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# *SOCIAL CHOICE IN A REPRESENTATIVE DEMOCRACY*

BY

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## Abstract

In a representative democracy, citizens stand at two removes from legislation. First, they do not deliberate and vote directly on legislation. Rather they elect assemblies that enact such legislation in their stead. Second, and less commonly remarked, citizens do not vote directly for assemblies. Rather they vote for individual candidates, with the candidates receiving the most votes elected. This paper examines the efficiency properties of these voting systems. We show, first, that in general these procedures are inefficient. Second, we identify a condition on assembly preferences (called k-blockness) that insures the election of a pareto-optimal assembly. We then prove two negative results. The first theorem shows that inefficiency may recur if all but two of the voters have k-block preferences. The second theorem shows that whatever neutral restriction is imposed on preferences, an "almost inefficient" assembly may be elected.

KEYWORDS: social choice, voting scheme, sincere voting, committees, representative democracy, separable preferences
JEL Nos.: C70, D71, D72.

# Social Choice in a Representative Democracy*

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Jean-Pierre Benoit** and Lewis A. Kornhauser***

## 1. Introduction.

In a representative democracy, citizens stand at two removes from legislation. First, they do not deliberate and vote directly on legislation. Rather they elect assemblies that enact such legislation in their stead. Second, and less commonly remarked, citizens do not vote directly for assemblies. Rather they vote for individual candidates. The winning candidates then constitute the assembly.

These candidate-based procedures merit more attention than they have received because they come in many forms that, together, dominate our political institutions. City councils and school boards are typically elected through "at-large" procedures, in which each voter casts one or more votes for candidates and the top individual vote-getters are united to form the winning assembly. The United States Congress and the legislatures of the states all result from "districted" procedures in which candidates declare for a specific seat in a particular region; citizens vote only for candidates running in their

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district. Executive city governments (e.g. the mayor, comptroller, and district attorney) are chosen by "numbered-seat" procedures in which each candidate runs for a specific post, and citizens are entitled to vote for each post. In all of these elections, although voters have preferences over the assemblies being chosen, they are called upon to vote for candidates and these candidates have their votes tallied as individuals.

In this paper we analyze these commonly used voting procedures. Our main conclusions are simply stated. First, we show that in general these procedures are inefficient. Second, we identify a condition on assembly preferences (called k-blockness) that insures the election of a Pareto optimal assembly. We then prove two negative results. The first theorem shows that inefficiency may recur if all but two of the voters have k-block preferences. The second theorem shows that whatever neutral restriction is imposed on preferences, an "almost inefficient" assembly may be elected.

Our discussion proceeds as follows. In the next section, we briefly set out the basic structure and preliminary definitions. Section 3 presents some results from Benoit and Kornhauser [1991]. In section 4, we establish our main results for at-large assemblies. In section 5, we extend these results with some modifications to designated-seat assemblies. Section 6 considers voting procedures in which the size of the assembly is endogenous.

## 2. Notation and Preliminary Definitions.

As in Benoit and Kornhauser [1991], we classify procedures for election assemblies along two dimensions: (a) whether candidates must declare which seat they contest and (b) the scope of the electorate that votes for each position in the assembly. We call assemblies in which candidates do not declare which seat they contest at-large assemblies. When candidates declare the seat they contest, we have designated-seat assemblies. If the entire electorate votes on each seat, the assembly is a numbered-seat one; if selected portions of the electorate vote for candidates contesting specific seats, we have a districted assembly. We begin with the case of at-large assemblies.

Let $N = \{a_1, a_2, \ldots a_n\}$ be the set of candidates. Let $Æ_m$ be the set of all subsets A of N with cardinality m. $Æ_m$ is the set of all m-sized at-large assemblies. Let v be the number of voters. Let $\Gamma(N)$ be the set of all possible orderings of the elements of N; let $\Gamma(Æ_m)$ be the set of all possible orderings over $Æ_m$. We may now define the following:

Definition: An assembly-based procedure is a function f from $\Gamma(Æ_m)^v \dashrightarrow Æ_m$.

Definition: A candidate-based procedure is a function g from $\Gamma(N)^v \dashrightarrow Æ_m$.

We assume that each voter i has a well-defined preference ordering $R_i(Æ_m) \in \Gamma(Æ_m)$, over the outcomes, and that this ordering admits only strict preference.

## 3. Simple Voting.

Every analysis of a voting system requires an assumption about the behavior of voters. Initially, social choice theorists assumed that individuals vote sincerely, i.e., that they truthfully revealed their preferences. Sincerity is not fully satisfying, however, because truthful revelation of one's preferences may not be an "optimal" strategy for a voter. This assumption faces a further obstacle in the study of candidate-based procedures. Votes for candidates are not, in any obvious way, sincere expressions of a voter's preferences over assemblies.

A sincere voter acts in a simple, straightforward manner. In Benoit and Kornhauser [1991], we introduced the idea of "simple" voting, which extends the notion of sincere voting to candidate-based procedures. There is an intuitive sense in which a person's behavior is not simple when she votes for a candidate who is elected to an assembly, but there is a candidate for whom she did not vote and whom she would have preferred on the assembly. Hence we ask that a simple voter always prefer those candidates for whom she voted to candidates for whom she did not vote. More precisely, simple voting requires that, on any assembly, an individual never (strictly) want to replace candidates for whom she voted with ones who did not receive her vote, or, equivalently, that she always (weakly) want to replace candidates for whom she did not vote with ones for whom she did vote.

The following definition is discussed at greater length in

Benoit and Kornhauser [1991]:

Definition: A voter i votes simply if and only if for every X(k) contained in V, for every Y(k) contained in C/V and for every A(m-k) such that $A(m-k) \cap [X(k) \cup Y(k)] = \phi$, i weakly prefers X(k) U A(m-k) to Y(k) U A(m-k).

Voting simply, like voting sincerely, may not be an optimal strategy for a voter faced with a candidate-based procedure. A suitable alternative to simple voting, however, is difficult to identify. A straightforward analysis of the game induced by the voting procedure is unsatisfactory because the most prevalent game theoretic solution concept, Nash equilibrium, has no power in the analysis of voting with large numbers of people. In these games any candidate can be elected in a Nash equilibrium. Indeed, if all individuals vote for the same alternative, no one will have an incentive to change her vote since no single vote influences the outcome. Refinements of the Nash equilibrium concept such as perfection will not dramatically reduce the set of equilibria particularly when there are several candidates for each seat.[1]

One might impute to each voter i both a complex set of beliefs concerning the distribution of the votes of others over the candidates and preferences over uncertain actions. Under appropriate assumptions, voter i's choices would then be repre-

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[1]The most compelling notion of behaviour is that of following a dominant strategy. Unfortunately, Fishburn [1981] shows that the conditions necessary for the existence of a dominant strategy are extremely strong. When these conditions are met, simple voting is dominant.

sentable by an expected utility function. This procedure may yield a unique outcome but it is open to the criticism that we have no non-arbitrary way to select beliefs to impute to each voter.

In what follows we assume that voters in candidate-based procedures vote simply. We make this assumption for four reasons. First, many of our results indicate that prevalent voting procedures are inefficient. The choice of sincere voting is not restrictive for such results because strategic action on the part of voters is unlikely to improve the efficiency of any given procedure. Second, an assumption of strategic voting, e.g., sophisticated voting, imputes much foresight and insight on the part of voters. Simple voting, on the other hand, allows consideration of less sophisticated voters who nonetheless exhibit optimizing properties. Third, in many situations, simplicity may reasonably describe voter behavior. Finally, an understanding of simple voting is a necessary first step to the understanding of the candidate-based procedures which predominate our political institutions.

The next two definitions identify the restrictions on assembly preferences that are necessary and sufficient for simple rankings of candidates and simple voting respectively. These definitions are discussed at greater length in Benoit and Kornhauser [1991]:

Definition: An individual has separable assembly preferences if and only if, for every pair of (m-1) subassemblies

$A(m-1)$ and $B(m-1)$ such that $A(m-1)\cap(\{x\}\cup\{y\})$ and $B(m-1)\cap(\{x\}\cup\{y\})$ are empty, $\{x\}\cup A(m-1) > \{y\}\cup A(m-1)$ implies $\{x\}\cup B(m-1) \geq \{y\}\cup B(m-1)$.

Definition (at large): An individual has k-top-separable preferences over at-large assemblies if and only if there exist a set of k candidates, called the individual's top candidates, such that if $x_i$ is a top candidate and yi is not a top candidate, then $x_i A(m-1) \geq y_i A(m-1)$ for all m-1-sized sets $A(m-1)$ such that $A(m-1)\cap(x_i \cup y_i) = \phi$.

Theorems 1 and 2 are proved in Benoit and Kornhauser [1991].

Theorem 1: Preferences are separable if and only if they are k-top-separable for every $k < n$.[2]

Theorem 2: Consider a voting procedure in which each voter must select k candidates. Then simple voting is possible if and only if a voter has k-top-separable preferences over assemblies.

A strict separable assembly ranking generates a unique candidate ranking in the obvious way; $x_1 > y_1$ if and only if $\{x_1\}\cup A(m-1) \geq \{y_1\}\cup A(m-1)$ for all assemblies $A(m-1)$ that exclude $x_1$ and $y_1$. However, many assembly rankings are consistent with any single candidate ranking, i.e., a single candidate ranking could be generated from many different separable assembly preferences. Similarly, many assembly rankings are consistent with a specific top-set of k candidates. The following two propositions from Benoit and Kornhauser [1991] elaborate these ideas:

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[2]This is proposition 3 in the companion paper.

Proposition 1: Let $a_1 > a_2 > \ldots > a_n$ be a candidate ranking and A and A' be two assemblies. If there exist i and j such that the ith most highly ranked candidate in A is more highly ranked than the ith most highly ranked candidate in A' but the jth most highly ranked candidate in A' is more highly ranked than the jth most highly ranked candidate in A, then A is ranked above A' by some consistent assembly rankings, and below A' by others.

Proposition 2: Let $T_k$ be a top set of k candidates and let A and A' be two assemblies. Let $S = A/(A \cap A')$ and $S' = A'/(A \cap A')$. If [S is not contained in $T_k$ or $S' \cap T_k$ is not empty] and [S' is not contained in $T_k$ or $S \cap T_k$ is not empty], then there exist assembly rankings consistent with $T_k$ that rank A above A' and others that rank A' above A.[3]

Let m equal the size of the assembly to be chosen, n equal the number of candidates, k the number of votes each voter casts, and v the number of voters. We maintain the following assumption throughout:

Assumption 1 : Each voter has k-top-separable assembly preferences, all voters vote simply, and the electorate is sufficiently heterogeneous that at least m candidates receive votes.

Unless otherwise stated, all our theorems are valid for all m, n, k, and v satisfying the following assumption:

Assumption 2: $m \geq 2$, $n \geq m + 3$, $k \leq m$, and $v \geq \max(m,6)$.

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[3]Proposition 1 above is proposition 2(b) of Benoit and Kornhauser [1991] and Proposition 2 in the text is proposition 4(b) of the earlier paper.

This is not the most general assumption possible but it is easily met by the situations we have in mind and it permits a simple statement of the theorems.

## 4. Constant Scoring Systems.

As already noted, most political systems elect assemblies using procedures which tally the votes received by candidates, not assemblies. In this section, we study the efficiency properties of these voting procedures. An example shows that with such procedures a Pareto inferior assembly may be elected.

Suppose there are nine candidates running for three seats, and the voters divide into three groups of equal size with the following preference profile:

| | | Group | |
|---|---|---|---|
| Rank | I | II | III |
| 1 | $a_1$ | $a_2$ | $a_3$ |
| 2 | $a_2$ | $a_3$ | $a_1$ |
| 3 | $a_4$ | $a_5$ | $a_6$ |
| 4 | $a_7$ | $a_8$ | $a_9$ |
| 5 | $a_8$ | $a_9$ | $a_7$ |
| 6 | $a_9$ | $a_7$ | $a_8$ |
| 7 | $a_3$ | $a_1$ | $a_2$ |
| 8 | $a_5$ | $a_6$ | $a_4$ |
| 9 | $a_6$ | $a_4$ | $a_5$ |

Whether voters have 1,2, or 3 votes, the assembly $\{a_1,a_2,a_3\}$ is elected; indeed when they have 1 or 2 votes, only these three candidates receive any votes. In addition, $\{a_1,a_2,a_3\}$ forms a Condorcet set, i.e., each member of the assembly defeats each candidate not in the assembly in a pairwise contest. These candidate preferences, however, are consistent with the electorate unanimously preferring the assembly $\{a_7,a_8,a_9\}$, even though no candidate in this Pareto-preferred assembly receives a single vote.[4]

Note that this difficulty is not due to the assumption of simple voting. If the voters managed to get together and agree to vote for $a_7$, $a_8$ and $a_9$, the first group would have a strong incentive to deviate from the agreement and vote for $a_1$, since each member of this group prefers $\{a_1,a_7,a_8\}$, $\{a_1,a_7,a_9\}$ or $\{a_1,a_8,a_9\}$ to $\{a_7,a_8,a_9\}$. Similarly, the last two groups would deviate to get $a_2$ and $a_3$ elected, respectively.[5]

To generalize this example we require the following definitions:

Definition: A constant scoring system for an at-large assembly of size m is one in which each voter is given some number, k, of votes to be distributed among k individual candidates, and the assembly elected consists of those m candidates receiving the m greatest total number of votes.

Definition: A voting procedure is efficient over a prefer-

[4]Throughout we assume that voters mark their ballots for all candidates simultaneously. One might believe that a sequential voting procedure, in which voters filled seat i with knowledge of the representatives in seats 1 through i-1, would avoid many inefficiency properties. A slight modification of the example in the text shows that sequential voting does not avoid all such problems.

Suppose that group I has a few more voters than group II which is slightly larger than group III. Then, with sequential voting by plurality or by plurality with runoff, the assembly $\{a_1,a_2,a_3\}$ is still elected.

[5]With a procedure based on a complete candidate ranking, Pareto optimality could be guaranteed by a social choice rule which selected say $\{a_1,a_2,a_4\}$, as the outcome. A rule which chose $\{a_7,a_8,a_9\}$ would not work since the above rankings of candidates are consistent with a unanimous preference of the assembly $\{a_1,a_2,a_3\}$ to the assembly $\{a_7,a_8,a_9\}$. An efficient procedure that selects $\{a_7,a_8,a_9\}$ must be assembly-based.

ence domain if, for all preference profiles in the domain, simple voting results in the election of a Pareto optimal assembly. Otherwise, the system is inefficient.

The most commonly encountered voting procedures are constant voting systems. Our next theorem is true also true for general scoring systems.[6]

<u>Theorem 3</u>: A constant scoring system is inefficient over the domain of separable preferences.

<u>Proof</u>: The proof consists of constructing preference profiles such that the assembly $\{a_1, a_2, \ldots, a_{m-1}, a_m\}$ gets elected even though everyone prefers $\{a_1, a_2, \ldots, a_{m-2}, a_{m+1}, a_{m+2}\}$. The proof considers two cases:

Case 1: $k < m$. Let the population be divided into m equal sized groups (plus or minus one, since all groups must be integral numbers) with the following preferences:

| | Group | | | | |
|---|---|---|---|---|---|
| Rank | 1 | 2 | | m-1 | m |
| 1 | $a_1$ | $a_2$ | ... | $a_{m-1}$ | $a_m$ |
| 2<br>·<br>·<br>·<br>·<br>m-2 | (In positions 2 through m-2, the first m-2 groups rank candidates from $a_1$ through $a_{m-2}$ arbitrarily (making sure no one lists the same candidate twice, of course). Groups m-1 and m arbitrarily rank candidates $a_2$ through $a_{m-2}$) | | | | |
| m-1 | $a_m$ | $a_m$ | ... | $a_1$ | $a_1$ |
| m | $a_{m+1}$ | $a_{m+1}$ | ... | $a_{m+1}$ | $a_{m+1}$ |
| m+1 | $a_{m+2}$ | $a_{m+2}$ | ... | $a_{m+2}$ | $a_{m+2}$ |
| m+2 | $a_{m-1}$ | $a_{m-1}$ | ... | $a_m$ | $a_{m-1}$ |
| | · | · | | · | · |
| | · | · | | · | · |
| n | $a_n$ | $a_n$ | ... | $a_n$ | $a_n$ |

[6]The definition of simple voting must be modified so that a simple voter always assigns higher weight to more highly ranked candidates.

Note that for k<m, candidates $a_1$ through $a_m$ each receive at least one vote, and no other candidate receives any votes. Therefore the assembly $\{a_1, a_2, \ldots, a_m\}$ is chosen. However, we can assume that everyone would like to replace $\{a_{m-1}, a_m\}$ by $\{a_{m+1}, a_{m+2}\}$; i.e., these candidate rankings are consistent with assembly preferences which rank $\{a_1, \ldots, a_{m-2}, a_{m+1}, a_{m+1}\}$ over $\{a_1, \ldots, a_{m-1}, a_m\}$.

Case 2: k=m. Let the population be divided into four groups of two different sizes (plus or minus 1), p and q, p < q, as follows:

| | Group Size | | | |
|---|---|---|---|---|
| Rank | p | p | q | q |
| 1 | $a_1$ | $a_1$ | $a_1$ | $a_1$ |
| 2 | $a_2$ | $a_2$ | $a_2$ | $a_2$ |
| . | . | . | . | . |
| . | . | . | . | . |
| m-2 | $a_{m-2}$ | $a_{m-2}$ | $a_{m-2}$ | $a_{m-2}$ |
| m-1 | $a_{m-1}$ | $a_m$ | $a_m$ | $a_{m-1}$ |
| m | $a_{m+1}$ | $a_{m+1}$ | $a_{m+2}$ | $a_{m+3}$ |
| m+1 | $a_{m+3}$ | $a_{m+3}$ | $a_{m+3}$ | $a_{m+1}$ |
| m+2 | $a_{m+2}$ | $a_{m+2}$ | $a_{m+1}$ | $a_{m+2}$ |
| m+3 | $a_m$ | $a_{m-1}$ | $a_{m-1}$ | $a_m$ |
| . | . | . | . | . |
| . | . | . | . | . |
| n | $a_n$ | $a_n$ | $a_n$ | $a_n$ |

Again, $\{a_1, a_2, \ldots, a_m\}$ is elected although we can assume that everyone would prefer to replace candidates $a_{m-1}$ and $a_m$ with $a_{m+1}$ and $a_{m+2}$. ▲

Pareto optimality is generally considered to be a weak requirement and is easily satisfied by most voting systems in which individuals vote directly for the outcomes. However, when assemblies are being chosen but only candidates are being voted upon this requirement is too strong.

In our first example, as well as in the proof of the theorem, the inefficiency arises because all voters would be willing to compromise and forgo the election of a highly ranked candidate in order to avoid the election of a lowly ranked candidate, but the voting system does not recognize this. Thus, in the example, all voters would gladly sacrifice the election of their top two candidates so as not to have their bottom ranked candidates elected as well. This willingness to compromise on highly ranked candidates is not the root of the problem, however. In the following example, each voter has lexicographic preferences[7] and an inefficient assembly is nonetheless elected:

| | Voter | | |
|---|---|---|---|
| Rank | I | II | III |
| 1 | $a_4$ | $a_5$ | $a_6$ |
| 2 | $a_1$ | $a_3$ | $a_2$ |
| 3 | $a_2$ | $a_1$ | $a_3$ |
| 4 | $a_3$ | $a_2$ | $a_1$ |
| 5 | $a_5$ | $a_4$ | $a_4$ |
| 6 | $a_6$ | $a_6$ | $a_5$ |

When each person gets three votes, $\{a_1,a_2,a_3\}$ is elected but, because preferences are lexicographic, $\{a_4,a_5,a_6\}$ is unanimously preferred.

We now offer a condition on individual preferences that is sufficient to guarantee that a constant scoring system yields a Pareto optimal assembly. We let $|S|$ be the number of elements in S.

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[7]A voter has lexicographic preferences when (a) she has separable assembly preferences and (b) A is preferred to A' if and only if the most highly ranked candidate in A/A' is preferred to the most highly ranked candidate in A'/A.

Definition: An individual has k-block preferences over assemblies if and only if there exists a set B(k) of candidates such that, for any assemblies A and A', if $|A \cap B(k)| > |A' \cap B(k)|$, then A is strictly preferred to A'.[8]

We will say that an individual has block preferences if her preferences are k-block for some k.

An individual with block preferences has favorite candidates and is most concerned with getting the maximum number of these candidates elected. Thus, someone with 1-block preferences has a most preferred candidate and prefers any assembly with this candidate to any assembly without her. A person with 2-block preferences has two preferred candidates and prefers any assembly with one of these candidates to any assembly without either, and any assembly with both these candidates to one with fewer than both. No further restrictions are imposed. An individual with strong dislikes who is concerned with keeping certain candidates out of an assembly is not likely to have k-block preferences.

k-blockness implies k-top-separability and hence is sufficient to guarantee that simple voting over candidates is possible. k-blockness, however, is a significantly stronger assumption than k-top-separability. For instance, if i has 1-block preferences with $a_1$ her top candidate, she must prefer

[8]Fishburn [1981] assumes that A is strictly preferred to A' iff $|A \cap B(k)| > |A' \cap B(k)|$. This implies blockness and is in fact a much stronger assumption. For instance, suppose $B(k) = \{a_1, a_2\}$. Let $A = \{a_1, a_3\}$ and $A' = \{a_2, a_4\}$. Fishburn's definition requires indifference between A and A' but our definition permits $A > A'$, $A < A'$ or $A = A'$. Fishburn restricts the assembly ordering to one with k+1 indifference classes.

$\{a_1,a_5\}$ to $\{a_2,a_3\}$ though this preference is not compelled by 1-top-separability (where $a_1$ is again the preferred candidate).

K-blockness does not imply (k-1)-blockness. Furthermore, while, k-blockness is consistent with (k-1)-blockness, it is inconsistent with (k-2)-blockness. For instance, let $\{a_1,a_2,a_3\}$ be a 3-block set. $\{a_1\}$ cannot form a 1-block set because 3-blockness implies that $\{a_1\}UA(m-2)U\{a_4\} < \{a_2,a_3\}UA(m-2)$. Similarly, no other singleton can form a 1-block set.

Finally, k-blockness neither implies nor is implied by separability. If, however, a separable assembly ranking is k-block, then the block set B(k) consists of the k most highly ranked candidates.

As the next theorem indicates, k-blockness guarantees Pareto optimality in constant scoring systems. In this theorem, we ignore the possibility of two candidates receiving the same number of votes. Inevitably, ties may cause an inefficiency.

<u>Theorem 4</u>: If every voter has k-block preferences and k votes, then a Pareto optimal assembly is elected.

<u>Proof</u>: (reductio) Suppose that A is elected but B is Pareto superior. Let A* = A/(AnB) and B* = B/(AnB). Consider any individual. Since she likes B at least as much as A, B* must have at least as many elements which are in her block set as A* does. Since she has k votes, she casts at least as many votes among the elements of B* as among A*. This is true for all individuals, so the total number of votes received by the members of B* is at least as great as the total received by the members

of A*. In particular, the maximum number of votes received by an element of B* must be greater than the minimum number received by an element of A* (assuming no ties) so that an element of B* must have been elected. A contradiction. ▲

While this is a positive theorem for voting over candidates, several comments are in order. First, the theorem requires not only that voters have block preferences, but also that all these preferences be k-block for the same k, and that this k be the number of votes which each person is to cast. This is a lot to ask. However, if individuals are given either too few or too many votes, efficiency cannot be guaranteed.

Approval voting may offer a solution to this last difficulty. Under approval voting, each person can cast as many votes as he or she likes. One reasonable way for a person with k-block preferences to vote in such a system would be to cast k votes for her block set.[9] This assumption gives the following:

Theorem 5: If every voter has block preferences then approval voting (with each individual voting for a block set) elects a Pareto optimal assembly. If preferences are only separable, a Pareto inferior assembly maybe elected.

The proof is virtually identical to the proofs of theorems 3 and 4.

So if all individuals have block preferences, there are

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[9] Voting in this manner is simple, but it is not the only simple way to vote. Fishburn [1981] shows that under a stronger assumption than blockness (see footnote 8), voting in this manner is a dominant strategy.

efficient candidate-based procedures. Blockness is quite a strong assumption. Nevertheless, it is not easily relaxed. As the next theorem indicates, if even three people have non-block preferences a Pareto inferior assembly may be elected.

<u>Theorem 6</u>: Let k = m. For v-3 ≥ t ≥ 3, there exist preference profiles in which all individuals have k-block preferences, except for t people whose preferences are separable, and an inefficient assembly is elected.

<u>Proof</u>: Suppose the population divides into four groups with the following top sets:

| I | II | III | IV |
|---|---|---|---|
| (2+y) | (x) | (x+1) | (1) |
| $a_1$ | $a_{k+1}$ | $a_{k+2}$ | $a_{k+1}$ |
| $a_2$ | $a_1$ | $a_{k+3}$ | $a_{k+3}$ |
| $a_3$ | $b_1$ | $b_1$ | $b_1$ |
| . | . | . | . |
| . | . | . | . |
| . | . | . | . |
| $a_k$ | $b_{k-2}$ | $b_{k-2}$ | $b_{k-2}$ |

The size of each group appears in parentheses; if v is even, x = (v-4)/2, y = 0 and, if v is odd, x = (v-5)/2 and y = 1.

Let the first group have preferences which are not k-block but which are separable. In particular, assume that those two voters prefer the assembly $\{a_{k+1}, a_{k+2}, b_1, \ldots, b_{k-2}\}$ to the assembly $\{a_1, a_{k+3}, b_1, \ldots, b_{k-2}\}$. We can further assume that all remaining voters share this preference and that any number up to all of them have k-block preferences. The inefficient assembly $\{a_1, a_{k+3}, b_1, \ldots, b_{k-2}\}$ is elected. ▲

The proof also yields the following theorem.

<u>Theorem 7</u>: Suppose m = k = 2. A constant scoring rule is

inefficient over any preference domain strictly weaker than 2-blockness.

<u>Proof</u>: The fact that the restriction is weaker than 2-block means that some violation of blockness is permitted. Without loss of generality, we can assume that this violation is having $\{a_1,a_2\}$ as block set but preferring $\{a_{k+1},a_{k+2}\}$ to $\{a_1,a_{k+3}\}$. The preference profile from the above proof (using just the first two elements of each set) establishes the theorem. ▲

The two previous theorems are for voting systems which give each voter the same number of votes as there are seats to be filled. Rather than extend these results, we present the next theorem, which is perhaps the most discouraging for standard voting procedures. It indicates that no matter what restriction is placed upon individual preferences, voting over candidates will be "nearly inefficient"; that is, an assembly may be elected although all but two voters prefer a different assembly. To state the result we need some preliminaries.

A restriction is on individual preferences (as opposed to across preferences), if it can be defined on a single individual.

A restriction is <u>neutral</u> if it is independent of the labeling of candidates. That is, if a preference ordering P satisfies a neutral restriction, then P' will satisfy the same restriction, where P' is defined as follows: for all assemblies $A_i$, let $A_i'$ equal $A_i$ with $a_r$ replaced by $a_m$ and $a_m$ replaced by $a_r$ (if $a_r$ or $a_m$ occurs in $A_i$. Otherwise $A_i'$ equals $A_i$). Then in the ordering P', $A_i'$ is ranked ahead of $A_j'$ iff $A_i$ is ranked ahead of $A_j$ in the ordering P.

All the restrictions we have considered so far are neutral restrictions on individual preferences. An example of a non--neutral restriction on individual preferences would be that the assembly A be ranked first.

The next theorem indicates that no matter what neutral restriction on individual preferences is imposed, an assembly which is nearly inefficient may be chosen by a standard voting method.

<u>Theorem 8</u>: Consider any neutral restriction on individual rankings beyond top-separability. There exist preference profiles in which all individual orderings satisfy this restriction, simple voting elects an assembly, A, and all but one or two voters prefer a different assembly, A'.

<u>Proof</u>: First suppose that k=m. We assume that the population divides into three groups, and that each individual preference ordering satisfies some neutral restriction. Without loss of generality, we can assume that each individual in the first group has as top set $\{a_1,a_2\} \cup B(m-2)$, where $B(m-2)$ is of size m-2 and does not contain $a_1$, $a_2$, $a_3$, or $a_4$. Furthermore we can assume that $\{a_1,a_3\} \cup B(m-2)$ is ranked above $\{a_2,a_4\} \cup B(m-2)$. (Suppose instead that $\{a_2,a_4\} \cup B(m-2) > \{a_1,a_3\} \cup B(m-2)$. By neutrality (interchanging $a_1$ and $a_2$) a group with top set $\{a_1,a_2\} \cup B(m-2)$ and $\{a_1,a_4\} \cup B(m-2) > \{a_2,a_3\} \cup B(m-2)$ also satisfies the restriction. Again by neutrality (interchanging $a_3$ and $a_4$), a group with top set $\{a_1,a_2\} \cup B(m-2)$ and $\{a_1,a_3\} \cup B(m-2) >$

$\{a_2,a_4\}$UB(m-2) also satisfies the restriction). Similarly, we can assume that the second group has as top set $\{a_3,a_4\}$UB(m-2) and that they all also rank $\{a_1,a_3\}$UB(m-2) above $\{a_2,a_4\}$UB(m-2). Finally, the third group has top set $\{a_2,a_4\}$UB(m-2) (and hence ranks this set first).

We assume that the three groups are, respectively, of size (V-1)/2, (V-1)/2, and 1 if V is odd, and (V-2)/2, (V-2)/2, 2 if V is even. The assembly $\{a_2,a_4\}$UB(m-2) is elected, but all but two people prefer $\{a_1,a_3\}$UB(m-2).

Now suppose that k is strictly less than m. The population divides into m groups. The first group has one voter, whose top set is $\{a_m, a_1,..., a_{k-1}\}$. The remaining m-1 groups are of (approximately) equal size. In each group each individual has a top set whose members are drawn from $T=\{a_1,a_2,...,a_{m-1}\}$ in such a way that each element of T appears at least once. For all these voters, we can assume that the assembly $\{a_1,...,a_{m-1},b_m\}$ is prefered to the assembly $\{a_1,...,a_{m-1},a_m\}$, where $b_m$ is not equal to $a_m$. The assembly $\{a_1,...,a_m\}$ is chosen, but all save one voter (possibly) prefers $\{a_1,...,a_{m-1},b_m\}$. ▲

## 5. Designated Seats

In a designated seat assembly, the set of candidates N is partitioned into m sets $N_1$, $N_2$, . . . $N_m$. In a candidate-based procedure, therefore, each voter submits a ranking for each seat. The set of possible designated-seat assemblies $DÆ_m = N_1 x N_2 x . . . x N_m$ is strictly contained in the set of possible at-large assemblies $Æ_m$ so that the range of both candidate-based and assembly-based procedures is restricted.

In a standard voting system for a numbered-seat assembly, each voter casts a single vote for each seat to be filled. In a standard voting system for a districted seat assembly, each voter casts a single vote for $k < m$ specified seats. For each seat, the candidate who receives the most votes is elected. In section 3, we saw that constant scoring systems could result in the election of Pareto inferior at-large assemblies. Our next theorem indicates that standard voting systems may elect Pareto inferior designated-seat assemblies.

<u>Theorem 9</u>: (a) Suppose that the number of voters $v \geq \max\{m,6\}$ and that there are at least three seats that are contested by at least two candidates. Then, for every $m > 2$, a standard voting system in a numbered-seat election is inefficient over the domain of separable preferences and (b) Suppose there are at least two seats i and j contested by at least two candidates. In addition, there are at least two voters who vote in district i but not in district j and at least two voters who vote in district j but not in i. For every $m \geq 2$, a standard voting system in a districted election is inefficient over the domain of separable preferences.[10]

<u>Proof</u>: Without loss of generality we may assume that m is the number of seats that are contested by at least two candidates. (a) Let the population be divided into m equal groups

[10]The assumptions on v are slightly more restrictive than necessary.

(plus or minus one since all groups must be integral numbers). Suppose group i's most preferred candidate for seat i is $i_1$ and, for j not equal to i, group i's most preferred candidate for seat j is $j_2$. A standard voting procedure for a numbered-seat assembly elects the assembly $\{a_2,b_2,c_2,\ldots,m_2\}$ even though we can assume that every group prefers the assembly $\{a_1,b_1,c_1,\ldots,m_1\}$.

(b) For simplicity, the proof considers only the case in which each voter votes in a single district. Let the population be divided into m equal groups (plus or minus 1). Suppose that everyone who votes in district 1 most prefers the candidates $\{a_1,b_1,c_1,\ldots,m_1\}$ in each district; everyone who votes in district 2 most prefers the candidates $\{a_2,b_2,c_1,\ldots, m_1\}$; and everyone who votes in districts 3 through m most prefers the candidates $\{a_2,b_1,c_1,\ldots, m_1\}$. Thus $\{a_1,b_2,c_1, \ldots, m_1\}$ is elected but we can assume that everyone prefers $\{a_2,b_1,c_1,\ldots,m_1\}$. ▲

To obtain sufficient conditions for efficiency, we consider numbered-seat and districted-seat assemblies separately.

In a numbered-seat election, suppose a voter assigns an order of importance to the seats (e.g., the mayoralty is most important, followed by district attorney and so forth...).

<u>Definition</u>: Let A and A' be two assemblies with the seats listed in declining order of importance and suppose that the two assemblies first differ in the jth position. Then preferences are said to be <u>top-lexicographic</u> if an individual always prefers A to A' when the jth candidate in A is the individual's most

preferred candidate for seat j.

Theorem 10: Suppose each individual agrees on the order of importance of the seats in a numbered-seat assembly, and that preferences are top-lexicographic. Then a standard voting system always elects a Pareto optimal assembly

Proof: Let A be the elected assembly and let A' be a different assembly. Suppose that the two assemblies first differ in position j. The candidate chosen to fill that seat must be someone's most preferred candidate for that seat, and that individual prefers A to A' since preferences are top lexicographic. ▲

For simplicity, in our discussion of districted procedures, we consider only those elections in which each individual votes in a single district. In this context, an individual's preferences are 1-block if there is a district and a candidate in that district, such that the individual prefers any assembly with that candidate to any assembly without that candidate.

Theorem 11: If each individual votes in a single district and has 1-block preferences with respect to that district, then a Pareto optimal assembly will be elected.

Proof: Obvious. ▲

From the proof of theorem 9, one can see that if voters in a numbered-seat election have top-lexicographic preferences, but disagree on the order of importance of the seats, then an inefficient assembly may be elected. Similarly, if voters in a districted election have 1-block preferences but not with respect to the district in which they vote, then an inefficient assembly may be elected. On the other hand, if all but a few voters' preferences satisfy the assumptions of theorems 10 and 11, the elected assembly will still be efficient; compare this to theorem 6).

The assumptions on preferences in theorems 10 and 11 are both non-neutral since they are tied to specific seats. In a numbered-seat assembly where the different seats have different titles and responsibilities, a non-neutral assumption may make sense. If the differently numbered seats are identical in duties and numbered merely for electoral purposes, a non-neutral assumption is unwarranted. In such a case there are neutral assumptions which will guarantee the election of an efficient assembly.[11] However, this efficiency can only be assured with respect to those assemblies which can be formed respecting the seats for which each candidate has declared. It cannot be guaranteed with respect to the at-large assemblies which could be formed. In the neutral case, this broader notion of efficiency would seem appropriate.

The non-neutrality in the assumption of theorem 11 may be warranted when voters elect local representatives whose responsibilities are primarily to their district. Again, if all representatives have identical duties, a non-neutral assumption is not justified. In districted elections, no neutral assumption

[11]For instance if assembly A has more first choices than assembly B, then A is preferred to B.

can guarantee the eleection of a Pareto optimal assembly, even among the restricted set of assemblies. This can be seen from the proof of theorem 9, since the preferences used satisfy any neutral restriction.

6. Assemblies of varying sizes.

In some elections the size of the assembly to be chosen need not be predetermined. For instance, in the annual election of former baseball players to the Baseball Hall of Fame no specific number of candidates need be selected. Indeed, an assembly of size zero may be chosen. In this context Barbera, Sonnenschein, and Zhou [1991] show that any strategy-proof election method is inefficient. They argue that the reason stems from the constraints imposed by incentive-compatibility. Our results suggest rather that the problem lies in the candidate-based nature of the election procedures. Thus, one way to choose the assembly would be in fact to fix the assembly size at m and select the top m vote-getters. This is just at-large voting, which is not strategy-proof but, as we have seen, is inefficient.

Barbera, Sonnenschein, and Zhou show further that the only strategy-proof method of selecting an assembly is voting by quota: each voter lists as many candidates as she desires and any candidate appearing on at least q lists is chosen to the assembly.[12] We now recast this problem as a special case of choosing a numbered-seat assembly.

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[12]More precisely, voting by quota is the only neutral, anonymous strategy-proof method.

Let there be n candidates labeled $a_1$, $b_1, \ldots,$ $n_1$. We introduce n "pseudo-candidates" labeled $a_2, b_2, \ldots, n_2$. The numbered-seat assembly is of size n. Candidate j runs for seat j; candidate $j_1$ requires at least q votes to be elected, otherwise the pseudo-candidate is elected, i.e., the seat remains empty. It is easily seen that this formulation is equivalent to voting by quota in a variable-sized at-large election.

Simple voting in this numbered-seat election is strategy-proof since there are only two candidates running per seat. If q is half the population size, theorem 9 establishes that voting by quota is inefficient over the domain of separable preferences. The proof of theorem 9 is easily modified to encompass the case of other q's, yielding Barbera, Sonnenschein, and Zhou [1991]'s result. We now identify sufficient conditions for efficiency.

Neutrality requires that all "real" candidates be treated equally. However, we might allow that preferences treat the pseudo-candidates differently than the real candidates. Indeed, the prevalence of supra-majority quotas in this type of elections suggests that people are more interested in keeping "bad" candidates (real candidates ranked second) off assemblies than having "good" candidates (real candidates ranked first) elected. For instance, a two thirds quota for appointing a candidate might indicate that having a bad candidate elected is twice as bad as having a good candidate elected is good.

A person with weighted preferences evaluates an assembly by taking a weighted sum of the members she wanted elected to the assembly and subtracting off those members she did not want elected. In our numbered-seat formulation this gives:

Definition: Let $A_j$ be any assembly and let an individual assign it a score in the following manner. First each seat i is given a score as follows. If seat i contains the individual's favorite candidate for that seat and that candidate is labeled 1, $s_i=s$; if seat i has the individual's preferred candidate and that candidate is labeled 2, then $s_i=1$; if i does not contain the individual's preferred candidate then $s_i=0$. The assembly's score is then $S_j = \Sigma s_i$. An individual's preferences are said to be s-weighted if $A_j > A_k$ when $S_j > S_k$ or $S_j = S_k$ and $A_j$ has more candidates labeled 1 than does $A_k$.

Thus, a person with 1/2-weighted preferences requires the presence of two good candidates to counter-balance the presence of each bad candidate.

Definition: In a numbered-seat election with quota q there are two candidates $i_1$ and $i_2$ running for each seat i, each voter casts one vote for each seat, and candidate $i_1$ is elected if and only if she receives at least q votes.

Remark: Suppose there are v voters. Then a numbered seat election with quota q is equivalent to an election in which each vote cast for a candidate labeled 1 is multiplied by (v-q)/q, and the candidate is elected if and only if the weighted total of votes she receives is at last as great as the number of votes her opponent receives.

Theorem 12: In a numbered-seat election with quota q and v voters, suppose that each voter has (v-q)/q-weighted preferences. Then a Pareto optimal assembly will be chosen.

Proof: (reductio): Suppose that A is elected but B is Pareto superior. Let A* = A/(A∩B) and B* = B/(A∩B). Consider any individual. Since she likes B at least as much as A, B* must have at least as high a score as A*. Therefore the total number of weighted votes (in accordance with the remark) she distributes over B* must be at least as great as the number distributed over A*. This is true for all individuals, so the total number of weighted votes received by the members of B* is at least as great as the total received by the members of A*. Therefore, either some member of B* receives at least as many weighted votes as her opponent in A*, and is elected, or all members of B* receive exactly as many weighted votes as their opponents, but B* consists entirely of label 2 candidates. In this second case A is preferred to B. In both cases we have a contradiction. ▲

Weightedness treats candidates labeled 1 and 2 asymmetrically. This asymmetric treatment is necessary to guarantee efficiency when the quota is not simple majority rule. Furthermore this asymmetric treatment must be precise.

Theorem 13: Consider a numbered-seat election with quota q. For each $s \neq (v-q)/q$, there exists a number of seats n and a number of voters v such that this voting system is inefficient over s-weighted preferences. Furthermore, if the quota is not simple majority rule then for all $n \geq 2$, $v \geq 2$, this voting system is inefficient over all neutral individual preference restrictions.

Proof: First suppose that $s > (v-q)/q$ (the quota is "too big"). Write $s = w_2/w_1$ (if s is not rational, let $w_1/w_2$ be the largest rational number smaller than s), and let $n = v = w_1+w_2$. For seats i through $i + (w_1-1) \bmod n$ let individual i rank candidate 1 first; for the other seats let candidate 2 be ranked first. Then 1) each individual ranks $w_1$ label 1-candidates first and 2) each label 1-candidate is ranked first by $w_1$ individuals. The first fact implies that all voters prefer the assembly $\{a_1,b_1,\ldots,n_1\}$ to the assembly $\{a_2,b_2,\ldots,n_2\}$, while the second fact implies that the assembly $\{a_2,b_2,\ldots,n_2\}$ is chosen.

Now suppose that $s < (v-q)/q$. Write $(v-q)/q = w_2/w_1$ (or $w_2/w_1$ is the largest rational smaller than $(v-q)/q$), and again let $n = v = w_1+w_2$. We proceed as before, but now $\{a_1,b_1,\ldots,n_1\}$ is elected although everyone prefers $\{a_2,b_2,\ldots,N_2\}$.

Now suppose that q is not simple majority rule. This means that for v odd $q \neq (v+1)/2$, and for even $q \neq v/2$. Let the population divide into two (almost) equal groups (depending on whether v is odd or even) with top assemblies $\{a_1,b_2\}$ and $\{a_2,b_1\}$. Then for supermajority rules $\{a_2,b_2\}$ is elected but we can assume that all voters prefer $\{a_1,b_1\}$, while for submajority rules $\{a_1,b_1\}$ is elected but we can assume that all voters prefer $\{a_2,b_2\}$. ▲

## 7. Conclusion.

Jurisdictions generally adopt candidate-based procedures, in particular constant scoring rules, to elect assemblies. This common practice poses a problem since such procedures possess minimal efficiency properties only under severe preference restrictions. At least three explanations might be offered for this practice.

First, the design of assembly-based procedures presents great difficulties. For instance, an at-large election with five candidates running for ten seats has 252 possible assemblies, necessitating a huge ballot merely to list all the possibilities. It is unclear that any practical assembly-based procedure will perform well. Elections in which candidates run as slates do not resolve the problem since the entire electorate may favor an assembly which has not formed as a slate.

Second, although an inefficient or nearly inefficient assembly might be elected, perhaps such an occurrence is unlikely, or perhaps an inefficient assembly will not be "too" inefficient. Third, although efficiency is the economist's favorite property, there might be other virtues possessed by constant scoring rules which more than compensate for their inefficiency. These last two explanations merit further investigation.

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