ECONOMIC RESEARCH REPORTS

SOCIAL CONFLICT, GROWTH AND INCOME DISTRIBUTION

BY

Jess Benhabib
and
Aldo Rustichini

RR # 91-22 **April, 1991**

C. V. STARR CENTER FOR APPLIED ECONOMICS

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

Social Conflict, Growth and Income Distribution

by

Jess Benhabib
Department of Economics
New York University
New York, NY 10003
and

Aldo Rustichini
Department of Economics
Northwestern University
Evanston, IL 60208

March 1991

We would like to thank V.V. Chari, Ehud Kalai, Pat Kehoe, Roy Radner and Chuck Wilson for very helpful comments. We also would like to thank the C.V. Starr Center for Applied Economics at New York University for technical assistance.

Abstract

In this paper we study the relationship between wealth, income distribution and growth in a game-theoretic context in which property rights are not completely enforcable. We consider equilibrium paths of accumulation which yield players utilities that are at least as high as those that they could obtain by appropriating higher consumption at the present and suffering retaliation later on. We focus on those subgame perfect equilibria which are constrained Pareto-efficient (second best). In this set of equilibria we study how the level of wealth affects growth. In particular we consider cases which produce classical traps (with standard concave technologies): growth may not be possible from low levels of wealth because of incentive constraints while policies (sometimes even first-best policies) that lead to growth are sustainable as equilibria from high levels of wealth. We also study cases which we classify as the "Mancur Olson" type: first best policies are used at low levels of wealth along these constrained Pareto efficient equilibria, but first best policies are not sustainable at higher levels of wealth where growth slows down.

We also consider the unequal weighting of players to trace the subgame perfect equiliria on the constrained Pareto frontier. We explore the relation between sustainable growth rates and the level of inequality in the distribution of income.

"It is consequent also to the same condition, that there be no property, no dominion, no mine and thine distinct; but only that to be every man's, that he can get; and for so long as he can keep it."

Hobbes, Leviathan.

1. Introduction.

In this paper we explore the relation between wealth, growth and income distribution when property rights are not fully defined or are not completely enforcable. We have in mind a situation where organized groups have the power to assure for themselves a share of the income by direct appropriation, by manipulating the political system or by rent-seeking behavior to effectuate favorable transfers and regulations. Depending on the context these groups may represent, among others, organized labor, industrial groups and occupational associations, the military, the bureaucracy and ethnic or racial groups. The redistributive power of such groups necessarily imparts an element of joint ownership to the resources of society and may reduce the incentives to accumulate wealth.[1] We propose to study these issues in the context of a game-theoretic model.

We study a dynamic game in which each player independently chooses a consumption level and the residual output, if any, becomes the capital or the productive resource in the following period. Stationary equilibria in such games have been studied by Lancaster [1973], Levhari and Mirman [1980], Majumdar and Sundaram [1991], and many others. (See also Tornell and Velasco [1990].) We consider equilibrium paths of accumulation in which players receive utilities that are at least as high as those that they could obtain by appropriating higher

[1] The role of the enforcement of property rights by the state to internalize social gains and promote growth has been discussed by D. North [1981], [1991] in a historical context. The effects of rent-seeking behavior by organized groups on the economic efficiency of mature economies has been studied by M. Olson [1982].

immediate consumption levels and suffering some retaliation later on. (For a related framework of analysis, see Marcet and Marimon [1990]; see also Chari and Kehoe [1990].) We focus, however, on those subgame-perfect equilibria which are second-best, that is on those particular equilibria among all that are subgame-perfect which lie on the constrained Pareto frontier. Within this set we analyze the effects of wealth (or the stock of capital) on growth and on steady state income levels. In particular we also consider examples which produce classical "growth traps" with standard concave production technologies.[2] Even though first-best policies lead to growth, along second-best equilibria growth may not be possible from low levels of wealth because of incentive constraints: the accumulation of wealth by one player can lead to appropriation and to high consumption levels by other players, and therefore may not be sustainable as an equilibrium strategy. This possibility of negative or low growth outcomes from low levels of wealth may be applicable to some of the stagnant or contracting economies in Latin America and in Africa that have been plagued by political instability and that have often experienced capital flight (see Tornell and Velasco [1990]). Baumol, Blackman, and Wolff ([1989], see chapter 5) provide some empirical support for the wealth dependence of growth rates; it is the more affluent of countries that are able to join what they call a "convergence club", with the poorer LDC's being left behind. The sample of all countries shows no convergence in growth rates.

Another possibility is for incentive constraints to bind at high wealth levels and not at low ones. Capital may be too precious at low levels and players may follow first-best policies of accumulation. Inefficiency may set in

[2] For a modern version of a standard classical trap based on non-concave, "threshold" technologies, see Azariadis and Drazen [1990].

at higher levels of wealth and first-best policies may have to be abandoned as the incentives for appropriation grow and redistributive pressures increase. The possibility that inefficiencies are associated with stable and wealthy economies in which organized groups have had the time to mature and to exert redistributive pressures has been suggested by Mancur Olson [1982]. We illustrate this possibility in section 5 below.

Our brief discussion so far has made little mention of income distribution. Indeed the possibilities mentioned above can occur and social conflict can arise, even under complete equality. The desire to secure a higher share of output is not restrained by an equal distribution of income. As our examples and analysis demonstrate, redistributive pressures and incentive constraints can result in less than efficient accumulation policies, even when incomes are equally distributed. However, unequal allocations may further increase the inefficiencies because those who are worse-off have a higher incentive to appropriate output and to exert redistributive pressures. Maintaining an unequal distribution of income may then further slow down growth by further reducing the incentives for accumulation. Some empirical documentation of the negative influence of income inequality on growth has recently been given by Persson and Tabellini [1991]. (See also Alesina and Rodrik [1991].) In section 6 we study this question and provide a parametric example that illustrates the effect of inequality on growth. Figure 4 illustrates the range of possibilities.

In much of our analysis the "second-best" problem is formulated as a dynamic programming problem. The nature of the incentive constraints that depend on the optimal value, however, causes some non-trivial difficulties. Value functions are no longer necessarily concave or even continuous, and the usual contraction mapping theorem does not go through. From an economic point of view, if growth is possible only from stocks above a critical level because of incentive constraints, then it is optimal to decumulate the stocks to a lower steady state (maybe to zero) when the initial stocks are below that critical level. This "growth trap" can create a discontinuity in the value function. We work out an example of a value function with a discontinuity in section 4.2.

The paper is organized as follows. The next section sets up the problem and provides existence results in a general framework. Section 3 works out a simple and illustrative second best problem where incentive constraints retard growth but accumulation rates do not depend on wealth. Numerical examples illustrate how growth is influenced by the incentive constraints along the symmetric (egalitarian) equilibrium. Section 4.1 provides some general conditions under which a political "growth trap" occurs without having to explicitly compute the "second best". Again a numerical example is provided. Section 4.2 computes an explicit example of a growth trap with a discontinuous value function. Section 5 illustrates the "Mancur Olson" case, that is the case where first best policies are optimal at low stock levels but cannot be sustained at high stock levels. Finally section 6 illustrates, with a parametric example, the effect on growth of introducing an unequal distribution of income in the presence of incentive constraints. Section 7 contains final remarks. We should note that in sections 3 and 4.1 we study cases where incentive constraints result in permanently lower growth rates. In the subsequent sections we analyze cases where incentive constraints produce asymptotically lower levels of income, rather than permanently lower growth rates.

Our further and continuing research explores the effects of introducing sanctions against "defecting" from second-best policies, and of asymmetric appropriation and defection abilities across players. We also note that our analysis applies to the case of a firm where workers set the wage to capture a share of the output and capitalists decide the level of investment, as in Lancaster [1973]. (See also Benhabib and Ferri [1987].)

2. The Second Best Problem.

We consider two players characterized by two concave and strictly increasing utility functions U_i, $i = 1, 2$ and a common discount factor $\beta \in (0,1)$. k_t represents the capital stock at time t. The production function $f(k_t)$ is concave, increasing and $f(0) \geq 0$. The feasible paths of the consumption sequences must satisfy $f(k_t) - c_t^1 - c_t^2 \leq k_{t+1}$, and $c_t^1, c_t^2 \geq 0$, $t = 0, 1, \ldots$ In our game, histories at time t are sequences of consumption pairs $h_t = (c_1^1, c_1^2, \ldots, c_t^1, c_t^2)$ and strategies are maps from histories to consumptions. For a given initial stock k, the second best value is defined by

$$v_{sb}(k) \equiv \sup \sum_t {}_0^{\infty} \beta^t [\alpha_1 U_1(c_t^1) + \alpha_2 U_2(c_t^2)] \tag{2.1}$$

where the supremum is taken over the sequences $(c_t^1, c_t^2)_{t \geq 0}$ of subgame perfect equilibrium outcomes and $\alpha_1, \alpha_2 \geq 0$.

The purpose of this section is to prove that the second best is achieved over a smaller set of SPE. We start with a few definitions.

To avoid ambiguities, we describe in detail how the <u>allocation of consumption</u> is regulated. In the following it will be useful to distinguish between attempted consumption and consumption (the first is the consumption a player is trying to get, the second is what the allocation rule gives him). For a given capital stock k and two attempted consumptions c_1 and c_2, the allocated consumption is

$$A_1(c_1, c_2, k) = \begin{cases} c_1 & \text{if} \quad c_1 + c_2 \leq f(k) \quad \text{or} \quad c_1 \leq f(k)/2 \\ f(k) - c_2 & \text{if} \quad c_1 + c_2 \geq f(k) \quad \text{and} \quad c_1 \geq f(k)/2 \geq c_2 \\ f(k)/2 & \text{if} \quad c_1, c_2 \geq f(k)/2 \; . \end{cases}$$

and similarly for a_2. Note that if $c_2 \leq f(k)/2$, then $a_1(c_1, c_2, k) = \min(c_1, f(k) - c_2)$.

This allocation rule seems natural, although our subsequent analysis can be carried out under alternative rules that may be appealing as well.

Remark. Note that the utility function of both players is strictly increasing in consumption. This in particular implies that the following pair of strategies is an equilibrium, independently of the capital stock k:

$$c_1 = c_2 = f(k)$$

Note in fact that the allocation rule gives $A_1(c_1, k, c_2) = A_2(c_2, k, c_1) = f(k)/2$ to both players. If $c_2 = f(k)$, for any choice of c_1 the capital stock in the next period is zero. So by reducing c_1 the first player can only reduce his payoff.

The fast consumption strategy is the stationary strategy defined by:

$$\bar{c}_i(k) = f(k).$$

As noted above, it is clear that the pair $(\bar{c}_1, \bar{c}_2)$ is a SPE, since the utility functions of the players are strictly increasing.

The value of this equilibrium to player i is given by:

$$\bar{v}_i(k) = \sum_t {}_0^\infty \beta^t U_i(\bar{c}^i(k_t)), \quad i=1,2$$

where $k_0 = k$, $k_t = f(k_{t-1}) - \bar{c}^1(k_{t-1}) - \bar{c}^2(k_{t-1})$, $t \geq 1$. Of course if $f(0) = 0$, the above summation reduces to $U_i(f(k)/2)$.

A trigger strategy pair is described by an agreed consumption path $(c_t^1, c_t^2)_{t\geq 0}$ and the threat of a shift to a fast consumption equilibrium after the first defection is detected.

The individual rationality constraint on an outcome path is the condition:

$$\sum_t \beta^t U(c_t^i) \geq \bar{v}_i(k) .$$

Clearly, in a SPE, the equilibrium outcome of the equilibrium of any subgame satisfies this inequality.

Consider now a trigger strategy equilibrium. For any capital stock k and equilibrium consumption c of the other player the value of defection is the value for a player of deviating optimally, that is:

$$v_i^D(k,c) \equiv \operatorname{Max} \left\{ \sup_{c' \geq 0} \{U_i(A_i(k,c,c')) + \sum_{t=1}^{\infty} \beta^t U_i(\bar{c}^i(k_t))\}, \quad \bar{v}_i(k) \right\}$$

where

$$k_1 = f(k) - c_0 - A_i(k, c, c'),\quad k_{t+1} = f(k_t) - \bar{c}_1(k_t) - \bar{c}_2(k_t),\quad t \geq 1.$$

Note that this optimization problem can be expressed without the maximization operator in defining v_i^D by simply adding the constraint $v_i^D(k, c) \geq \bar{v}_i(k)$. We denote by $c_i^D(k, c)$ the optimal deviation. In the games we consider such optimal consumption exists and is unique, so no ambiguity is possible there.

The following lemma is clear. We state and prove it for completeness.

Lemma 2.1. Let $(c_t^1, c_t^2)_{t\geq 0}$ be the outcome of a SPE, ξ say. Then the trigger strategy pair with this agreed consumption path is an SPE, ξ' say.

Proof. For any history h_t, we denote $v_i(h_t)$ the value to the i^{th} player of the equilibrium in ξ starting with h_t. We only need to consider equilibrium histories $h_{t-1} = (c_1^1, c_2^1, \ldots, c_{t-1}^1, c_{t-1}^2)$. Let k_t be the capital stock. We claim that c_t^2 is an optimal choice for player 2 next period, in ξ'. The best alternative choice is $c_2^D(k_t, c_t^1) \equiv c^D$. In the equilibrium ξ such a choice would give him a payoff of $U_2(c^D)$ plus the equilibrium value of the subgame starting at (h_t, c_t^1, c^D). In the equilibrium of this subgame, the individual rationality constraint is satisfied, so

$$U_2(c_t^2) + \beta v_2(h_t, c_t^1, c_t^2) \geq U_2(c^D) + \beta v_2(h_t, c_t^1, c^D) \geq U_2(c^D) + \beta \bar{v}_2(f(k_t) - c_t^1 - c^D) = v_2^D(k_t, c_t^1)$$

and our claim follows. ∎

It follows that the supremum in the definition of second best is the same as the supremum over trigger strategy equilibria.

This reduction allows us also to prove that the second best value is in fact achieved. We turn to this now. Let α_1, $\alpha_2 \geq 0$ be weights attached to the players. From what we have seen, the second best is the solution of the problem

$$v_{sb}(k) = \sup_{\{(c_t^1, c_t^2)_{t\geq 0}\}} \sum_t \beta^t [\alpha_1 U_1(c_t^1) + \alpha_2 U_2(c_t^2)] \tag{2.2}$$

subject to $f(k_t) - c_t^1 - c_t^2 \geq k_{t+1}$,

$$\sum_t {}_0^\infty \beta^{t+i} U_j(c_{t+i}^j) \geq v^D(k_i, c_i^{j'}) \qquad i = 1, 2, \ldots \quad j = 1, 2.$$

In the following we shall refer to this as the second best problem.

We assume now that the production function f and the discount factor β satisfy:

$$\lim_{k\to+\infty} f'(k)\beta < 1. \tag{A0}$$

Then we have

<u>Lemma 2.2</u>

1. A solution to the second best problem exists.
2. The function v_{sb} is uppersemicontinuous.

Proof. For every capital stock k the set of admissible paths is the set of sequences $\{(k_t, c_t^1, c_t^2)_{t\geq 0}\}$ such that (1) and (2) above are satisfied. In a properly chosen weighted space this set is compact (because it is a closed subset of an order interval). Now existence follows immediately from the continuity of the function $\{(c_t^1, c_t^2)_{t\geq 0}\} \mapsto \sum \beta^t[\alpha_1 U_1(c_t^1) + \alpha_2 U_2(c_t^2)]$.

For the second statement, note that the correspondence defining the set of admissible paths has a closed graph, and since the image space is compact, it is also uppersemicontinuous. Now apply the Maximum Theorem. ■

3. A Simple Example of Second Best Equilibrium with no Wealth Dependence.

We will start by exploring a simple example of a second best equilibrium in detail to illustrate how growth rates may differ between first and second best equilibria. This first example is simple because growth rates on equilibrium paths will turn out to be independent of the levels of wealth, that is of the capital stock. More interesting and complex cases will be studied later.

Each of the two identical players in this example have utility functions given by

$$U_i = \sum_0^\infty \beta^t \left(\frac{1}{1-\epsilon}\right)(c_t^i)^{1-\epsilon} \tag{3.1}$$

where $0 < \beta < 1$ and $0 < \epsilon < 1$.[3] Production is linear, and is given by

[3] With some caution it is possible to extend the analysis to the cases where ϵ is less than or equal to zero.

$$y = ak + b\ell \tag{3.2}$$

where $k \geq 0$ is the capital stock, and a, b and ℓ are non-negative constants.

We will consider in this example symmetric equilibria (setting $\alpha_1 = \alpha_2 = 1$ for the problem given by (2.2) above), where both players get equal consumption levels. In section 6 we discuss the implications of the unequal weighting of the players, and therefore of unequal consumption allocations, on the growth rates along second-best equilibria.

In the symmetric case, the total utility of each player along the first-best equilibrium can be described by a dynamic program:

$$\hat{v}(k) = \max_{0 \leq c \leq \frac{y}{2}} \left(\frac{1}{1-\epsilon}\right) c^{1-\epsilon} + \beta \hat{v}(y - 2c) \tag{3.3}$$

where $y = ak + b\ell$. The solution to the program is given by the consumption function:

$$\hat{c} = \text{Min}(\hat{\lambda} y + \hat{\eta},\ y/2) \tag{3.4}$$

where[4]

[4] A sufficient interiority condition for $\hat{c} = \hat{\lambda} y + \hat{\eta} \leq y/2$ for all $k \geq 0$ is easily computed to be $\beta a \geq 1$. This condition will be satisfied in all our examples below.

$$\hat{\lambda} = \frac{1}{2}(1 - \beta^{\frac{1}{\epsilon}} a^{\frac{(1-\epsilon)}{\epsilon}}) \geq 0 \tag{3.5}$$

$$\hat{\eta} = \frac{\hat{\lambda} b \ell}{a-1} \geq 0 \tag{3.6}$$

and where we have imposed the restrictions $a > 1$, $\beta^{1/\epsilon} a^{(1-\epsilon)/\epsilon} < 1$ to avoid negative consumption levels and to assure a well-defined value function. For any $\lambda \geq 0$ such that $c = \lambda y + \eta \leq y/2$ along the equilibrium path, with $\eta = (\lambda b \ell)/(a-1)$, the value function is given by[5]

$$v(k) = s\left[y + \frac{b\ell}{a-1}\right]^{1-\epsilon} \tag{3.7}$$

where

$$s(\lambda) = \frac{(1/(1-\epsilon))\lambda^{1-\epsilon}}{1 - \beta(a(1-2\lambda))^{1-\epsilon}} . \tag{3.8}$$

We note for further use that s is derived here for arbitrary $\lambda \geq 0$, not only for the first-best $\hat{\lambda}$. We will use this fact in deriving the second-best value function later on.

---

[5] We can also express v(k) as

$$\begin{aligned} v(k) &= ((1-\epsilon)(1-\beta(a(1-2\lambda))^{1-\epsilon}))^{-1}(\lambda y + \lambda b\ell/(a-1))^{1-\epsilon} \\ &\leq ((1-\epsilon)(1-\beta(a(1-2\lambda)^{1-\epsilon}))^{-1}(y/2)^{1-\epsilon} \end{aligned}$$

since $v(k)$ above is defined for $c = \lambda y + \lambda b\ell/(a-1) \leq y/2$ and $c \geq 0$.

When a player defects against first-best play by his opponent, he must choose his consumption in the current period taking into account that trigger strategies will be enacted subsequently. Optimal defection value is therefore given by

$$v^D(k, \hat{c}(k)) = \max_{0 \le c_D \le (1-\hat{\lambda})y - \hat{\eta}} \left\{ \frac{1}{1-\epsilon} c_D^{1-\epsilon} + \beta \left(\frac{1}{1-\epsilon} \right) \left(\frac{1}{2} [a((1-\hat{\lambda})y - \hat{\eta} - c_D) + b\ell] \right)^{1-\epsilon} + \left(\frac{\beta^2}{1-\beta} \right) \left(\frac{1}{1-\epsilon} \right) \left[\frac{1}{2} b\ell \right]^{1-\epsilon} \right\} \quad (3.9)$$

where $\hat{c}(k) = \hat{\lambda}y = \hat{\eta} \le y/2$.

This value reflects the trigger strategy equilibrium for which following a defection, all output is consumed in equal shares by the two players. In the period after defection takes place, all capital is exhausted and subsequently the only output produced each period is $b\ell$.

In general, the optimal defection policy for consumption is given by

$$c_D(k, \lambda y + \eta) = \text{Min}\left(M \left((1-\lambda)y + \left(\frac{a-1}{a} - \lambda \right) \left(\frac{b\ell}{a-1} \right) \right), (1-\lambda)y - \eta \right) \quad (3.10)$$

whenever the other player's consumption policy is $\lambda y + \eta \le y/2$, for $\eta = \dfrac{\lambda b\ell}{a-1}$ and any λ and where $M = \left[\left(\frac{\beta a}{2}\right)^{1/\epsilon}\left(\frac{2}{a}\right) + 1\right]^{-1} < 1$. (Of course when the other player chooses $\lambda = \hat{\lambda}$, he is following the first-best strategy.) If, in addition, c_D is interior, that is if $M\left((1-\lambda)y + \left(\frac{a-1}{a} - \lambda\right)\frac{b\ell}{a-1}\right) \leq (1-\lambda)y - \frac{\lambda b\ell}{a-1}$, the value of optimal defection becomes

$$v^D(k, (1-\lambda)y + \eta) = s_D\left[y + \left(\frac{((a-1)/a)-\lambda}{1-\lambda}\right)(b\ell/(a-1))\right]^{1-\epsilon} + \left(\frac{\beta^2}{1-\beta}\right)\left(\frac{1}{1-\epsilon}\right)\left(\left(\frac{1}{2}\right)b\ell\right)^{1-\epsilon} \quad (3.11)$$

where $s_D = (1-\lambda)^{1-\epsilon}(1-\epsilon)^{-1}[M^{1-\epsilon} + \beta((1-M)a/2)^{1-\epsilon}] = (1-\lambda)^{1-\epsilon}(1-\epsilon)^{-1}M^{-\epsilon}$. We note that $v^D(k; c)$ is the value of optimal defection against a player with consumption policy $c = \lambda y + \eta \leq y/2$, for an arbitrary λ, which can of course be the first-best policy if $c(k) = \hat{\lambda}y + \hat{\eta} \leq y/2$.

For first best policies that constitute an equilibrium, the values that they generate for each player must dominate the values of defection at each point on the equilibrium path, that is $v(k) \geq v^D(k, \hat{c}(k))$ for all k on the equilibrium path. As we illustrate in later examples however $v(k)$ and $v^D(k)$ can intersect, so that first-best outcomes can be enforced from some k's, but not others. This "state" or "wealth" dependence of equilibria was explored in Benhabib and Radner

[1987].

In this first example of second best equilibria we eliminate the state dependence by simplifying the production function. Setting $b = 0$, which implies $\eta = 0$, yields

$$v^D(k,c) = M^{-\epsilon}(1-\lambda)^{1-\epsilon}(ak)^{1-\epsilon} \equiv s_D(\lambda)(ak)^{1-\epsilon} \tag{3.12}$$

$$\hat{v}(k) = s(ak)^{1-\epsilon} \tag{3.13}$$

$$\hat{c}(k) = \hat{\lambda}y = \frac{1}{2}(1-\beta^{1/\epsilon}a^{(1-\epsilon)/\epsilon})y \le y/2 \tag{3.14}$$

$$c^D = M(1-\lambda)y \equiv \lambda_D y \le (1-\lambda)y \tag{3.15}$$

Both $v(k)$ and $v^D(k,\ \hat{c}(k))$ start at the origin but do not intersect if $s \neq s_D$. Clearly if $s(\hat{\lambda}) \ge s_D(\hat{\lambda})$, the second best equilibrium is also the first-best at every k.

A symmetric second-best equilibrium (the non-symmetric case will be discussed later in section 6) with incentive compatibility constraints will be given by the solution to the following problem:

$$v_{sb}(k) = \max_{0 \le c \le \frac{y}{2}} (1-\epsilon)^{-1}c^{1-\epsilon} + v_{sb}(ak - 2c)$$

subject to $v_{sb}(k) \ge v^D(k,c)$. Alternatively, if σ is a Lagrange multiplier, the problem can be defined as

$$v_{sb}(k) = \underset{0 \le c \le \frac{y}{2}}{\text{Max}} \; (1-\epsilon)^{-1} c^{1-\epsilon} + \beta v_{sb}(ak-2c) + \sigma(v_{sb}(k) - v^D(k,c)) \qquad (3.16)$$

We now characterize the solution to (3.16):

<u>Proposition 3.1</u>. Let $U(c) = (1-\epsilon)^{-1}c^{1-\epsilon}$ and $y = ak$ where $0 < \epsilon < 1$, $\hat{\lambda} = \frac{1}{2}(1-\beta^{1/\epsilon}a^{(1-\epsilon)/\epsilon}) \ge 0$. Then the symmetric second-best consumption policy is given by

$$\text{(a)} \quad c_{sb} = \hat{\lambda} y \quad \text{if} \quad s(\hat{\lambda}) \ge s_D(\hat{\lambda}) \qquad (3.17a)$$

$$\text{(b)} \quad c_{sb} = \lambda_s y \quad \text{if} \quad s(\hat{\lambda}) < s_D(\hat{\lambda}) \qquad (3.17b)$$

where $\lambda_s = \text{Min}\,\{\lambda \,|\, \lambda \in [\hat{\lambda}, z],\ s(\lambda) = s_D(\lambda)\} \ne \emptyset$, $z = \frac{M}{1+M} \le \frac{1}{2}$ and

$$M = \left[1 + \left(\frac{\beta a}{2}\right)^{1/\epsilon}\left(\frac{2}{a}\right)\right]^{-1} \le 1\,.[6]$$

<u>Proof</u>: See Appendix A.1.

---

[6] We note that when λ_s is determined from $s(\lambda_s) = s_D(\lambda_s)$, we also obtain $\lambda_s \le 1/2$ which is required to hold in the analysis above. Furthermore we have $\lambda_s \le z = M(1+M)^{-1}$ which implies $\lambda_s \le M(1-\lambda_s) = \lambda_D$.

Figure 1 below illustrates the second-best solution. The solution is to find λ_s which equates the value for each player of following the consumption policy $c_{sb} = \lambda_s y$ with the value of defecting from it. This requires equating $s_s \equiv s(\lambda_s) = s_D(\lambda_s)$. In other words, consumption rates must be increased and accumulation slowed down up to the point where defection is no longer attractive. The situation is more complicated in the non-symmetric case, as we shall see later. $\hat{V}^D$ in Figure 1 is the value of defecting against a player following first best strategies.

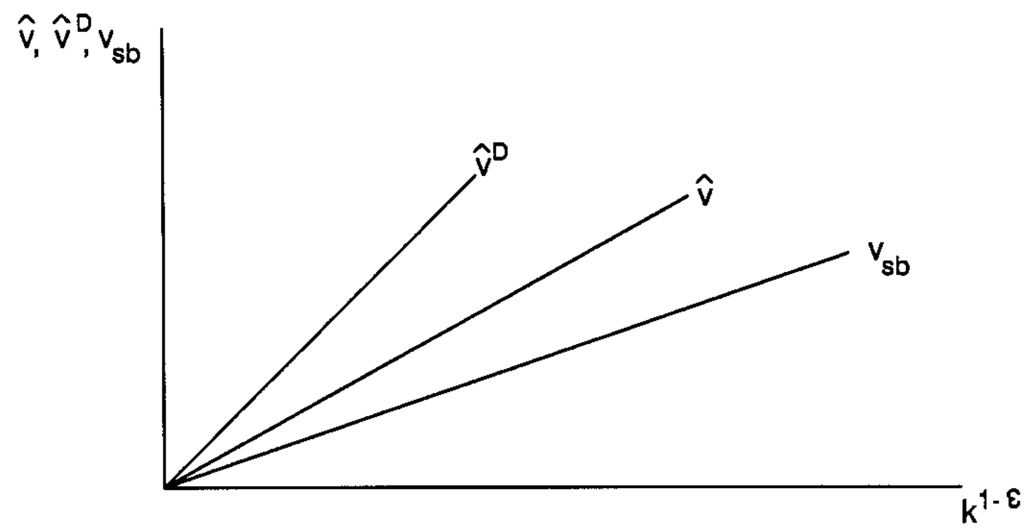

Figure 1

The following numerical values illustrate the effects of incentive compatibility constraints on economic growth along the symmetric equilibrium for the proposition above. We set a = 3.3, b = 0, $\beta = 0.325$[7] (implausibly high

---

[7] Note that $\beta a > 1$ so that $c = \hat{\lambda} y + \hat{\eta} < y/2$ for all $k \geq 0$, as pointed out in the previous footnote.

discounting of course) and $\epsilon \doteq .5$. For these values $v^D(k, \hat{c}(k)) > \hat{v}(k)$ for all $k > 0$, where $\hat{v}(k)$ corresponds to the first-best values with policies $c = \hat{\lambda}y$. We compute $\hat{\lambda} = 0.326$, $\lambda_s = 0.349$. These magnitudes imply that if the first best could be sustained, the capital stock would perpetually grow (since we have an (a-k) technology) at 15%. On the second-best path however the economy grows at -0.0015%, that is it contracts. Of course parameters were chosen to make this stark point. Other parameter values would allow positive growth along the second best equilibrium, but at a slower rate then the first best. Of course in some cases the first-best may be enforcable as an equilibrium from all stocks so that incentive constraints will not bind. Finally, we note that for the parameters above, it is easy to check that $\sigma = 0.6032$, and that $y - 2c_{sb}(k) \geq 0$ and $y - \hat{c}(k) - c_D(k) \geq 0$ for all $k \geq 0$, as assumed in the computations.

In the above example the second-best equilibrium is sustained by a grim trigger strategy, that is a trigger strategy where players exhaust the stock if a defection occurs. We can also compute the best symmetric equilibrium that is sustained by a weaker trigger strategy, that is a strategy where players revert to an interior stationary Markov equilibrium if a defection occurs. Such an equilibrium solves for player 1 the problem given by

$$v_1^M(k) = \underset{c_1}{\text{Max}}\ (1-\epsilon)c_1^{1-\epsilon} + \beta v_1^M(ak - c_1 - c_2(k))$$

where in the symmetric case the solution satisfies $c_1(k) = c_2(k)$, and $c_2(k)$ is also a best response for the second player. This problem is easily solved, with $c_i(k) = \lambda_M ak$, where $\lambda_M \geq 0$ solves $(1 - 2\lambda_M)^\epsilon = \beta a^{1-\epsilon}(1 - \lambda_M)$ for $i = 1, 2$. It is easy to show that $\lambda_M > \hat{\lambda}$.

Now we can compute the best sustainable symmetric equilibrium with trigger strategies where players revert to the above stationary Markov equilibrium after a defection. On such an equilibrium, using the parameters above, each player consumes $\lambda_{sM}(ak)$ with $\lambda_{sM} = 0.435$. This yields a contraction rate of about 43%, much higher than the contraction rate for the second-best equilibrium under grim triggers. This is not surprising since the "grimmer" the trigger strategy, the closer the best enforceable equilibrium will be to the first-best. The point is that even along symmetric second-best equilibria sustained by grim strategies, growth may not be possible. Of course it is easy to construct examples where positive growth occurs, at different rates, for first-best and second-best equilibria, as well as for equilibria obtained by trigger strategies that are weaker than grim strategies.

4. Wealth Dependent Growth

4.1 General Conditions for a Growth Trap

When $b > 0$ for the example of the previous section, it is in general not possible to find a constant λ_s to equate $v(k)$ and $v^D(k, c)$. In particular for $\lambda = \hat{\lambda}$, $\hat{v}(k)$ and $v^D(k, \hat{c})$ may intersect at some k. If $\hat{v}(k) \geq v^D(k, \hat{c})$ for $\lambda = \hat{\lambda}$ and $k \geq \underline{k}$, first-best policies will be sustainable as equilibria for $k \geq \underline{k}$. From initial conditions below $\underline{k}$ where $\hat{v}(k) < v^D(k, \hat{c}(k))$, it may be possible to construct "switching" equilibria (which are not necessarily second-best), along which growth occurs at a rate slower than first-best rates until $\underline{k}$ is reached, and first best policies are followed once $\underline{k}$ is attained. This was demonstrated in Benhabib and Radner [1987]. In this section we will derive conditions under which the second-best growth rates will be wealth dependent: in particular we will find conditions under which first-best growth rates are sustainable from high stocks while growth is not at all possible from low stocks because of incentive compatibility constraints. The intuition for the result is simple: relative to first best levels, consumption rates must be increased and accumulation slowed down to prevent defection. When stocks are low, consumption must be increased so much to prevent defection that growth is no longer possible. Examples will follow.

The general proposition below will allow us to show how growth rates are affected by wealth levels.

<u>Proposition 4.1.1</u>. Assume that

(i) for some k, $\hat{v}(k) < v^D(k, \hat{c}(k))$

(ii) there exists $\tilde{c}$ such that $U(\tilde{c}) + \beta\hat{v}(f(k) - 2\tilde{c}) = v^D(k, \tilde{c})$ and $f(k) - 2\tilde{c} \leq k$.

Then $f(k) - 2c_{sb}(k) \leq k$.

<u>Proof</u>. We may assume without loss of generality that $\tilde{c}$ is the least c such that condition (ii) is satisfied. Assume now that $f(k) - 2c_{sb}(k) > k$. Then clearly $c_{sb}(k) < \tilde{c}$. But then

$$v^D(k, c_{sb}(k)) \leq v_{sb}(k) \tag{4.1.1}$$

$$\leq U(c_{sb}(k)) + \beta\hat{v}(f(k) - 2c_{sb}(k)) \tag{4.1.2}$$

$$< v^D((k), c_{sb}(k)) \tag{4.1.3}$$

(4.1.1) holds by the definition of the second-best. (4.1.2) holds since $\hat{v} \geq v_{sb}$.

(4.1.3) holds by the fact that $U(c') + \beta\hat{v}(f(k) - 2c') < v^D(k, c')$ for every $c \in (\hat{c}(k), \tilde{c})$; this interval is non-empty because of assumptions (i) and (ii). Therefore we have a contradiction, which concludes the proof. ■

We can now construct an example to apply the proposition above. We assign new parameter values to the example in the previous section as follows: $a = 1.875$, $b = 0.2$, $\beta = .55$,[8] $e = .45$. With these values, for $k \geq 1$ we have $\hat{v}(k) > v^D(k, \hat{c}(k))$, while for $k \leq .9$, $\hat{v}(k) < v^D(k, \hat{c}(k))$. It is easily shown that for $k > 0.001$, the first-best strategies lead to growth at the rate of about 7%. Thus for $k \geq 1$ the 7% growth rate can be sustained as a first-best equilibrium. However, for k in [0.1, 0.4] conditions of the above proposition apply. For $k = 0.4$ ($= 0.1$) $\tilde{c}$ defined in the proposition is given by 0.19111 (0.10112) and $y - 2\tilde{c} - k < 0$. Therefore the second-best equilibrium cannot generate growth for $k \in [0.1, 0.4]$. Of course as in the previous example, we can check that for $k \geq 0.1$ we have $y - 2\hat{c}(k) > 0$ and $y - \hat{c}(k) - c_D(k) > 0$.

The above example and proposition allow us to starkly establish how growth rates can depend on wealth because incentive constraints can be strongly binding at some wealth levels and weakly binding at others. In fact, for our example, incentive constraints are not binding at all for $k \geq 1$ but binding strongly enough to deter positive growth for $k \in [0.1, 0.4]$. The proposition above allowed us to construct the example without explicitly calculating the second-best equilibrium, which in general is quite difficult to compute. Nevertheless, in the next sub-section we provide a fully characterized example of a a second-best equilibrium for which growth is possible from high levels of wealth but not

---

[8] Once again $\beta a > 1$ so that $\hat{c} = \hat{\lambda}y + \hat{\eta} < y/2$ for all $k \geq 0$, as pointed out in footnote 4.

from low levels of wealth and for which the value function is discontinuous.

4.2 A Numerical Example of a Growth Trap Equilibrium with a Discontinuous Value Function.

This example derives explicitly a second-best policy for which growth towards a high steady state occurs from large stocks but not from low stocks. First best policies which are not incentive compatible always lead to the high steady state. The value function for this example is discontinuous although technology and preferences are convex and continuous. In fact the first best policies lead to a unique positive steady state. As in the previous section, the players have identical preferences and are equally weighted.

We consider

$$f(k) = \begin{cases} Ak & k \le 1 \\ A + B(k-1) & k \ge 1, \quad \frac{B}{2} < 1 \end{cases} \qquad (4.2.1)$$

with $A = 5/2$ and $\beta = 1/2$;

$$U(c) = \begin{cases} c & \text{if } c \le 1 \\ 1 + b(c-1) & \text{if } c \ge 1 \,. \end{cases}$$

Since $A\beta > 1 > B\beta$, $k = 1$ is a steady state capital stock for the optimal growth problem with $c = 3/4$ as the steady state state consumption.

We assume b is small:

$$B\beta < b < \frac{A\beta}{2} < 1 \,.$$

The first-best policies, the associated value function and the value of optimal defection are derived in appendix A2.1.

In the following proposition we compute the second best value and to policy function for values of $k \le 1$. The second best value will be piecewise linear. To lighten the notation, we let

$$k_0 = 1,\ k_1 = \frac{14}{15},\ k_2 = \frac{68}{75},\ k_3 = \frac{1018}{1125},\ k_4 = \frac{4}{5},\ k_5 = 0\ ;$$
$$a_0 = \frac{5}{4},\ a_1 = \frac{25}{16},\ a_2 = \frac{175}{32},\ a_3 = \frac{25}{16},\ a_4 = \frac{5}{4}\ ;$$
$$b_0 = \frac{1}{4},\ b_1 = -\frac{1}{24},\ b_2 = -\frac{43}{12},\ b_3 = -\frac{1}{4},\ b_4 = 0\ .$$

Then we have

Proposition 4.2.1. The second best value function is given by

$$V_{sb}(k) = a_i k + b_i \quad \text{for } k \in [k_{i+1}, k_i)\ .$$

v_{sb} is continuous and concave on $[k_3, k_0]$; $v_{sb}(k_3-) < v_{sb}(k_3+) = v_{sb}(k_3)$ and v_{sb} is convex and continuous on $[0, k_3]$.

The second best policy is

$c_{sb}(k)$	$=$	$\hat{c}(k) = 5/4\ k - 1/2$	$k \in [k_1, k_0]$
	$=$	$2/3$	$k \in [k_2, k_1]$
	$=$	$-25/4\ k + 19/3$	$k \in [k_3, k_2]$
	$=$	$c_D(k, c_{sb}(k)) = 1$	$k \in [k_4, k_3]$
	$=$	$c_D(k, c_{sb}(k)) = (Ak)/2$	$k \in [0, k_4]$

The path of capital stock for the second best equilibrium outcome is: if $k \in [k_{i+1}, k_i]$, $i \le 2$ then the capital stock converges to the steady state in i+1 periods,

if $k \in (k_4, k_3)$ ($k \in [0, k_4]$ respectively), then the capital stock converges to zero in two (respectively one) period.

Proof: See Appendix A2.2.

Figure 2a shows the value function and figure 2b shows the consumption function described in the above proposition. The figures are not drawn to scale.

Remark. The second best consumption policy is the same as the first-best policy for $k \geq k_1$. Then for $k \in [k_3, k_1]$ it is the minimum consumption which makes second best value equal to the value of defection. Consumption is decreasing over the range where the incentive compatibility constraint is binding, and then increasing when the second best solution is the first best. Overall, the second best consumption is non-monotonic, even in the region where we have steady growth. Note that over $[k_3, k_1)$ the first-best consumption is lower than the second-best. As k increases the incentive constraint becomes less binding and second-best consumption decreases with k along the equilibrium. The intermediate phase ($[k_2, k_1]$) has the lowest consumption above k_4. Finally observe that the consumption policy is continuous except at k_3.

Remark: The reason for the discontinuity of the consumption policy and the value function may be understood as follows. As k decreases, higher levels of consumption are needed in order to make the value of second best and the value of defecting from it equal to each other. To higher levels of consumption corresponds a reduction in the continuation value and a reduction in the post-defection value. The rate at which these two second-period values change is the critical factor. The rate for the defection value is constant at 5/8; the rate

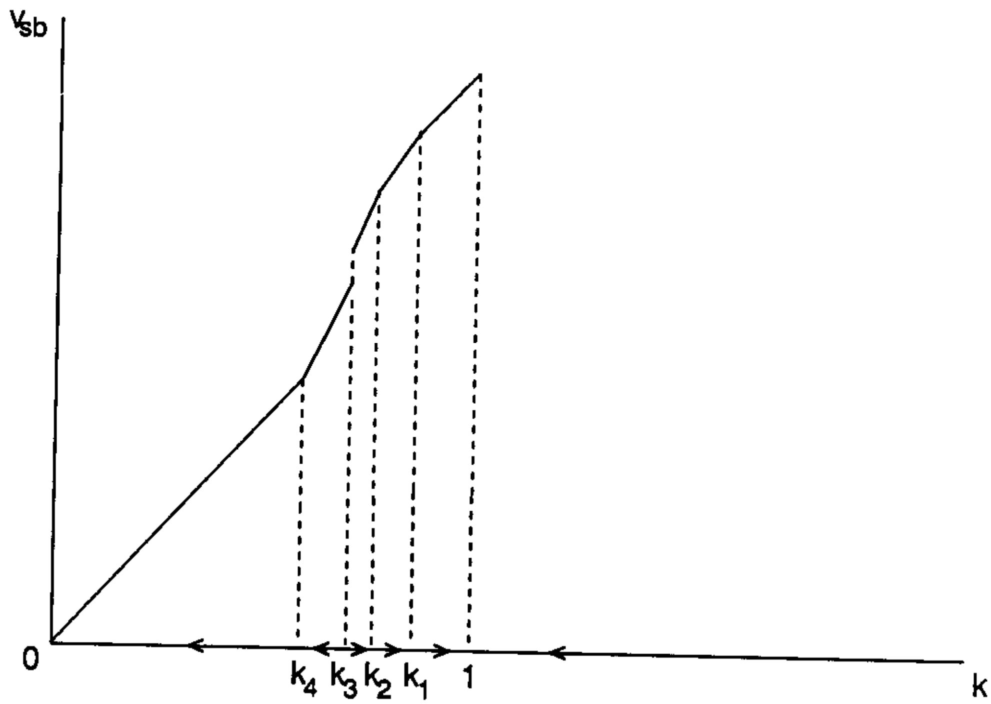

Figure 2a

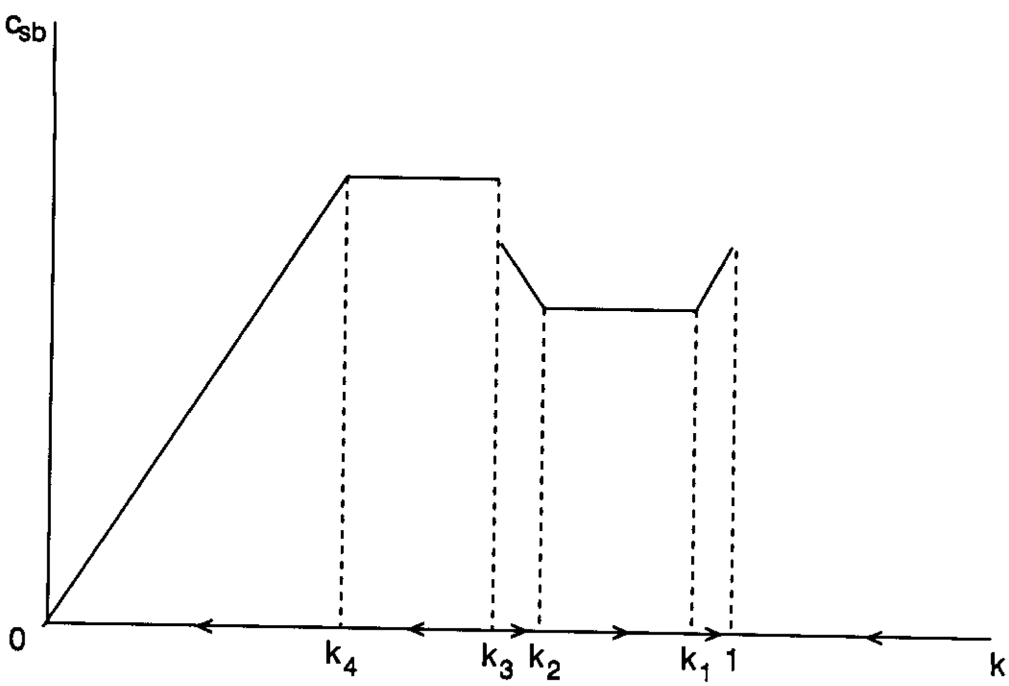

Figure 2b

for the second-best continuation value is changing with k, because the second best value is concave. The difference between these two rates is 1 - a(Ak-2c) + 5/8. When Ak-2c is less than k_2 this difference changes sign (and becomes negative), so no increase in consumption can equate the second best and the value of defection, and at the same time allow the capital stock not to decline.

5. The Cobb-Douglas Production, Linear Utility (The "Mancur Olson" Case).

Previous examples showed how incentive constraints could result in equilibria for which growth occurs from high stocks but not from low stocks. In the following example the opposite is true. When stocks are low defection is not attractive and players follow first best policies to accumulate precious stocks. As stocks get larger defection becomes more attractive and accumulation has to slow down. First best policies are abandoned and the economy stays short of the first best steady state. In the spirit of the work of Mancur Olson [1982] (stretching it a bit) inefficiency emerges at high rather than low levels of wealth.

Let the production function be

$$f(k) = k^{\alpha}, \ \alpha \in (0,1); \tag{5.1}$$

and utility function

$$U(c) = c \tag{5.2}$$

The optimal solution has a steady state given by

$$k^* = (\alpha\beta)^{\frac{1}{1-\alpha}} \tag{5.3}$$

and the optimal policy is, as usual, i.e.,

$$\hat{c}(k) = \begin{cases} 0 & \text{if} \quad k \le k^{*\frac{1}{\alpha}} \\ \dfrac{k^\alpha - k^*}{2} & \text{if} \quad k \ge k^{*\frac{1}{\alpha}} \end{cases} \tag{5.4}$$

and

$$\hat{v}(k^*) = \frac{(\alpha\beta)^{\frac{1}{1-\alpha}}}{1-\beta}$$

Consider now a given level of capital stock k and consumption c_0 of one of the players. Then the value of defection for the other player is

$$v^D(k,c) \equiv \max_{c' \ge 0} \; c' + \frac{\beta}{2}(k^\alpha - c - c')^\alpha \tag{5.5}$$

The optimal defection consumption is clearly in the interval $[0,\ k^\alpha - c_0)$. In particular

$$c_D(k,c) = \begin{cases} 0 & \text{if } (k^\alpha - c)^{\alpha-1} > \dfrac{2}{\alpha\beta} \\ k^\alpha - c - \gamma & \text{otherwise} \end{cases} \tag{5.6}$$

with $\gamma \equiv \left(\dfrac{\alpha\beta}{2}\right)$, and (5.7)

$$v^D(k,c) = \begin{cases} \frac{\beta}{2}(k^\alpha - c)^\alpha & \text{if } (k^\alpha - c)^{\alpha-1} > \frac{2}{\alpha\beta} \\ k^\alpha - c + \varsigma & \text{otherwise} \end{cases} \tag{5.8}$$

where

$$\varsigma \equiv \gamma^\alpha \frac{\beta}{2} - \gamma > 0 \tag{5.8'}$$

Note that if the net stock left by the other player, k^α - c, is too low then the optimal defection is to consume nothing.

Before we proceed we define the set of incentive compatible steady states; formally, these are the values of k such that

$$\frac{f(k)-k}{2(1-\beta)} \geq v^D\left(k, \frac{f(k)-k}{2}\right)$$

holds. These are therefore the values of k such that the value for each player of keeping k as a steady state dominates the value of defecting from this pair of k and consumption. This set will be useful in determining the second best value and policy.

For any value of α, β the set of values of k which satisfy the above inequality is the interval

$$\underline{k} \leq k \leq \bar{k} \tag{5.9}$$

which may be empty. This follows from the fact that the inequality is equivalent to $\beta k^\alpha - (2-\beta)k - 2\varsigma(1-\beta) \geq 0$ whose left hand side is concave.

<u>Proposition 5.1</u>. Let f, U, k^*, ζ, $\underline{k}$, $\bar{k}$ be defined as in (5.1), (5.2), (5.3), (5.8′), (5.9); then on the interval [max { $\underline{k}$, $\bar{k}^{1/\alpha}$ }, $\bar{k}$] we have:

(a) if $k^* < \bar{k}$, then $v_{sb}(k) = \hat{v}(k)$, $c_{sb}(k) = \hat{c}(k)$

(b) if $\bar{k} \le k^*$, then $v_{sb}(k) = (k^\alpha + \bar{k})/2 + \zeta$, $c_{sb}(k) = (k^\alpha - \bar{k})/2$.

If $\underline{k} < \bar{k}^{1/\alpha}$, $v_{sb}(\bar{k}^{1/\alpha}) > v^D(\bar{k}^{1/\alpha}, c_{sb}(\bar{k}^{1/\alpha}))$. Then for an interval $[k_1, \bar{k}^{1/\alpha}]$

$$v_{sb}(k) = \beta \frac{k^{\alpha^2} + \bar{k}}{2} + \beta\zeta \quad , \quad c_{sb}(k) = 0$$

<u>Proof</u>: See Appendix A3.

In the next table we report values of k^*, $\bar{k}$, $\bar{k}^{1/\alpha}$, for different values of α and β. The value $\underline{k}$ is not shown because in each case $\underline{k} < \bar{k}^{1/\alpha}$. Both cases where $\bar{k} < k^*$ and $k^* < \bar{k}$ appear. In the first, we know that the second best policy over $[\bar{k}^{1/\alpha}, \bar{k}]$ is to consume as much as needed to go to $\bar{k}$ in one step. However, on $[k_1, \bar{k}^{1/\alpha}]$ the second best consumption is the same as the first best consumption, which is zero. The second best accumulation path then stops at $\bar{k}$, while the first best grows to k^*. For higher k, second-best consumption is higher than the first best. Therefore on $[k_1, \bar{k}^{1/\alpha}]$, when stocks are low, players follow first-best strategies but stop doing so above $\bar{k}^{1/\alpha}$. Note that $d = v_{sb}(\bar{k}^{1/\alpha}) - v^D(\bar{k}^{1/\alpha}, c_{sb}(\bar{k}^{1/\alpha}))$. Therefore $d > 0$ implies $\underline{k} < \bar{k}^{1/\alpha}$.

$k^* < \bar{k}$	$k^* > \bar{k}$	α	β	k^*	$\bar{k}$	$\bar{k}^{1/\alpha}$	d
	✓	.975	.97	.1074	.0906	.0852	.00013499
✓		.9142	.92	.1329	.1542	.1294	.00211
	✓	.80	.65	.0380	.0251	.0099	.0026
✓		.70	.65	.0724	.0820	.0280	.0132

Figure 3 below illustrates the case where $\underline{k} < \bar{k}^{1/\alpha}$.

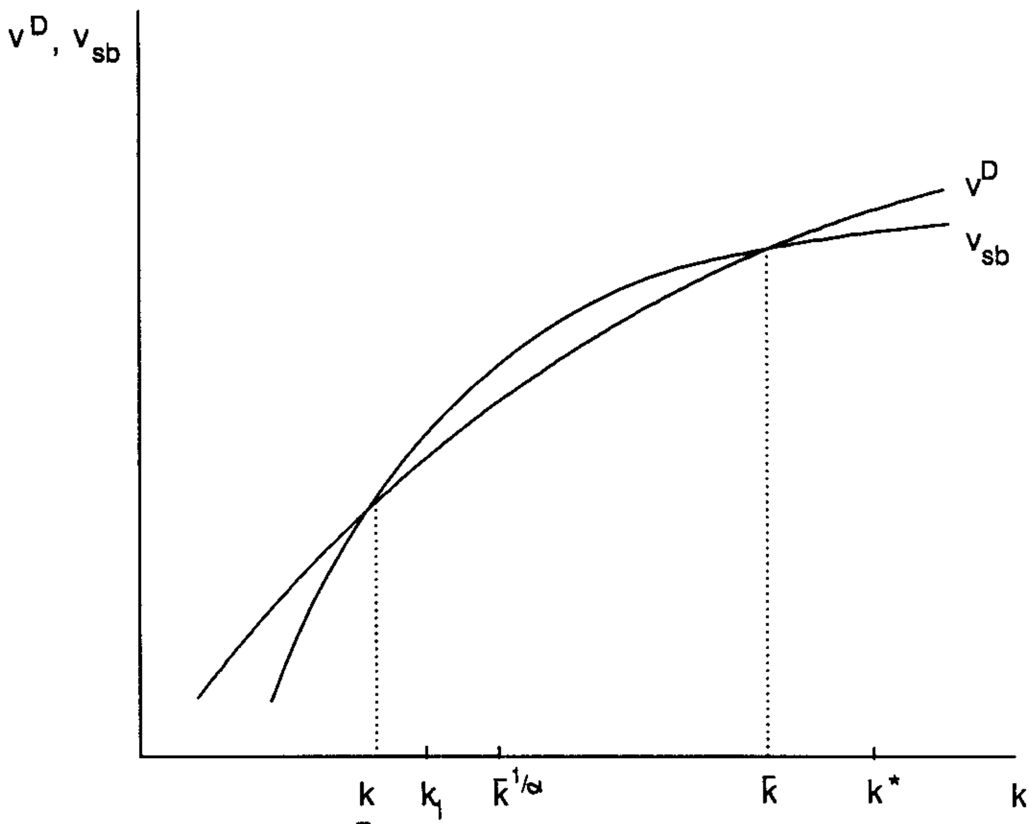

Figure 3

6. Growth and Inequality.

So far we have studied the symmetric cases when utilities and incomes of the players are equal in equilibrium. In this section we study unequal income distribution or asymmetric equilibria in the presence of incentive constraints. In particular we are interested in the effects of inequality on accumulation. In the next few Propositions we illustrate the possibilities for a particular example. Figure 4 at the end of this section summarizes our results and the range of possibilties for the production and time preference parameters.

We consider the case where

$$U_i(c) = c \qquad i = 1, 2$$

$$f(k) = \begin{cases} ak & k \le 1 \\ a + b(k-1) & k \ge 1 . \end{cases}$$

We always assume $a\beta > 1 > b\beta$, where β is the discount factor. As in the sections above, we will solve the second-best problem given by (2.2) (or (2.1)). In this case however, to characterize the set of constrained Pareto-Optima we will allow the weights α_1 and α_2 to differ between the players.

We shall consider the two cases $\alpha_1 = \alpha_2 > 0$ and $\alpha_1 > \alpha_2 \ge 0$. It is easy to imagine that in equilibrium the player with the higher weight will have a higher consumption: so the case $\alpha_1 \ne \alpha_2$ is in fact a model of inequality in income distribution.

Since $a\beta > 1$, the symmetric consumption policy for the first best is

$$\hat{c}(k) = \left\{ \frac{f(k)-1}{2} \quad \text{if } k \ge \frac{1}{a}, \qquad 0 \quad \text{if } k \le \frac{1}{a} \right\}.$$

Consider first the case where $a\beta < 1$. Whether $\alpha_1 = \alpha_2$ or not, the second best equilibrium pair of consumption policies is given by

$$c_1(k) = c_2(k) = \hat{c}(k) = \frac{ak}{2} .$$

This follows immediately from the fact that $v_{sb} \le \hat{v}$, and that the pair of consumptions is an equilibrium pair. But the interesting case is, of course,

$a\beta > 1$. We begin the analysis of this case by computing the defection value.

Recall that $v^D = \max\left\{w_D(k,c),\ \frac{f(k)}{2}\right\}$, where

$$w_D(k,c) \equiv \max_{c' \leq f(k)-c} \quad c' + \frac{\beta}{2} f(f(k) - c - c')$$

Denote by $c_w(k, c)$ the solution to this problem. The first order condition which characterizes $c_w(k, c)$ is

$$0 \in 1 - \frac{\beta}{2} f'(f(k) - c - c')$$

which gives the defection value and policy as follows.

Case 1: $a\beta > 2$:

$$c_w(k,c) = \begin{cases} \frac{a\beta}{2}(f(k) - c) & \text{if } f(k) - c \leq 1 \\ f(k) - c - 1 + \frac{a\beta}{2} & \text{if } f(k) - c \geq 1 \end{cases}$$

and

$$v^D(k,c) = \begin{cases} \max\left\{\frac{a\beta}{2}(f(k) - c),\ \frac{f(k)}{2}\right\} & \text{if } f(k) - c \leq 1 \\ \max\left\{f(k) - c - 1 + \frac{a\beta}{2},\ \frac{f(k)}{2}\right\} & \text{if } f(k) - c \geq 1\ . \end{cases}$$

Case 2: $a\beta < 2$:

$$c_w(k,c) = f(k) - c$$

$$v^D(k,c) = \max\left\{f(k) - c,\ \frac{f(k)}{2}\right\}$$

It turns out that in the case of equal income distribution the efficient solution is an equilibrium for a larger set of the parameter values.

<u>Proposition 6.1</u>: Let $\alpha_1 = \alpha_2 = 1/2$ and $a\beta > 2 - \beta$. Then the symmetric first best pair of consumption policies is an equilibrium pair, that is: $c^1_{sb}(k) = c^2_{sb}(k) = \hat{c}(k)$, and $v_1(k) = v_2(k) = \hat{v}(k)/2$, for every $k \geq 0$.

<u>Proof</u>: See Appendix A4.1.

When there is inequality in income distribution, and $a\beta > 2$, the consumption of the second player is reduced until he is indifferent between the equilibrium and defection. In particular, for low values of the capital stock his consumption is forced down to zero. At the same time the difference in value is higher: the ratio $v_1(k)/v_2(k)$ tends to infinity as k tends to zero. These are the main results of the following proposition.

<u>Proposition 6.2</u>. Let

1. $\alpha_1 > \alpha_2 \geq 0$,
2. $a\beta > 2$.

Then the functions below define a second best pair of equilibrium consumption and value functions for the first and second player respectively.

$v_1(k) = \frac{b}{2}(k-1) + \frac{a}{2} + \frac{a\beta-1}{1-\beta}$	$c_1(k) = \frac{b}{2}(k-1) + \frac{a(1-\beta)}{2} + a\beta - 1$	if $k \geq 1$
$v_1(k) = \frac{a}{2}k + \frac{a\beta-1}{1-\beta}$	$c_1(k) = \frac{a}{2}k + \frac{a\beta}{2} - 1$	if $\beta \leq k \leq 1$
$v_1(k) = s_n k + c_n$ for $k \in [\beta^{n+1}, \beta^n]$	$c_1(k) = (a-\frac{1}{\beta})k$	if $0 \leq k \leq \beta$

where $s_n = n\left(a-\frac{1}{\beta}\right) + \frac{a}{2}$, $c_n = \beta^n \frac{a\beta-1}{1-\beta}$, and

$v_2(k) = \frac{b}{2}(k-1) + \frac{a}{2}$	$c_2(k) = \frac{b}{2}(k-1) + \frac{a(1-\beta)}{2}$	if $k \geq 1$
$v_2(k) = \frac{a}{2}k$	$c_2(k) = \frac{a}{2}k - \frac{a\beta}{2}$	if $\beta \leq k \leq 1$
$v_2(k) = \frac{a}{2}k$	$c_2(k) = 0$	if $0 \leq k \leq \beta$

<u>Proof</u>: See Appendix A4.2.

The following proposition takes care of the egalitarian case.

<u>Proposition 6.3</u>. Let

1. $\alpha_1 = \alpha_2$,
2. $1 < a\beta < 2 - \beta$.

Then the only second best equilibrium pair of strategies is the pair

$$c_1(k) = c_2(k) = \left\{ \frac{ak}{2} \text{ if } k \le 1, \; bk + a - b \text{ if } k \ge 1 \right\}$$

with values

$$v_1(k) = v_2(k) = \left\{ \frac{ak}{2} \text{ if } k \le 1, \; bk + a - b \text{ if } k \ge 1 \right\}.$$

<u>Proof</u>: See Appendix A4.3.

<u>Proposition 6.4</u>: Let

1. $\alpha_1 > \alpha_2 \ge 0$,
2. $a\beta < 2$.

Then the only second best equilibrium pair of consumption policies is given by

$$c_1(k) = c_2(k) = c(k) = \left\{ \frac{ak}{2} \text{ if } k \le 1, \; \frac{bk}{2} + \frac{a-b}{2} \text{ if } k \ge 1 \right\}$$

and

$$v_1(k) = v_2(k) = v(k) = \left\{ \frac{ak}{2} \text{ if } k \le 1, \; \frac{bk}{2} + \frac{a-b}{2} \text{ if } k \ge 1 \right\}$$

<u>Proof</u>: See Appendix A4.4.

We summarize the results of this section with the following figure:

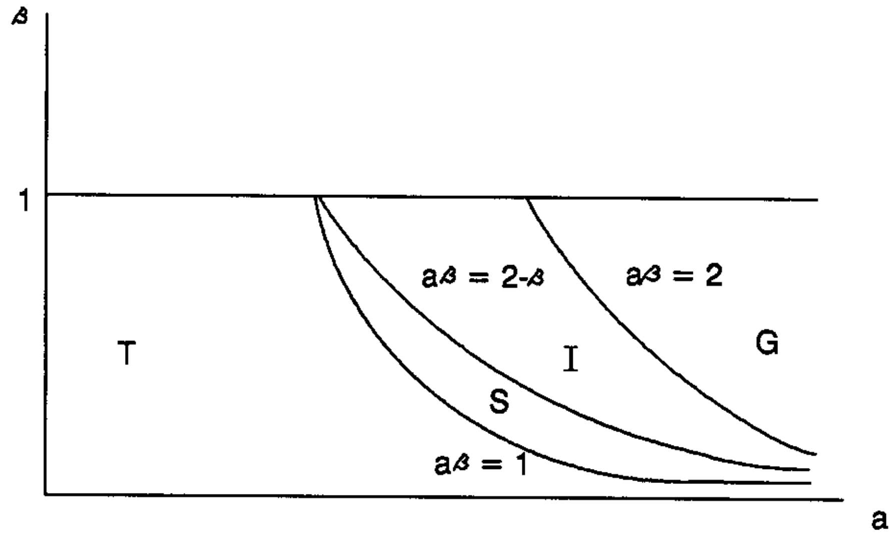

Figure 4

If $(a, \beta) \in T$, then the first best solution and the second best policies, for both cases of $\alpha_1 = \alpha_2$ and $\alpha_1 \neq \alpha_2$ give convergence to zero.

In S, I, G the first best solution grows to the steady state. In I, growth to the steady state under the second best solution is possible if $\alpha_1 = \alpha_2$, but not otherwise. In S the second best solution, for both cases $\alpha_1 = \alpha_2$ and $\alpha_1 \neq \alpha_2$ converges to zero. In G all solutions converge to the steady state.

If we think of the pairs (a, β) as possible economies, characterized by technology and a discount factor, then the set I are those economies where growth is not an equilibrium if there is inequality in income distribution; and the set S are those economies where growth is not an equilibrium even with equal income distribution because of incentive constraints or the presence of "social conflict".

Although we expect the spirit of these results to continue to hold with non-linear utility, the regions above may not be as starkly delineated in that case.

7. Final Remarks.

Our basic model can be modified in a number of ways to bring it closer to reality. The amount that each player can consume when he chooses to defect may be bounded, with the bounds possibly differing across players. This not only introduces an asymmetry between the players but can reduce the value of defection. A more complicated modification would allow players to control, at some cost, the upper bound of the consumption of their opponent. This would introduce a second policy variable for each player. Alternatively, players may institute a mechanism to impose sanctions that are costly to the defectors, but which require resources to establish and to maintain.

The disincentives for accumulation that arises from strategic behavior in our model leads to overconsumption by players. An alternative that is in the same spirit would permit players to divert resources to another productive activity that is safe from appropriation, but which provides a lower return (Switzerland). This modification would allow us to directly model capital flight, as in Tornell and Velasco [1990], but would require us to keep track of both the private assets and the capital stock whose output is subject to appropriation.

<u>References</u>.

Alesina, A. and D. Rodrik (1991), "Distributive Politics and Economic Growth," working paper, Harvard University.

Azariadis, C. and A. Drazen (1990), "Threshold Externalities and Economic Development," <u>Quarterly Journal of Economics</u>, 501-526.

Baumol, W.J., S.A.B. Blackman and E.N. Wolff (1989), <u>Productivity and American Leadership: The Long View</u>, MIT Press, Cambridge, Mass.

Benhabib, J. and G. Ferri (1987), "Bargaining and the Evolution of Cooperation in a Dynamic Game," <u>Economics Letters</u> 24, 107-111.

Benhabib, J. and R. Radner (1987), "The Joint Exploitation of a Productive Asset: A Game Theoretical Approach," C.V. Starr Center Working Paper #88-17, New York University, forthcoming, <u>Economic Theory</u>.

Chari, V.V. and P.J. Kehoe (1990), "Sustainable Plans," <u>Journal of Political Economy</u>, 98, 783-802.

Lancaster, K. (1973), "The Dynamic Inefficiency of Capitalism," <u>Journal of Political Economy</u>, 81, 1098-1109.

Levhari, D. and L.J. Mirman (1980), "The Great Fish-War: An Example Using the Cournot-Nash Solution," <u>Bell Journal of Economics</u>, 11, 322-334.

Majumdar, M. and R.K. Sundaram (1991), "Symmetric Stochastic Games of Resource Extraction: The Existence of Non-Randomized Stationary Equilibrium," in T.E.S. Raghavan, et. al. (eds.) <u>Stochastic Games and Related Topics</u>, Kluwer Academic Publishers, Netherlands, 175-190.

Marcet, A. and R. Marimon (1990), "Communication, Commitment and Growth," working paper, Universitat Pompeu Fabra, Barcelona.

North, C.D. (1981), <u>Structure and Change in Economic History</u>, New York: Norton.

North, C.D. (1991), "Institutions," <u>Journal of Economic Perspectives</u>, 5, 97-112.

Olson, M. (1982), <u>The Rise and Decline of Nations</u>, New Haven: CT, Yale University Press.

Persson, T. and G. Tabellini (1991), "Is Equality Harmful for Growth," U.C. Berkeley Working Paper # 91-55.

Tornell, A. and A. Velasco [1990], "Why Does Capital Flow from Poor to Rich Countries?: Interest Groups and Dynamic Games in Poor Countries," C.V. Starr Center for Applied Economics working paper #90-39, New York University.

Appendix A1: Proof of Proposition 3.1.

The first-order condition for (3.16) is given by (assuming interiority)

$$c_{sb}^{-\epsilon} = 2\beta v_k^s(ak-2c_{sb}) - \sigma v_c^D(k,c_{sb}) \tag{A1.1}$$

where $v_c^D(k, c_{sb})$ is the derivative of $v^D(k, c)$ with respect to c. Let $v_k^D(k, c)$ denote the derivative of $v^D(k, c)$ and $v_k^{sb}(k)$ the derivative of $v_{sb}(k)$ with respect to k. We obtain

$$v_k^{sb}(k) = a\beta v_k^s(ak-2c_{sb}) + \sigma v_k^{sb}(k) - \sigma v_k^D(k,c_{sb}) \ . \tag{A1.2}$$

Let $c_{sb} = \lambda_s y$ so that $c_D = \lambda_D y$ with $\lambda_D = (1-\lambda_s)M$. Then using

$$v_c^D(k,c) = -\left(\frac{\beta a}{2}\right)\left[\left(\frac{a}{2}\right)(ak-c_{sb}-c_D)\right]^{-\epsilon} = -a^{-1}v_k^D(k,c) \tag{A1.3}$$

and (A1.1) in (A1.2), updating and substituting $c_{sb} = \lambda_s y$ reduces (A1.3) to

$$\lambda_s^{-\epsilon} = \left(\frac{\beta a^{1-\epsilon}}{1-\sigma}\right)\left[\lambda^{-\epsilon} - \sigma\left(\frac{\beta a}{2}\right)\left[\frac{a}{2}\left((1-\lambda_s-\lambda_D)\right)\right]^{-\epsilon}\right](1-2\lambda_s)^{-\epsilon} + \frac{\sigma}{1-\sigma}\left[\frac{\beta a}{2}\left[\frac{a}{2}(1-\lambda_S-\lambda_D)\right]^{-\epsilon}\right]$$

which is independent of k. We solve for λ_s from the constraint $v_{sb}(k) = v^D(k, c_{sb})$ when $\hat{v}(k) < v^D(k, \hat{c}(k))$ since otherwise the second-best is identical to first best. If $c_{sb} = \lambda_s y$, then $v_{sb}(k) = s(\lambda_s)y^{1-\epsilon}$ where $s(\lambda_s) = ((1-\epsilon)^{-1}\lambda_s^{1-\epsilon})/(1-\beta(a(1-2\lambda_s))^{1-\epsilon})$. Since $v^D(k, c_{sb}) = s_D(\lambda_s)y^{1-\epsilon}$, we need to consider the λ_s which solves $s(\lambda_s) = s_D(\lambda_s) = (1-\epsilon)^{-1}(1-\lambda_s)^{1-\epsilon}M^{-\epsilon}$.

We note the following. By construction we are considering $s(\hat{\lambda}) < s_D(\hat{\lambda})$ since $\hat{v}(k) < v^D(k, \hat{c}(k))$. It can be shown by computation that $s(\lambda)$ attains a maximum at $\lambda = \hat{\lambda}$ with $\frac{ds(\lambda)}{d\lambda} < 0$ for $\lambda > \hat{\lambda}$, $\frac{ds(\lambda)}{d\lambda} > 0$ for $\lambda < \hat{\lambda}$. Since $s_D(\lambda)$

is decreasing in λ for $\lambda > \hat{\lambda}$, $s(\lambda_s) = s^D(\lambda_s)$ for $\lambda_s > \hat{\lambda}$. We now show that $\lambda_s \in [\hat{\lambda}, \frac{M}{1+M}]$ exists.

To show that $\lambda_s \in [\hat{\lambda}, \frac{M}{1+M}]$ it is enough to show that $s_D\left(\frac{M}{1+M}\right) \leq s\left(\frac{M}{1+M}\right)$.

We consider $s_D(\lambda) - s(\lambda)$:

$$s_D(\lambda) - s(\lambda) = M^{-\epsilon}(1-\lambda)^{1-\epsilon} - \lambda^{1-\epsilon}(1-\beta(a(1-2\lambda))^{1-\epsilon})^{-1} \leq 0$$

if

$$M^{-\epsilon}(1-\beta(a(1-2\lambda))^{1-\epsilon})(1-\lambda)^{1-\epsilon} - \lambda^{1-\epsilon} \leq 0$$

or

$$M^{-\epsilon}(1-\lambda)^{1-\epsilon} - M^{-\epsilon}(\beta a^{1-\epsilon}(1-3\lambda+2\lambda^2)^{1-\epsilon}) - \lambda^{1-\epsilon} < 0.$$

If $\lambda = \frac{M}{1+M}$, this inequality becomes

$$M^{-\epsilon}\left(\frac{1}{1+M}\right)^{1-\epsilon} - M^{-\epsilon}\beta a^{1-\epsilon}\left(\frac{1-M}{(1+M)^2}\right)^{1-\epsilon} - \left(\frac{M}{1+M}\right)^{1-\epsilon} \leq 0$$

or

$$((1+M)^{1-\epsilon})^{-2}\left[M^{-\epsilon}(1+M)^{1-\epsilon} - M^{-\epsilon}\beta a^{1-\epsilon}(1-M)^{1-\epsilon} - M^{1-\epsilon}(1+M)^{1-\epsilon}\right] \leq 0 \ .$$

This inequality will hold if

$$(1+M)^{1-\epsilon}(1-M) - \beta a^{1-\epsilon}(1-M)^{1-\epsilon} \leq 0$$

or

$$(1+M)^{1-\epsilon} - \beta a^{1-\epsilon}(1-M)^{-\epsilon} \leq 0$$

But $M = \left[1 + \left[\beta a^{1-\epsilon}\left(\frac{1}{2}\right)^{1-\epsilon}\right]^{1/\epsilon}\right]^{-1}$ implies $1 + M = \left[2 + \left[\beta a^{1-\epsilon}\left(\frac{1}{2}\right)^{1-\epsilon}\right]^{1/\epsilon}\right]M$

and $(1 - M) = \left[\beta a^{1-\epsilon}\left(\frac{1}{2}\right)^{1-\epsilon}\right]^{1/\epsilon} M$. Therefore the above inequality becomes

$$\left[2 + \left[\beta a^{1-\epsilon}\left(\frac{1}{2}\right)^{1-\epsilon}\right]^{1/\epsilon}\right]^{1-\epsilon} M^{1-\epsilon} - \beta a^{1-\epsilon}\left[\beta a^{1-\epsilon}\left(\frac{1}{2}\right)^{1-\epsilon}\right]^{-1} M^{-\epsilon} \leq 0$$

or

$$\frac{\left[2 + \left[\beta a^{1-\epsilon}\left(\frac{1}{2}\right)^{1-\epsilon}\right]^{1/\epsilon}\right]^{1-\epsilon}}{1 + \left[\beta a^{1-\epsilon}\left(\frac{1}{2}\right)^{1-\epsilon}\right]^{1/\epsilon}} \leq 2^{1-\epsilon}$$

This will hold (for $0 \leq \epsilon \leq 1$) since

$$\frac{\left[2 + \left[\beta a^{1-\epsilon}\left(\frac{1}{2}\right)^{1-\epsilon}\right]^{1/\epsilon}\right]^{1-\epsilon}}{1 + \left[\beta a^{1-\epsilon}\left(\frac{1}{2}\right)^{1-\epsilon}\right]^{1/\epsilon}} \leq \left(\frac{\left[2 + \beta a^{1-\epsilon}\left(\frac{1}{2}\right)^{1-\epsilon}\right]}{1 + \left[\beta a^{1-\epsilon}\left(\frac{1}{2}\right)^{1-\epsilon}\right]^{1/\epsilon}}\right)^{1-\epsilon} \leq 2^{1-\epsilon}$$

and $$\frac{\left[2 + \beta a^{1-\epsilon}\left(\frac{1}{2}\right)^{1-\epsilon}\right]}{1 + \beta a^{1-\epsilon}\left(\frac{1}{2}\right)^{1-\epsilon}} \leq 2 .$$

We can now obtain σ from (3.19). This yields $v_{sb}(k) = s(\lambda_s)(ak)^{1-\epsilon}$ which is strictly concave in k. This concludes the proof. ■

Appendix A2.1: Computation of the value of the first-best program and the value of defection for section 4.2.

For technology given by (4.2.1) and utility (4.2.2), the first best optimal policy is given by:

$$\hat{c}(k) = \begin{cases} 0 & \text{if} \quad k \le \frac{2}{5} \\ \frac{f(k)-1}{2} & \text{if} \quad k \ge \frac{2}{5} \end{cases} .$$

The value of the first-best optimal program has a more complicated form. We only need an explicit solution for values of $k \ge 2/5$, which is given by

$$\hat{v}(k) = \begin{cases} \frac{f(k)}{2} + \frac{1}{4} & \text{if} \quad k\text{: } f(k) \le 3 \\ b\frac{f(k)}{2} + \left(\frac{7}{4} - b\frac{3}{2}\right) & \text{if} \quad k\text{: } f(k) > 3 \end{cases} .$$

$\hat{v}$ is easily computed as:

$$U\left(\frac{f(k)-1}{2}\right) + \frac{\beta}{1-\beta} U\left(\frac{f(1)-1}{2}\right) = U\left(\frac{f(k)-1}{2}\right) + \frac{3}{4} \quad \text{if} \quad \frac{f(k)-1}{2} \le 1 \quad (\text{or } f(k) \le 3)$$

Therefore,

$$U\left(\frac{f(k)-1}{2}\right) + \frac{3}{4} = \frac{f(k)}{2} + \frac{1}{4} \qquad \text{for } f(k) \le 3 .$$

$$U\left(\frac{f(k)-1}{2}\right) + \frac{3}{4} = 1 + b\left(\frac{f(k)-1}{2} - 1\right) + \frac{3}{4} = \left(\frac{7}{4} - b\frac{3}{2}\right) + \frac{f(k)}{2}b \qquad \text{for } f(k) \ge 3.$$

Define net output as $y = f(k) - c$. In the cases that we will be considering, the value of defection from a net output y, left after the consumption of the other player, is given by

$$\max_{c \geq 0} \left\{ U(c) + \beta U\left[\frac{f(y-c)}{2}\right]\right\}.$$

Differentiating with respect to c we find the derivative of this expression to be positive, except for $y - c < 4/5$, when it is $b - \frac{A\beta}{2} < 0$.

Therefore, for $y \equiv f(k) - c$,

$$c_D(k, c) = \begin{cases} y = f(k) - c & \text{if} \quad \frac{4}{5} \leq f(k) - c \leq 1 \\ 1 & \text{if} \quad 1 \leq f(k) - c \leq 1 + \frac{2}{A} = \frac{9}{5} \\ y - \frac{4}{5} = f(k) - c - \frac{4}{5} & \text{if} \quad f(k) - c \geq \frac{9}{5}. \end{cases}$$

The defection consumption is never larger than 1 when the net output is smaller than 9/5.[9] In this case the precise value of b is not relevant (as long as it is strictly positive).

Let us consider now the value of defection from the optimal program. We only need this for values of $k \geq 2/5$ and this is the only case we consider here. The net output here is $\frac{Ak+1}{2}$.

We first consider the case of $k \leq 1$. The net output here is

$\frac{Ak+1}{2} = \left[Ak - \frac{Ak-1}{2}\right]$. Since $\frac{Ak+1}{2} > 1$ for $k > 2/5$, the consumption for an

[9] When $y = 9/5$, $c_D(k, c) = 9/5 - 4/5 = 1$.

optimal defection is $c_D(k, \hat{c}(k)) = 1$ for $k \in [2/5, 1]$,[10] and so

$$v^D(k, \hat{c}(k)) = \left(1 - \frac{A\beta}{4}\right) + \frac{A\beta}{4}Ak = \frac{25}{32}k + \frac{11}{16}.$$

Now note that for large values of k the value of the optimal solution is larger than the value of defection from the optimal policy. In fact we have:

$$v(k) \geq v^D(k, \hat{c}(k)) \quad \text{for} \quad k \geq \frac{14}{15},$$

with a strict inequallity except at $k_1 = 14/15$.[11]

We need briefly to check that $v^D(k, \hat{c}(k)) < \hat{v}(k)$ for every $k \geq 1$. Here

---

[10] Note that

$$y = Ak - \frac{Ak-1}{2} = f(k) - \hat{c}(k) = \frac{Ak+1}{2} < \frac{A+1}{2} = \frac{7}{4} < \frac{9}{5} \quad \text{for } k \leq 1.$$

$$Ak - \frac{Ak-1}{2} = f(k) - \hat{c}(k) = \frac{Ak+1}{2} \geq \frac{\frac{5}{2}\cdot\frac{2}{5}+1}{2} = 1 \quad \text{for } \frac{2}{5} \leq k \leq 1.$$

So when $c_D = 1$,

$$v^D(k, \hat{c}(k)) = U(1) + \beta U\left(\frac{1}{2}\left[A\left(\frac{Ak+1}{2} - 1\right)\right]\right).$$
$$= \left(1 - \frac{A\beta}{4}\right) + \frac{A\beta}{4}Ak .$$

[11] We compute $c_D(k, \hat{c}(k))$ for $k \geq 1$. Since

$$\hat{c}(k) = \frac{f(k) - 1}{2}; \quad y = f(k) - \frac{f(k) - 1}{2} = \frac{f(k) + 1}{2},$$

$$c_D(k, \hat{c}(k)) = \begin{cases} 1 & \text{if } \frac{f(k)+1}{2} \in [1, \frac{9}{5}] \\ y - \frac{4}{5} = \frac{f(k)}{2} - \frac{3}{10} & \text{if } \frac{f(k)+1}{2} \geq \frac{9}{5}. \end{cases}$$

$\hat{c}(k) = \dfrac{f(k) - 1}{2}$, so the net output if $\dfrac{f(k) + 1}{2}$. At k = 1, this is larger than one, so the following is a complete description of the optimal defection policy:[12]

$$c_D(k, \hat{c}(k)) = \begin{cases} 1 & \text{if } \dfrac{f(k)+1}{2} \in [1, \dfrac{9}{5}] \quad \left(\text{i.e., } \dfrac{2}{5} < k < 1 + \dfrac{1}{B \cdot 10}\right) \\ \dfrac{f(k)}{2} - \dfrac{3}{10} & \text{if } \dfrac{f(k)+1}{2} \in [\dfrac{9}{5}, +\infty) \quad \left(\text{i.e., } k > 1 + \dfrac{1}{10 \cdot B}\right) \end{cases}.$$

Note that for large k (i.e., $k \geq 1 + 1/(10 \cdot B)$), $c_D(k, \hat{c}(k)) < \hat{c}(k)$, so we only need to prove $v^D(k, \hat{c}(k)) < \hat{v}(k)$ for $k \in [\,1, 1+(1/10 \cdot B)]$. We already proved that claim on [4/5, 1]. Here $c_D(k, \hat{c}(k)) = 1$, and so

$$v^D(k, \hat{c}(k)) = \left(1 - \frac{A\beta}{4}\right) + \frac{A\beta}{4} f(k) = \frac{11}{16} + \frac{5}{16} f(k),$$

while $\hat{v}(k) = \dfrac{1}{4} + \dfrac{1}{2} f(k)$. Since $v^D(1, \hat{c}(1)) < \hat{v}(1)$, and

$\hat{v}' = \dfrac{1}{2} f'(k) > \dfrac{5}{16} f'(k) = v^D(\cdot, \hat{c}(\cdot))'$. Thus we have proved our claim. A similar argument also holds for $k \geq 1 + (10 \cdot \beta)^{-1}$.

---

[12] We can compute the value of $\bar{k}$ at which c_D changes as follows:

$$\frac{f(\bar{k}) + 1}{2} = \frac{9}{5} \Leftrightarrow f(k) = \frac{13}{5} > \frac{5}{2}.$$

Therefore

$$A + B(k - 1) = \frac{13}{5} \Leftrightarrow B(k - 1) = \frac{1}{10}; \quad \bar{k} = \frac{1}{B \cdot 10} + 1.$$

Appendix A2.2: Proof of Proposition 4.2.1. The proof will consist of a verification that the function v_{sb} satisfies the Bellman equation, subject to the incentive compatibility constraints, i.e., prove that

$$v_{sb}(k) = \max_{c} \{ U(c) + \beta v_{sb}(f(k) - 2c) \mid c: v_{sb}(k) \geq v^D(k, c) \} \text{ for every } k.$$

We begin anyway with a description of the derivative of v_{sb}. We already know that for $k \geq k_1$ the value of the optimal solution is larger than the value of defecting from the optimal policy. This gives the second best value and policy for the interval $[k_1, k_0]$. The remaining part of the value function is derived by a simple iteration.

Consider now any $k \leq k_1$. We use the fact (which we shall prove later) that the second best consumption is the lowest consumption the incentive compatibility constraint allows.

If we let k' be the next period capital stock, and a second best policy is adopted, we have

$$v_{sb}(k) = \frac{Ak - k'}{2} + \frac{1}{2}(a_0 k' + b_0) \qquad a_0 = \frac{5}{4}, \quad b_0 = \frac{1}{4}$$

$$v^D(k, c_{sb}(k) = 1 + \frac{A}{4}\left(\frac{Ak + k'}{2} - 1\right)$$

Since the incentive compatibility constraint is binding, we can determine k' by equating $v_{sb}(k)$ with $v^D(k, c_{sb}(k))$. We obtain:

$$k' = \frac{15\,k}{26 - 16a} + \frac{8b - 6}{13 - 8a} = k'(k, a_0, b_0)\,.$$

Substituting in the expression for $v_{sb}(k)$

$$v_{sb}(k) = \left[\frac{A}{2} + \frac{a_0 - 1}{2} \frac{15}{26 - 16a_0} \right] k + \left[\frac{b_0}{2} + \frac{a_0 - 1}{2} \frac{8b_0 - 6}{13 - 8a_0} \right]$$

$$\equiv a_1 k + b_1 \ .$$

This is the second best value for $k \in [k_2, k_1]$, where k_2 satisfies

$$k'(k_2, a_0, b_0) = \frac{14}{15}, \quad \text{i.e.,} \quad k_2 = \frac{68}{75} \ .$$

Iterating the above procedure we derive the values of a_i, b_i, k_i as in the statement of the proposition.

A clarification of this point: a mechanical application of the iteration scheme of the third step gives a critical $k_4' > k_3$. In fact it is an equilibrium, starting at k_4', to consume the capital stock down to k_3, and then grow to the steady state, k_0. Further iterations produce a path with a complicated dynamic sequence, non-monotonic, of capital stocks.

This is still an equilibrium outcome: by construction, the value of following this path is at least as large as the value of defecting from it. It is not, however, a second best outcome.

We now turn to the case $k < k_3$. Here a direct computation shows that the constrained problem

$$v_{sb}(k) = \max_{c} \{ U(c) + \frac{1}{2} v_{sb}(Ak - 2c) + \mu(v_{sb}(k) - v^D(k, c)) \}$$

has solution

$$v_{sb}(k) = \begin{cases} \frac{5}{4}k & \text{if} \quad k \in [0, k_4] \\ \frac{25}{16}k - \frac{1}{4} & \text{if} \quad k \in [k_4, {}'k_3] \end{cases}$$

$\mu = 1/4$

$$c_{sb}(k) = \begin{cases} \frac{5}{4}k & \text{if} \quad k \in [0, k_4] \\ 1 & \text{if} \quad k \in [k_4, {}'k_3] \end{cases}$$

$v^D(k,\ c_{sb}(k)) = v_{sb}(k)$.

Of course, since on $[0,\ k_3)$ the v_{sb} function defined above is not concave, the first order condition on the Lagrangean are only necessary. But a direct check shows that the equation

$$v_{sb}(k) = \max \{ U(c) + \frac{1}{2} v_{sb}(Ak - 2c) \mid c:\ v_{sb}(k) \geq v^D(k, c) \} .$$

is indeed satisfied.

The function v_{sb} defined in the statement of the proposition is concave on the interval $[k_3,\ k_0]$, because is continuous and the derivative is defined except at $k_3,$ k_2, k_1 and decreasing. We can now prove our claim that the second best consumption is the lowest consumption which is incentive compatible. In fact the differentiation of

$$\frac{Ak - k'}{2} + \beta v_{sb}(k')$$

with respect to k' gives $-\frac{1}{2} + \frac{1}{2}v'_{sb}(k') > 0$ for every $k' \in [k_3,\ k_0]$, since $v_{sb}'(k')$

$\geq 5/4$ for every such k'; so, conditional on choosing a next period k' in the interval $[k_3, k_0]$, the best choice is the highest incentive compatible k'.

We also note that when computing the defection policy and value the net output left by the consumption of a player who is following the equilibrium described in the proposition is less than 2. So the defection consumption is never larger than 1.

We now proceed with the verification that the function v_{sb} in the statement of the proposition satisfies the Bellman equation, under the incentive compatibility constraint. We consider two cases.

<u>Case 1</u>: $k \geq k_3$.

Let k' denote the next period capital stock, and

$v(k, \frac{Ak-k'}{2}) \equiv U\left(\frac{Ak-k'}{2}\right) + \beta v_{sb}(k')$. We have just seen that

$$v_{sb}(k) = \max\left[v(k, \frac{Ak-k'}{2});\ k'\ :\ v\left(k, \frac{Ak-k'}{2}\right) \geq v^D\left(k, \frac{Ak-k'}{2}\right),\ k' \geq k_3\right].$$

To prove our claim it is now enough to show that no $k' \in [Ak - 2, k_3]$ exists such that:

$$v\left(k, \frac{Ak-k'}{2}\right) \geq v^D\left(k, \frac{Ak+k'}{2}\right). \tag{*}$$

Since the net output $\frac{Ak+k'}{2} \geq \frac{5}{4}k_3 > 1$, we have $c_D\left(k, \frac{Ak-k'}{2}\right) = 1$ and

$$v^D\left(k,\ \frac{Ak-k'}{2}\right) = \frac{3}{8} + \frac{25}{32}k + \frac{5}{16}k'.$$ Also

$$v\left(k,\ \frac{Ak-k'}{2}\right) = \begin{cases} -\frac{1}{8} + \frac{5}{4}k + \frac{9}{32}k' & \text{if} \quad k' \in [k_4, k_3) \\ \frac{5}{4}k + \frac{1}{8}k' & \text{if} \quad k' \in [0, k_4] \end{cases}$$

Now one directly checks that (*) does not hold on $k' \in [Ak - 2,\ k_3]$. If $k' \in [k_4, k_3]$, $v\left(k, \frac{Ak-k'}{2}\right) - v^D\left(k, \frac{Ak-k'}{2}\right) > 0$ implies $15k - 16 > 0$, or $k > 1$. If $k' \in (ak - 2,\ k_4]$, the same inequality holds. Finally if $k' = Ak - 2$, then $v(k, 1) = 5/4\ k < v_{sb}(k)$.

<u>Case 2</u>: $k < k_3$.

Again we know that the best choice of next period capital, under the condition $k' < k_3$, is the one stated in the proposition. We claim that no $k' \geq k_3$ exists such that (*) above holds. First, from the fact that $k \geq k_4$ we derive $\frac{Ak+k'}{2} > 2$, so $c_D\left(k, \frac{Ak-k'}{2}\right) = 1$, and

$$v^D\left(k, \frac{Ak-k'}{2}\right) = \frac{3}{8} + \frac{25}{32}k + \frac{5}{16}k'$$

again. Writing $v_{sb}(k) = a(k)k + b(k)$, we have

$$v\left(k, \frac{Ak-k'}{2}\right) = \frac{5}{4}k + \frac{a(k')-1}{2}k' + \frac{b(k')}{2}$$

Now the difference

$$\begin{aligned} v\left(k, \frac{Ak-k'}{2}\right) - v^D\left(k, \frac{Ak-k'}{2}\right) &= \frac{15}{32}k + \left(\frac{8(a(k')-1)-5}{16}\right)k' + \frac{4b(k')-3}{2} \\ &< \frac{15}{32}k_3 + \left(\frac{8(a(k')-1)-5}{16}\right)k' + \frac{4b(k')-3}{2} \\ &\le \max_{k'\ge k_3} \frac{15}{32}k_3 + \left(\frac{8(a(k_2)-1)-5}{16}\right)k_2 + \frac{4b(k_2)-3}{2} \\ &\le \frac{15}{32}k_3 + \left(\frac{8(a(k_2)-1)-5}{16}\right)k_2 + \frac{4b(k_2)-3}{2} \end{aligned}$$

(since $8(a(k) - 1) - 5 \ge(<) 0$ if $k \ge(<) k_2$)

$$\begin{aligned} &= c\left(k_3, \frac{Ak_3-k_2}{2}\right) - v^D\left(k_3, \frac{Ak_3-k_2}{2}\right) \\ &= v_{sb}(k_3) - v_D(k_3, c_{sb}(k_3)) \\ &= 0 \end{aligned}$$

as claimed. ∎

Appendix A3: Proof of Proposition 5.1.

For an initial k, consider the value of going to some steady state k', in one step. For this policy to give an equilibrium pair two inequalities have to be satisfied

$$\frac{k^{\alpha}-k'}{2} + \frac{\beta}{1-\beta}\frac{k'^{\alpha}-k'}{2} \geq k^{\alpha} - \frac{k^{\alpha}-k'}{2} + \zeta \tag{A3.1}$$

$$\frac{k'^{\alpha}-k'}{2(1-\beta)} \geq k'^{\alpha} - \frac{k'^{\alpha}-k'}{2} + \zeta \tag{A3.2}$$

The inequality (A3.2) requires k' to be in the interval $[\underline{k}, \bar{k}]$. A computation shows that (A3.1) is equivalent to (A3.2), and independent of k. It follows immediately that if $k^* < \bar{k}$ then the first best policy is an equilibrium, and we have proved our first statement.

We now turn to the case $\bar{k} < k^*$. If we differentiate the left hand side of (A3.1) with respect to k' one finds that this function is increasing on $[\underline{k}, \bar{k}]$: this suggests that the second best policy is to choose the highest incentive compatible steady state. We now prove this.

We consider the Lagrangean associated with the second best maximization problem, under the constraint

$$v_{sb}(k) \geq v^{D}(k, c_{sb}(k)) \;:$$

$$v_{sb}(k) \equiv \max_{c\geq 0} \{ c + \beta v_{sb}(f(k) - 2c + \mu(k)\cdot(v_{sb}(k) - v^D(k,c)) \} \qquad (A3.3)$$

We claim that the choice of v_{sb}, c_{sb} as in the statement of the proposition, and $\mu(k) \equiv \beta\alpha\bar{k}^{\alpha-1} - 1$ solve the concave problem (A3.3) for $k \in [\ \max\{\ \underline{k},\ \bar{k}^{1/\alpha}\ \},\ \bar{k}]$.

$$1 - \beta 2v'_{sb}(f(k)-2c) - \mu v^D_c(k,c) = 0 \qquad (A3.4)$$

and

$$v'_{sb}(k) = \beta v'_{sb}(f(k)-2c)f'(k) + \mu(v'_{sb}(k) - v^D_k(k,c)) = 0 \qquad (A3.5)$$

Substituting (A3.4) into (A3.5), and using the fact that $v_k{}^D(k,\ c) = f'(k)$, $v_c{}^D(k,\ c) = -1$, we obtain

$$v'_{sb}(k) = \frac{f'(k)}{2} \qquad (A3.6)$$

Notice that the mulitplier μ does not enter into (A3.6). Therefore $v_{sb}(k) = (f(k)/2) + D$ where D is a constant.

From the boundary condition

$$v_{sb}(\bar{k}) = \frac{f(\bar{k}) - \bar{k}}{2(1-\beta)} \qquad (A3.7)$$

we obtain $D = \dfrac{\beta f(\bar{k}) - \bar{k}}{2(1-\beta)}$. (A3.8)

The value of $c_{sb}(k)$ is found by equating

$$v_{sb}(k) = v^D(k, c_{sb}(k)) \tag{A3.9}$$

i.e., $f(k)/2 + D = f(k) - c_{sb}(k) + \zeta$ which gives $c_{sb}(k) = f(k)/2 - D + \zeta$. Since $\frac{\beta}{1-\beta}\frac{(f(\bar{k})-\bar{k})}{2} = \bar{k} + \zeta$, from (A3.8) we derive $D = \bar{k}/2 + \zeta$, and so

$$c_{sb}(k) = \frac{f(k)-\bar{k}}{2}, \quad v_{sb}(k) = \frac{f(k)}{2} + \frac{\bar{k}}{2} + \zeta = v^D(k, c_{sb}(k)) \tag{A3.10}$$

Finally from (A3.4) we derive $1 + \mu - 2\beta v'_{sb}(\bar{k}) = 0$, and since $k^* > \bar{k}$, we also have

$$\mu = \alpha\beta\bar{k}^{\alpha-1} - 1 > 0 \tag{A3.11}$$

It is clear that the values of v_{sb}, c_{sb}, v^D, μ defined in (A3.10), (A3.11), satisfy the equations (A3.3), (A3.4), (A3.5), (A3.7).

Notice finally that the inequalities (A3.1) and (A3.2) hold in the case $k_0 \in (\bar{k}, k^*)$, $k = \bar{k}$. So the second best policy is c_{sb}:

$$c_{sb}(k) = \frac{f(k)-\bar{k}}{2} \qquad \text{for } k \geq \max\left[\bar{k}^{\frac{1}{\alpha}}, \underline{k}\right].$$

The last statement of the proposition follows from the fact that inequality

$$\max\{\frac{k^\alpha - \bar{k}}{2}, 0\} + \beta\{\frac{(k^\alpha)^\alpha - \bar{k}}{2} + \beta\frac{\bar{k}^\alpha - \bar{k}}{2(1-\beta)}\} \geq k^\alpha - \max\{\frac{k^\alpha - \bar{k}}{2}, 0\} + \zeta$$

holds by assumption as a strict inequality for $k = \bar{k}^{1/\alpha}$, and therefore holds for an interval below $\bar{k}^{1/\alpha}$. It is easy to show that this equilibrium policy is second best. ■

Appendix A4.1: Proof of Proposition 6.1.

Consider first the case where $a\beta < 2$. If $k > 1/a$, then $\hat{c}(k) = \frac{f(k) - 1}{2}$, so

$$v^D(k,\hat{c}(k)) = \max\left\{\frac{f(k) + 1}{2}, \frac{f(k)}{2}\right\} = \frac{f(k) + 1}{2}, \text{ and so}$$

$\hat{v}(k) - v^D(k,\hat{c}(k)) = \frac{(a+1)\beta - 2}{2(1-\beta)} > 0$, because $a\beta > 2 - \beta$. If $k < 1/a$, then $\hat{c}(k) = 0$, so $v^D(k, \hat{c}(k)) = ak$; but $\hat{v}$ is concave (in fact, $\hat{v}(k) = (a\beta)^i \frac{a}{2}k + \beta^i \frac{a\beta - 1}{2(1-\beta)}$ for $k \in [a^{-i-1}, a^{-i}]$) and non-negative. Since $\hat{v}(k) \geq ak$ at $k = 1/a$ and at $k = 0$, $\hat{v}(k) \geq ak$ for $k \in [0, 1/a]$. So our claim is proved if $a\beta < 2$.

We now consider the case where $a\beta \geq 2$. If $k \geq 1/a$, then $v^D(k,\hat{c}(k)) = \frac{f(k)}{2} + \frac{a\beta - 1}{2}$, and $\hat{v}(k) - v^D(k, \hat{c}(k)) = (a\beta - 1)((1 - \beta)^{-1} - 1) > 0$, as claimed. If $k \leq 1/a$, then $v^D(k, \hat{c}(k)) = \frac{a^2\beta}{2}k$, and now an argument similar to the previous case satisifies $\hat{v}(k) \geq v^D(k, \hat{c}(k))$. ■

Appendix A4.2: Proof of Proposition 6.2.

The proof consists of verifying that the consumption stated in the proposition satisfy the constrained maximization problem

$$v_{sb}(k) = \max_{(c_1,\ c_2)} \ \alpha_1 c_1 + \alpha_2 c_2 + \beta v_{sb}(f(k) - c_1 - c_2)$$

subject to

$$v_1(k) = c_1 + \beta v_1(f(k) - c_1 - c_2) \geq v^D{}_1(k,\ c_2)$$

$$v_2(k) = c_2 + \beta v_2(f(k) - c_1 - c_2) \geq v^D{}_2(k,\ c_1)$$

$$v_{sb}(k) = \alpha_1 v_1(k) + \alpha_2 v_2(k).$$

In turn this is proved by verifying that these consumption rates satisfy the first order conditions in

$$\begin{aligned} v_{sb}(k) = \max \quad & \alpha_1 c_1 + \alpha_2 c_2 + \beta v_{sb}(f(k) - c_1 - c_2) \\ & + \sigma_1 \left(c_1 + \beta v_1(f(k) - c_1 - c_2) - w_D^1(k, c_2) \right) \\ & + \sigma_1{}' \left[c_1 + \beta v_1(f(k) - c_1 - c_2) - \frac{f(k)}{2} \right] \\ & + \sigma_2 \left(c_2 + \beta v_2(f(k) - c_1 - c_2) - w_D^2(k, c_1) \right) \\ & + \sigma_2{}' \left[c_2 + \beta v_2(f(k) - c_1 - c_2) - \frac{f(k)}{2} \right] \end{aligned}$$

with $\sigma_i \geq 0$, $\sigma_i{}' \geq 0$, and $v_i(k) = c_i + v_i(f(k) - c_1 - c_2)$, $i = 1,\ 2$.

In particular, we have for every value of k, $\sigma_1 = \sigma_1{}' = 0$; $\sigma_2{}' > 0$, $\sigma_2 > 0$. It is important to note that the two values v_1 and v_2, and therefore v_{sb} are concave functions but not differentiable. The supergradients of v_i and v_{sb} are denoted by v_i', v_{sb}'. Let w_{D,c_1} denote the derivative of w_D with respect to c_1.

The first order conditions, if $\sigma_1 = \sigma_1 = 0$, are given by

$$\alpha_1 - \beta v_{sb}' + \sigma_2(-\beta v_2' - w_{D,c_1}) + \sigma_2'(-\beta v_2') \ni 0 \tag{A4.1}$$

$$\alpha_2 - \beta v_{sb}' + \sigma_2(-\beta v_2' + 1) + \sigma_2'(-\beta v_2' + 1) \ni 0 \tag{A4.2}$$

Notice that they immediately imply $\alpha_1 - \sigma_2 \cdot w_{D,c_1} = \alpha_2 + \sigma_2 + \sigma_2'$.

We prove in the following that for $k \in [\beta, 1]$, the functions v_i and c_i in the proposition satisfy the conditions above. With the value $c_1(k) = \frac{a}{2}k + \frac{a\beta}{2} - 1$ and $\beta \le k \le 1$, we have $w_D^2(k, c) = ak - c_1 - 1 + \frac{a\beta}{2}$; also $f(k) = ak$.

The first order conditions now give

$\sigma_2' = \alpha_1 - \alpha_2 > 0$

so that the equality $v_2(k) = \frac{f(k)}{2}$ must hold for every $k \in [\beta, 1]$. At the (claimed) equilibrium values $c_1(k)$ and $c_2(k)$ we have $f(k) - c_1(k) - c_2(k) = 1$, for $k \in [\beta, 1]$, so that $v_i'(f(k) - c_1(k) - c_2(k)) \equiv v_i' \in \left[\frac{b}{2}, \frac{a}{2}\right]$.

We now claim that the two first order conditions (A4.1) and (A4.2) are satisfied. Setting $\sigma_2' = \alpha_1 - \alpha_2$ in (A4.1), we have that (A4.1) is equivalent to $\alpha_1(1 - \beta 2\theta) + \sigma_2(1 - \beta\theta) = 0$, $\theta \in v_1'$, $\sigma_2 \ge 0$, which holds for every pair

$$\theta \in \left[\frac{1}{\beta}, \frac{a}{2} \right], \quad \sigma_2 = \alpha_1 \frac{2\beta\theta - 1}{1 - \beta\theta} .$$

The analysis for other values of k is similar. ■

Appendix A4.3: Proof of Proposition 6.3.

In particular $a\beta < 2$, so $v^D(k,c) = \max\left\{f(k) - c,\ \frac{f(k)}{2}\right\}$. We consider first the case $k \le 1$. Differentiation of the Lagrangean with respect to k gives $v'(k) = \alpha f'(k)$, and by symmetry we derive $v_1'(k) = v_2'(k) = a/2$. The first order conditions in this case give $\alpha + \sigma_1' = \alpha + \sigma_2'$, or $\sigma_1' = \sigma_2'$. If $\sigma_1' > 0$, we are done, because $v_1(k) = f(k)/2$. Note that at least one of the constraints must be binding, since the efficient solution is not an equilibrium and we may assume $\sigma_1 = \sigma_2 > 0$. From the equality $v_1(k) = f(k) - c_2(k) = f(k) - c_1(k)$ we conclude $c_1'(k) = c_2'(k) = \gamma$, a constant, so $c_i(k) = c(k) = \gamma k + w$, $i = 1, 2$; also $v(k) = v_i(k) = (a/2)k + q$. Now the condition $c(k) + \beta v(f(k) - 2c(k)) = v(k)$ gives $\gamma = a/2$, $q = -w\frac{2 - a\beta}{\beta}$. Also $v(k) \ge f(k) - c(k)$ implies $q \ge -w$. Note from the fact that $f(k) - c(k) > \frac{f(k)}{2}$, it follows that $w < 0$.

We now have $-w\frac{a\beta - 1}{1 - \beta} \ge -w$, or, if $w \ne 0$, $a\beta \ge 2 - \beta$, a contradiction with the hypothesis of the proposition.

We now turn to the case where $k \ge 1$; again we may assume $f(k) - c(k) > \frac{f(k)}{2}$. We first claim $f(k) - 2c(k) < 1$; otherwise $c(k) < \hat{c}(k)$, so $v^D(k, c(k)) > v^D(k, \hat{c}(k)) \ge \hat{v}(k) \ge v_{sb}(k)$, a contradiction. But if $f(k) - 2c(k)$

< 1, then using the previous part we have $v(k) = c(k) + \beta v(f(k) - 2c(k)) = c(k) + \beta(a/2)f(k) - \beta ac(k)$. Also $v^D(k, c(k)) = f(k) - c(k)$, and therefore $v(k) - v^D(k, c(k)) = (a\beta - 2)(f(k) - 2c(k)) < 0$, unless $f(k) - 2c(k) = 0$. ∎

Appendix A4.4: Proof of Proposition 6.4.

Since $a\beta < 2$, $v^D(k, c) = \max\left\{ f(k) - c, \ \frac{f(k)}{2} \right\}$. The first order conditions applied to the Lagrangean are $\alpha_1 + \sigma_1' = \alpha_2 + \sigma_2'$ and therefore $\sigma_2' > \sigma_1' \geq 0$. This immediately implies $v_2(k) = f(k)/2$; since $v_2(k) \geq f(k) - c_1(k)$, $c_1(k) \geq f(k)/2$, which implies the result. ∎