 1] and drawn independently. A's expected value is

$$EW_A = EW_A(a,b) = \int_0^1 \int_0^1 y\,dy\,dx + \int_a^1 \int_b^1 (x-y)\,dy\,dx \qquad (3)$$

$$= (1/2)[1 + (a - b)(1 - a)(1 - b)].$$

The first term on the right-hand side of (3) is the expected value, y, to A, if there is an exchange. There will not be an exchange only when x and y are both greater than the thresholds, a and b, respectively, in which case A receives an additional amount, x-y (second term), which may be negative.

Fix b. Because

$$\frac{\partial EW_A}{\partial a} = (1/2)(1 - b)[1 - 2a + b],$$

it follows that if b < 1,

$$\frac{\partial EW_A}{\partial a} \begin{matrix} > \\ = \\ < \end{matrix} 0 \text{ according as } a \begin{matrix} < \\ = \\ > \end{matrix} (1 + b)/2.$$

Consequently, A maximizes EW_A by choosing $a = (1 + b)/2$. If $b = 1$, EW_A does not depend on a.

Clearly, any equilibrium must have either $b = 1$ or $a = (1 + b)/2$. Analogous consideration of EW_B indicates that, at any equilibrium, either $a = 1$ or $b = (1 + a)/2$. The intersection of these conditions shows that the only possible equilibrium is $a = b = 1$, which makes trading a certain event. (Hence, however high a number each player draws, the players will always offer to exchange.) At this always-trade equilibrium, $EW_A(1,1) = EW_B(1,1) = 1/2$.

The intuition behind this result is similar to that of the AND game, except that the unraveling based on best responses is in the opposite direction--the players will push trades up to the threshold of 1 (in the limit). Only at $a = b = 1$ are the players responding optimally to each other's strategies, making this the unique Nash equilibrium of OR.

The equilibrium results for both AND and OR can be justified in another way: ex ante--before each player draws a number--the equilibrium strategies in these games are the only strategies that guarantee each player their values of 1/2. In AND, this value is assured by never trading, which each player can effect by never offering, whatever the other player does. In OR, this value is assured by always trading, which each player can effect by always offering, whatever the other player does.

3. Are There Equilibrium Strategies That Are Not Threshold Strategies?

The dramatic difference between equilibrium strategies in AND and OR raises the question of whether there are other equilibrium strategies in these games--perhaps mixed--different from the thresholds of 0 and 1, respectively. The strategies we shall consider in this section allow trades to occur not with certainty but with some probability that is a function of the two numbers, x and y, that A and B draw.

Suppose that, in AND, A offers to exchange x with probability p(x), and B offers to exchange y with with probability q(y). Because both players must agree to exchange in order for a trade to occur, the value of the trade is now realized only probabilistically. A's value V_A is

$$V_A[p(x),q(y)] = \begin{cases} y \text{ with probability } p(x)q(y) \\ x \text{ with probability } 1 - p(x)q(y), \end{cases}$$

which translates into an expected value of

$$EV_A = y[p(x)q(y)] + x[1 - p(x)q(y)].$$

From A's vantage point, the game is one of incomplete information, in which A knows its own number x but does not know B's number y. A's expected value, given x and knowing the distribution of y values--assumed as before to be uniform over [0, 1]--is

$$EV_{A|x} = \int_0^1 yp(x)q(y)dy + \int_0^1 x[1 - p(x)q(y)]dy. \tag{4}$$

For convenience, let p = p(x), and consider A's optimal choice of p in (4). Rewriting (4), we obtain

$$EV_{A|x} = p\int_0^1 yq(y)dy + x - x\int_0^1 q(y)dy$$

$$= x + p\int_0^1 (y - x)q(y)dy.$$

If

$$\int_0^1 (y - x)q(y)dy \neq 0, \tag{5}$$

then the optimal value of p will equal 0 or 1, depending on whether (5) is negative or positive, which is to say that the only equilibrium strategies in this case are threshold strategies. On the other hand, if (5) is

$$\int_0^1 (y - x)q(y)dy = \int_0^1 yq(y)dy - x\int_0^1 q(y)dy = 0, \tag{6}$$

the value of p has no effect on $EV_{A|x}$, so every p is "optimal."

Now consider whether it is possible to find a q(y) so that (6) is true for more than one value of x (which would allow for the possibility of mixed strategies). Because (6) does not depend on x iff

$$\int_0^1 q(y)dy = 0, \tag{7}$$

(7) can hold only when q(y) = 0, given that q(y) is piecewise continuous. Hence, a mixed strategy p(x) is possible only when q(y) = 0.

In summary, A's only possible equilibrium strategy p(x) satisfies p(x) = 0 or 1, except perhaps at a single value of x, unless q(y) = 0. But

$q(y) = 0$ is equivalent to a threshold of $b = 0$, which makes A's expected payoff independent of A's actions. Hence, in the only case in which is is possible that $0 < p(x) < 1$ for more than one value of x, A's strategy choice is immaterial. It follows that the only possible equilibria involve thresholds.

Thus, allowing for probabilistic strategies does not expand the set of Nash equilibria in AND; a similar result can be shown for OR. But strategies are not the only aspect of these two games that might be probabilistic. The decision rule itself may be probabilistic, which gives rise to new equilibrium strategies in a game we call MIX (for "mixture").

4. MIX: A Probabilistic Exchange Game

Although the decision rule whereby a trade is consummated is usually fixed--and, as in AND, generally requires the assent of both players--we may well imagine situations in which only one player's offer is sufficient to effect a trade. For example, if one player makes an offer but the other player balks because the number it draws is above its threshold for exchange, it may be the case that on one set of issues the player who makes the offer can prevail, forcing the recalcitrant player to go along. On another set of issues, the roles of the players may be reversed. On a third set of issues, the assent of both may be required. Thus, the game is a mixture of OR (first two sets of issues) and AND (third set of issues).

More specifically, we postulate MIX to be a game in which exchange occurs (1) with certainty if both players offer and (2) with probability p if only one player offers. Thus, p is the conditional probability that if only one player offers, there will be an exchange, as in OR; otherwise the decision rule is that of AND, wherein both players must make offers to effect an exchange.

We assume that p is common knowledge and does not depend on which player makes the offer. Thus, each player knows that, if it does not reciprocate the other player's offer, there will be an exchange, anyway, with probability p.

To find equilibrium strategies in this hybrid game, we begin by computing A's expected value, $EM_A(p, a, b) = EM_A$, in the uniform case:

$$EM_A = \int_0^1 \int_0^1 x\,dy\,dx + p\int_a^1 \int_0^b (y-x)\,dy\,dx + p\int_0^a \int_b^1 (y-x)\,dy\,dx + \int_0^a \int_0^b (y-x)\,dy\,dx. \quad (8)$$

The first term on the right-hand side of (8) is the expected value of A's initial draw, x. If there is an exchange, A receives an additional amount, y-x, which may be negative. With probability p there will be an exchange if either $a \leq x \leq 1$ and $0 \leq y \leq b$ (second term), or $0 \leq x \leq a$ and $b \leq y \leq 1$ (third term). With certainty there will be an exchange if $0 \leq x < a$ and $0 \leq y < b$ (fourth term).

After evaluation and simplification, (8) becomes

$$EM_A(p, a, b) = 1/2 + ab^2/2 - a^2b/2 + (p/2)(a - b - a^2 + b^2 - 2ab^2 + 2a^2b).$$

It is easy to check that when $p = 0$, $EM_A(0, a, b) = EV_A$, and when $p = 1$, $EM_A(1, a, b) = EW_A$. In other words, MIX combines AND and OR, reducing to each when one player's offer alone is never sufficient ($p = 0$) or when it is always sufficient ($p = 1$) to effect a trade.

Like AND and OR, if $a = b$, the expected payoff to each player of MIX is 1/2. Unlike these games, however, in MIX, as we shall next show, there are nonextreme values of $a = b$ that are in equilibrium when $0 < p < 1$.

Fix b and p. Because

$$\frac{\partial EM_A}{\partial a} = \frac{b^2}{2} + \frac{p}{2} - pb^2 - a(b + p - 2pb),$$

it follows that

$$\frac{\partial EM_A}{\partial a} \begin{matrix} > \\ = \\ < \end{matrix} 0 \text{ according as } a \begin{matrix} < \\ = \\ > \end{matrix} \frac{b^2 + p - 2pb^2}{2(b + p - 2pb)}, \tag{9}$$

given that

$$b + p - 2pb > 0. \tag{10}$$

Inequality (10) is true because

$$(\sqrt{b} - \sqrt{p})^2 = b + p - 2\sqrt{bp} \geq 0$$

implies (10) since $\sqrt{bp} \geq bp$ when $b \geq 0$ and $p \leq 1$. When $p = 0$ or $p = 1$, it is easy to show that A maximizes EM_A by choosing the thresholds we derived in section 2 for AND ($a = b/2$) and OR ($a = [1+b]/2$).

Differentiating EM_B with respect to b,

$$\frac{\partial EM_B}{\partial b} \begin{matrix} > \\ = \\ < \end{matrix} 0 \text{ according as } b \begin{matrix} < \\ = \\ > \end{matrix} \frac{b^2 + p - 2pb^2}{2(b + p - 2pb)}. \tag{11}$$

To find the equilibrium strategies of A and B, define

$$f(x) = \frac{x^2 + p - 2px^2}{2(x + p - 2px)} \tag{12}$$

The two players' strategies will be in equilibrium when the partial derivatives given by (9) and (11) are both zero, or

$$a = f(b) \text{ and } b = f(a). \tag{13}$$

Consider first the situation in which the equilibrium is symmetric (i.e., $a = b$). Setting $x = f(x)$ yields

$$Q(x) = (1 - 2p)x^2 + (2p)x - p = 0, \tag{14}$$

whose solution is

$$x = x^* = \begin{cases} 1/2 & \text{if } p = 1/2 \\ \dfrac{-p + \sqrt{p - p^2}}{1-2p} & \text{otherwise,} \end{cases} \tag{15}$$

because $x \in [0, 1]$ is required.

We consider the two cases given by (15) in turn. If $p = 1/2$,

$$EM_A(1/2, a, b) = 1/2 + (a - a^2 - b + b^2)/4,$$

and

$$\frac{\partial EM_A}{\partial a} = \frac{1 - 2a}{4} \overset{>}{\underset{<}{=}} 0 \text{ according as } a \overset{<}{\underset{>}{=}} \frac{1}{2}.$$

Thus, $a = 1/2$ is a dominant strategy for A when $p = 1/2$: whatever B's choice of b, A maximizes EM_A by choosing $a = 1/2$, yielding

$$EM_A(1/2, 1/2, b) = 9/16 - (b - b^2)/4.$$

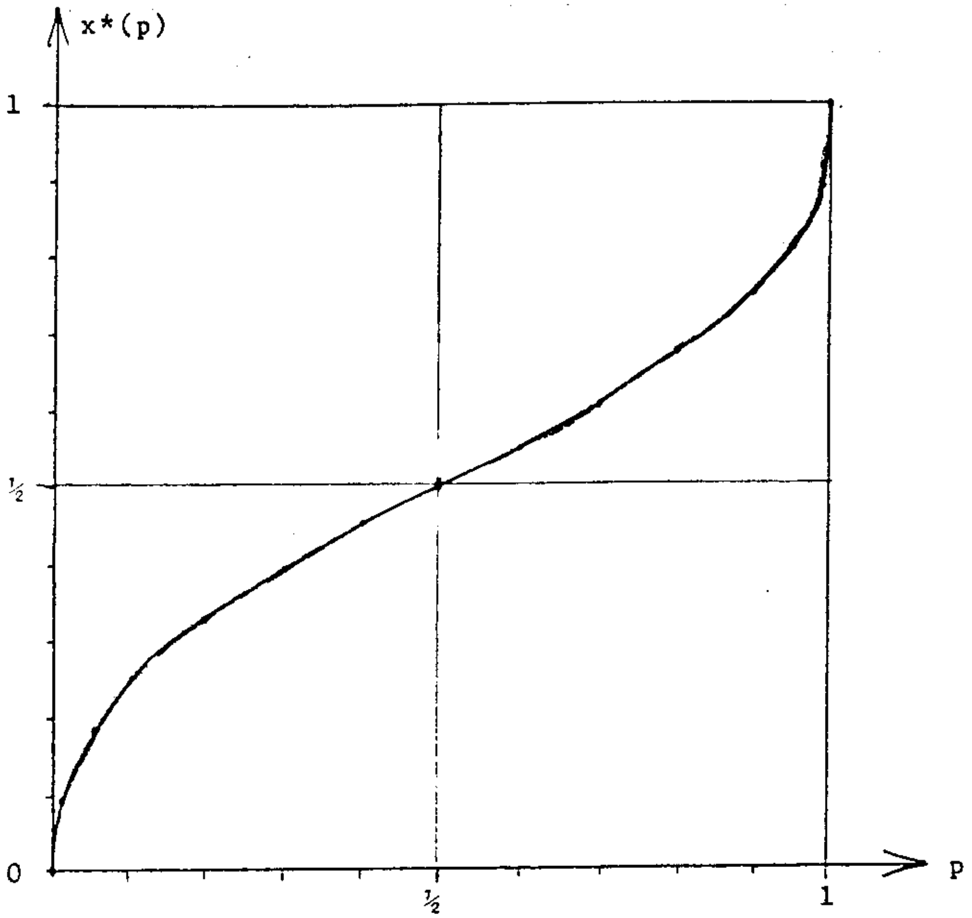
On the other hand, if $p \neq 1/2$, the equilibrium x^* given by (15) is a monotonically increasing signoid function of p, as illustrated in Figure 1. Observe that x^* increases at a decreasing rate until $p = .5$ (when $x^* = .5$) and then increases at an increasing rate, making $p = .5$, $x^* = .5$ an inflection point.

The main difference between the equilibria x^* when $p \neq 1/2$, and the equilibrium $x^* = 1/2$ when $p = 1/2$, is that the former are Nash equilibria, but the players' threshold strategies, $a = b = x^*$, are not dominant. Rather, given one player (say, A) chooses $a = x^*$ (for a given p), B's best response is always $b = x^*$. But should B choose some $b \neq x^*$, A's maximizing strategy against this choice may not be x^*, even though x^* is the unique symmetric equilibrium strategy for the postulated p.

We conclude that the choice of $x^* = 1/2$ for $p = 1/2$ is more compelling in MIX than the choice of any other x^* for $p \neq 1/2$. Perhaps more significant than this distinction between different types of Nash equilibria is the fact that there are as many nonextreme equilibria in MIX (including $x^* = 1/2$) as there are possible values of p between 0 and 1. The cases of $p = 0$ and $p = 1$, which define AND and OR, are special: these are the only values of p that give rise to extreme Nash equilibria ($x^* = 0$ and $x^* = 1$) and, therefore, lead to the complete unraveling discussed earlier.

FIGURE 1

THE FUNCTION $x^*(p)$



To illustrate the effects of nonextreme values of p, note that when $p = 1/5$, $x^* = 1/3$. The probability of an exchange in this case is

$$(1/3)(1/3) + 2[(1/3)(2/3)](1/5) = 1/5, \tag{16}$$

or the sum of the probabilities that either (1) both players offer or (2) one player offers and the other player does not (two cases--either player could make the offer) times the probability that a single offer is sufficient for an exchange.

It is no accident that this probability equals $p = 1/5$. In fact, (14) can be rewritten as

$$x^2 + 2[x(1-x)]p = p,$$

which is identical in form to (16); it says that the solution $x = x^*$ given by (15) ensures that when one player offers and the other player does not, the conditional probability that there will be an exchange, p, is equal to the unconditional probability of an exchange, $x^2 + 2[x(1-x)]p$. In other words, an equilibrium exists when there is equality between the conditional and unconditional probabilities--the former probability, in effect, does not depend on the conditioning factor.

Thus, p turns out to equal the unconditional probability of an exchange at the symmetric equilibrium. We now show, by contradiction, that there are never any asymmetric equilibria--that is, at every Nash equilibrium, $a = b$. In fact, this has already been shown for $p = 0$, $1/2$, and 1, so assume another mix. We search for solutions a and b to (13) such that $a \neq b$.

By differentiating (12), it can be shown that the sign of $f'(x)$ is the same as that of $(1-2p)Q(x)$, where $Q(x)$ is as defined in (14). It is easy to extend our previous solution of $Q(x) = 0$ to show that, if $x \in [0, 1]$, then

$$Q(x) \overset{>}{\underset{<}{=}} 0 \text{ according as } x \overset{>}{\underset{<}{=}} x^*(p),$$

where $x^*(p)$ is as given by (15).

Now assume that $0 < p < 1/2$. Then

$$f'(x) \overset{>}{\underset{<}{=}} 0 \text{ according as } x \overset{>}{\underset{<}{=}} x^*(p).$$

It follows that the absolute minimum of $f(x)$ over $[0, 1]$ occurs uniquely at $x = x^*(p)$, which from (14) is the unique solution of $x = f(x)$. Now consider solutions a and b to (13) such $a \neq b$. Clearly, $a \neq x^*(p)$ because, if it were, $b = x^*(p) = a$. Consequently, $a > x^*(p)$ and, similarly, $b > x^*(p)$, because $x^*(p)$ is an absolute minimum. But $x > x^*(p)$ iff $f(x) < x$, with $f(1) = 1/2$. Therefore, $a = f(b) < b$ and $b = f(a) < a$. This contradiction shows that (13) has no solution with $a \neq b$ for $0 < p < 1/2$. The argument for $1/2 < p < 1$ is analogous.

Although the only equilibria in MIX are symmetric, this is not the case if the players are distinguished. Define the conditional probability that there will be an exchange if only A offers to be p_A, and define the conditional probability that there will be an exchange if only B offers to be p_B. Then equilibrium conditions, analogous to (9) and (11), can be found for $a \neq b$ when $p_A \neq p_B$. For instance, if $p_A = 1$ (A alone can always force an exchange) and $p_B = 0$, it is not difficult to show that $a = .5$, b arbitrary, are in equilibrium. A, in effect, is a dictator who can ensure an expected payoff of 5/8, according to (1).

In section 5 we posit a nonconstant-sum exchange game, PLUS, in which the gains from trading mitigate the losses to the loser. In encouraging more trading, however, PLUS also creates a dilemma for the players in their choices of rational strategies.

5. PLUS: A Nonconstant-Sum Exchange Game

Consider AND, and assume that there is an exchange: A receives y and B receives x after the exchange. We now modify AND so that if, say, y is the smaller number, A does not receive y but $x - k(x - y)$, where $0 \leq k \leq 1$ is a constant. (Hence, A cannot receive less than y.) We call this game--in which the loser does not generally receive the smaller number but instead has some deduction, $k(x - y)$, made from the larger number--PLUS. If $k = 1$ (maximum deduction), PLUS is AND; if $k = 0$ (no deduction), both players receive the larger number, x.

In section 2 we showed that $a = b = 0$ (never trade) is the unique Nash equilibrium of AND. When $k = 0$, on the other hand, it is apparent that one Nash equilibrium in PLUS is $a = b = 1$ (always trade), because either player's deviation would prevent it from <u>always</u> receiving the larger of the two numbers drawn. Moreover, because the players never do worse, and sometimes do better, by always offering to trade when $k = 0$, their strategies associated with this equilibrium are dominant.

But what are the Nash equilibria when $0 < k < 1$? As in MIX, are there intermediate thresholds that are in equilibrium? The answer, surprisingly, is "no," even though PLUS is a nonconstant-sum game and both players can, on average, benefit by always offering to trade, at least ex ante.

More specifically, if $0 < k \leq 1$, the unique Nash equilibrium is $a = b = 0$. As we shall argue, however, this never-trade result is challengeable. To mount this challenge, we shall show via a nonequilibrium calculation that, even in a worst-case scenario, players may have good reason to offer to trade in PLUS.

As in the preceding games, assume that x and y are uniformly distributed over [0, 1] and drawn independently. We consider two cases in turn. First, assume $a \geq b$. To find equilibrium strategies in PLUS, we compute A's expected value, $EP_A(a, b)$, in the uniform case:

$$EP_A = EP_A(k, a, b) = \int_0^1 \int_0^1 x\,dx\,dy + \int_0^b \int_0^y (y - x)\,dx\,dy - k\int_0^b \int_y^a (x - y)\,dx\,dy. \qquad (17)$$

The first term on the right-hand side of (17) is A's expected value if there is never an exchange and A always receives x. However, there will be an exchange if $0 \leq y < b$ and $0 \leq x < a$, with two cases to consider: (1) if $x < y$ (second term), A ends up with the larger number y after the exchange and so has $(y - x)$ added to the already included x; (2) if $y < x$ (third term), A ends up with the smaller number y and so has $k(x - y)$ subtracted from the already included x.

After evaluation and simplification, (17) becomes

$$EP_A = 1/2 + b^3/6 - k(b^3/6 - ab^2/2 + a^2b/2). \qquad (18)$$

Fix b and k. Because

$$\frac{\partial EP_A}{\partial a} = (kb/2)(b - 2a) < 0,$$

because $a \geq b$, provided $k > 0$. Hence, when $a \geq b$, A should make a as small as possible (i.e., $a = b$) to maximize EP_A.

Now assume $a \leq b$. Calculations analogous to the preceding ones yield:

$$EP_A = \int_0^1 \int_0^1 x dx dy + \int_0^b \int_x^b (y - x) dx dy - k \int_0^b \int_0^x (x - y) dx dy$$

$$= 1/2 + ab^2/2 - a^2b/2 + a^3/6 - ka^3/6, \qquad (19)$$

and

$$\frac{\partial EP_A}{\partial a} = \frac{(b-a)^2}{2} - ka^2/2. \qquad (20)$$

Setting (20) equal to zero, it is easy to show that A maximizes EP_A by choosing

$$a = \frac{b}{1 + \sqrt{k}}. \qquad (21)$$

In AND ($k = 1$), for example, A maximizes EP_A by choosing $a = b/2$, as we showed in section 2. Similarly, B maximizes EP_B by choosing

$$b = \frac{a}{1 + \sqrt{k}}. \qquad (22)$$

Like AND, PLUS unravels: if $k > 0$, the only thresholds that satisfy (21) and (22) simultaneously are $a = b = 0$. This is true despite the fact that both players will, on average, gain ex ante from trading if $k < 1$ because the game is nonconstant-sum and some of the loser's losses are reimbursed.

Only when $k = 0$, and both players always receive the higher of the two numbers if there is an exchange, is the unraveling halted. Formally, if $k = 0$,

$$\frac{\partial EP_A}{\partial a} \begin{matrix} > 0 \\ = 0 \end{matrix} \text{ according as } a \begin{matrix} < b \\ \geq b, \end{matrix}$$

and similarly for B. Hence, the players' strategies are in equilibrium if $a = b = x$. Because

$$EP_A(x, x) = 1/2 + x^3/6,$$

and similarly for B, the equilibrium is Pareto-superior, as well as being dominant-strategy, when $a = b = 1$. At this equilibrium, $EP_A(1, 1) = EP_B(1,1) = 2/3$, which is the expected value of the higher of two numbers drawn independently from a uniform distribution over [0, 1].

Short of both players receiving the higher of the two numbers when $k = 0$, the unraveling of PLUS when $0 < k < 1$ seems unfortunate: the game is nonconstant-sum and both players could, if they agreed to an exchange, receive an expected payoff greater than 1/2. However, these strategies are not in equilibrium, so only before the players draw their numbers would it seem sensible for them to agree to an exchange in PLUS.

Yet, if a player can guarantee its expected payoff is greater than 1/2 by offering to exchange--whatever the other player does--the unique Nash equilibrium in PLUS seems not so compelling. After all, at this equilibrium, at which neither player offers to exchange, there will be no exchange and each player's expected payoff will be only 1/2.

How can a player guarantee more than 1/2? Assume B <u>publicly</u> announces b, offering to exchange any number below this threshold. Moreover, assume that the choice of this strategy can be enforced, perhaps by having B show its number to some third party and having that third party select or confirm B's strategy choice.

Assume A responds optimally to B's announcement by choosing, according to (21), $a = b/(1 + \sqrt{k})$. A's expected payoff, according to (19), for the case when $a \leq b$ is

$$EP_A = \frac{1}{2} + \left(\frac{1}{2}\right)\left[\frac{b}{1 + \sqrt{k}}\right]^3 \left(-2k + \sqrt{k} + 3\right), \tag{23}$$

which is greater than 1/2 except when $b = 0$.

We make an analogous calculation for B, based on (18):

$$EP_B = \frac{1}{2} + \left(\frac{1}{6}\right)\left[\frac{b}{1 + \sqrt{k}}\right]^3 \left(-3k^2 - 3k^{3/2} - k + 1\right). \tag{24}$$

The sign of $EP_B - 1/2$ is the same as that of

$$-3k^2 - 3k^{3/2} - k + 1,$$

which can be shown to have exactly one zero in [0, 1]. This zero occurs at $k \cong .288$. For $k < .288$, therefore, $EP_B > 1/2$, so B maximizes EP_B by setting $b = 1$, or always offering to exchange. On the other hand, if $k > .288$, B maximizes EP_B by setting $b = 0$, or never offering to exchange.

Thus, $k \cong .288$ is the threshold at which B can ensure $EP_B = 1/2$, even in the "worst case" when A knows b and responds optimally to it. Instead of the loser's receiving the smaller number at this threshold, it receives this number plus 71.2 percent of the gap separating the smaller number from the larger one.

This value added in PLUS is sufficient to ensure that, even when "exploited" by A, B can receive an expected payoff of 1/2--as much as it can achieve without trading--by always offering to trade. Therefore, even a very conservative B might offer to exchange when $k \leq .288$.

Note that when $k = 0$,

$$EP_A = EP_B = 1/2 + b^3/6,$$

according to (23) and (24). Hence, it is in both players' interest for B to choose $b = 1$ and for A, by responding optimally, to choose $a = 1$. Each player receives an expected payoff of 2/3, which is the expected value of the higher number, as we showed earlier.

As another illustration, assume $k = .1$, which closes the gap between the winning and losing numbers by 90 percent. If B chooses $b = 1$, A's best response is to choose $a \cong .760$ according to (21), so a trade will occur 76 percent of the time. The players' expected payoffs in this case are $EP_A \cong .657$ and $EP_B \cong .557$. On the other hand, if both players offered to trade all the time ($a = b = 1$), they would maximize their <u>combined</u> expected payoff (1.300, receiving .650 each), because collectively they would reap the benefits of being almost completely reimbursed when they lose. By contrast, if each player tried to exploit the other by choosing its best-response strategy ($a = b \cong .760$)--perhaps thinking that the other would always offer--both would suffer substantially ($EP_A = EP_B \cong .566$).

These results are summarized in the payoff matrix of Figure 2. This game is Prisoners' Dilemma, with the lower-right outcome the unique

FIGURE 2

PLUS AS A GAME OF PRISONER'S DILEMMA WHEN k = .1

		B	
		Cooperate (b = 1)	Exploit (b = .760)
A	Cooperate (a = 1)	(.650, .650)	(.557, .657)
	Exploit (a = .760)	(.657, .557)	(.566, .566)

Key: Underscored outcome is a Nash equilibrium.

Pareto-inferior Nash equilibrium. (However, this outcome is a Nash equilibrium only in the situation in which each player must choose either a cooperative or an exploitative strategy; when all other thresholds are admitted, the never-trade equilibrium is the only one, except when $k = 0$.)

Because the players' cooperative strategies of always trading give each a payoff only marginally worse (.650) than when they are successful exploiters (.657), each would have a strong incentive not to try to victimize the other player by choosing its exploitative strategy. Of course, this incentive would diminish as the game approaches constant-sum (i.e., $k \rightarrow 1$). But certainly for $k \leq .288$, the never-trade equilibrium seems ill-advised: by always offering to trade, each player can receive an expected payoff of at least .5, even if exploited. The fact that the players can do considerably better if they are both not exploitative creates a strong incentive for cooperative behavior, especially if play is repeated.

This logic, however, is really applicable only to ex ante calculations. Thus, if B always offers in our example, and A is exploitative, $EP_B = .557$. But if B draws a number greater than .557, its strategy of announcing $b = 1$ may give it a lower expected payoff (given A is exploitative) than not offering to exchange. Consequently, a blanket commitment on B's part to trade is sensible only before it draws its number, when its ex ante expected payoff without trading is .5.

Finally, the possible conflict between equilibrium and nonequilibrium strategies when $k < 1$ vanishes if the players' strategy choices in PLUS are interchanged in a new game, in which A chooses b and B chooses a. Like PLUS, this game is nonconstant-sum, but, curiously, it unravels upward rather than downward. Indeed, it is easy to show, by differentiating EP_A

with respect to b in (18) and (20), that A has a dominant strategy of selecting $b = 1$; likewise, B has a dominant strategy of selecting $a = 1$. Notice that these strategies are optimal for the players in the Figure 2 Prisoners' Dilemma as well.

6. Summary and Conclusions

We have analyzed four games, in each of which two players independently draw numbers that are uniformly distributed over [0, 1]. Each player can then choose to offer or not offer to exchange its number for the other player's number. A player's payoff is the number it holds after the players have made their choices and a possible exchange has occurred (except in PLUS). We summarize the rules that determine the conditions for an exchange, each player's payoff, and the equilibrium outcomes in the different games:

1. AND. Both players must offer to exchange in order for a trade to occur. The unique Nash equilibrium is that neither player will ever offer, no matter how low its number is.

2. OR. One player's offer is sufficient for a trade to occur. The unique Nash equilibrium is that each player will always offer, no matter how high its number is.

3. MIX. Both players must offer in order for a trade to occur with certainty; if only one player offers, a trade occurs with probability p. The unique Nash equilibrium is for the players to select a particular common threshold--below which they will always offer--that is monotonically (but not linearly) increasing in p. When $p = 1/2$, the players have dominant

strategies of choosing a threshold of 1/2. When $p = 0$, MIX is AND, and when $p = 1$ MIX is OR.

4. PLUS. Same as AND, except the payoff to the loser if a trade occurs is a number between the lower and the higher numbers drawn. Hence, the game is nonconstant-sum. Like AND, the unique Nash equilibrium in this game is for the players never to trade, except in the degenerate case when both receive a payoff equal to the higher number after the trade.

We demonstrated in PLUS that if the payoff to the loser after trading is sufficently close to the higher number, a player will do better always trading, even if exploited by the opponent. Thereby, PLUS highlights a possible conflict between the criterion of stability--as embodied in the concept of a Nash equilibrium--and the criterion of maximization of expected value. The latter, as we have shown, may prescribe choosing a nonequilibrium strategy.

Even in constant-sum games, stability seems quite unappealing as a criterion. In AND and OR, for example, should the number a player draws really be irrelevant to its choice of a strategy? MIX moderates the extreme equilibrium choices of the players in AND and OR by making intermediate choices dependent on p, which may be interpreted as a measure of the ability of players to implement exchanges by themselves.

Unraveling surely occurs in the world, in both upward and downward directions. AND suggests that requiring both parties to a conflict to agree to the terms of a settlement may hasten unraveling to the equilibrium strategy of no settlement. On the other hand, if only one party is sufficient to effect an exchange, OR suggests that unraveling may occur in the opposite direction, forcing a settlement that may be worse for the unwilling player than no settlement at all. MIX offers an intermediate case: parties offer to trade below certain thresholds, depending on how effective they are in being able to implement trades on their own.

Because AND, OR, and MIX are constant-sum games, they do not capture the essence of an exchange in which both players can benefit, at least ex ante. To be sure, in PLUS there is a loser after the exchange (except when $k = 0$), but on an expected-value basis both players can benefit from always trading (except when $k = 1$). Nevertheless, as we demonstrated, they will never trade in equilibrium.

We challenged this logic with an expected-value calculation, given $k \leq .288$ in the uniform case and the game, therefore, is quite strongly nonconstant-sum. In this situation, the nonequilibrium expected-value calculation in PLUS is in conflict with the equilibrium calculation, at least before the players have drawn their numbers. But there is no conflict in a new game, in which each player chooses the other player's threshold, because each player has a dominant strategy of selecting the maximum threshold of the other, so both players will always offer to trade.

In conclusion, the conditions for rational exchange are highly dependent on the game being played. The dramatic difference between AND and OR is based solely on the difference in the decision rule that determines the outcome in each game, with MIX reflecting a combination that indicates how both downward and upward unraveling may be arrested. PLUS casts doubt on unraveling in a nonconstant-sum extension of AND, especially if PLUS can benefit players in a worst-case scenario.

The games we have analyzed mirror simple and highly stylized exchanges. Nontheless, with refinements they may be useful in modeling real-life strategic processes, enhancing our understanding of conditions under which unraveling is likely to occur.

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