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BY

Yaw Nyarko

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C. V. STARR CENTER
FOR APPLIED ECONOMICS

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

Most Games Violate the Harsanyi Doctrine

Yaw Nyarko
New York University

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Abstract. The type of a player in a game describes the beliefs of that player about the types of others. We show that the subset vectors of such player-type beliefs which obey the Harsanyi Doctrine (or consistency condition) is of Lebesgue measure zero. Further, as the number of players becomes large the ratio of the dimension Harsanyi-consistent beliefs to the set of all beliefs tends to zero.

Mailing Address:
New York University
Department of Economics
269 Mercer St, #723
New York, N.Y. 10003
U.S.A.

1. Introduction. Consider a game with N players each of whom is one of a finite number of types. The type of a player describes the beliefs that player-type has about the types of the other players. A belief system is a specification of the beliefs of each player-type. A belief system is said to be consistent if there exists a probability measure, π, over the set of vectors of all types such that each player-type's beliefs equal the probability measure π conditional on that type. This consistency requirement is typically referred to as the "Harsanyi Doctrine" of Harsanyi (1968). Harsanyi (1968) argues that all differences in beliefs of players must be due to differences in the information they have received, and hence players must necessarily have consistent belief systems.

In a Harsanyi consistent game the type of a player fulfills two roles: First it specifies the beliefs of that player about the types of the others; and second it provides the signal used in conditioning the joint probability, π, to obtain beliefs about other agents. The beliefs obtained via these two roles must be the same. The results of this paper show that this puts a lot a restriction on the belief systems possible, and in particular "most" of them can not fulfill both roles simultaneously.

Let B denote the set of all belief systems and let H denote the subset of B which obeys the Harsanyi Doctrine. I show that the dimension of H is much lower than that of the set B, and hence the set H has Lebesgue measure zero in B. Among the set of all belief systems, "most" of them violate the Harsanyi Doctrine. I also show that relative to the dimension of B the dimension of H becomes arbitrarily small as the number of players tends to infinity.

That a belief system can violate the Harsanyi Doctrine is well-known among researchers in this area [see, e.g., Mertens and Zamir (1985, Section 4)]. However, to the best of my knowledge, this result on the size of the set of belief systems which violate the Harsanyi Doctrine is new. The result itself, once the problem has been set up to answer the appropriate question, is embarrassingly easy to prove, and involves counting the dimensionality of certain sets. The question that is being asked reduces to the following: When does there exist a joint probability measure with given conditional distributions? Our answer is: Almost never. (This is of course not to be confused with the question of the the existence of a joint distribution with given **marginals** [see e.g., Strassen (1965)].

One often hears the argument that if one adds some extra states or passive agents to the game one can transform any game into one for which the Harsanyi doctrine holds. The set up of this paper should show that this is not possible if such a transformation is done properly.

The Harsanyi Doctrine is of course is a crucial assumption in many papers in game theory and in other applied fields like industrial organization theory. The result of this paper shows that that assumption is very strong indeed. Elsewhere we have argued that the use of the Harsanyi Doctrine may be very restrictive in models of learning [see Nyarko (1990)].

2. An Illustration for the 2x2 case. Suppose there are two Players A and B. Player A can be one of two types a_1 or a_2, and Player B can be one of two types b_1 or b_2. Conditional upon each player's type that player will have beliefs about the types of the other player. Since each player can only be one of two types, each player's beliefs about the other is a number in [0,1] representing the probability that player assigns to the other player being of the first type.

In particular, Player A's belief system is a pair $p = (p_1,p_2) \epsilon [0,1]^2$, and Player B's is a pair $q = (q_1,q_2) \epsilon [0,1]^2$, with the following interpretation:

$$p_i = \text{Prob } (b_1|a_i) \quad \text{and} \quad q_i = \text{Prob } (a_1|b_i) \text{ for } i=1,2. \qquad (2.1)$$

The set of all beliefs systems, (p,q), is the set $B = [0,1]^4$.

If these beliefs are to obey the Harsanyi Doctrine there must exist a joint probability measure $\pi = \{\pi_{ij}\}^2_{i,j=1}$ over the four pairs of possible types $\{(a_1,b_1),(a_1,b_2),(a_2,b_1),(a_2,b_2)\}$, where π_{ij} denotes probability assigned to the vector (a_i,b_j) by the probability π, such that each player's beliefs are obtained via conditioning π on that player's own realized type (i.e., such that $p_i = \pi(b_1|a_i)$ and $q_i = \pi(a_1|b_i)$ for $i = 1,2$)

I will restrict attention to strictly interior belief systems, i.e., where $(p,q) \epsilon (0,1)^4$. In that case if a probability π with the above mentioned properties does exist, then from Bayes' rule we know that such a probability must satisfy the following relationships:

$$\Sigma_{i,j}\pi_{ij} = 1 \quad \text{where the sum is over } i,j=1,2 \tag{2.2}$$

$$p_1 = \pi_{11}/(\pi_{11}+\pi_{12}) \tag{2.3}$$

$$p_2 = \pi_{21}/(\pi_{21}+\pi_{22}) \tag{2.4}$$

$$q_1 = \pi_{11}/(\pi_{11}+\pi_{21}) \tag{2.5}$$

$$q_2 = \pi_{12}/(\pi_{12}+\pi_{22}) \tag{2.6}$$

Observe that in the above system of equations, if we consider the vector (p,q) as given, then there are four unknowns, $\{\pi_{ij}\}^2_{i,j=1}$ and five equations. This system of equations is over-identified and in general, the system of equations will not have a solution. Indeed from (2.3)-(2.6) it is easy to verify that

$$q_2 = [(1-p_1)/p_1]/\{[(1-p_1)/p_1]+[(1-p_2)/p_2][(1-q_1)/q_1]\} \tag{2.7}$$

Eq.(2.7) requires that q_2 be an (explicit) function of (p_1,p_2,q_1). The subset of vectors (p_1,p_2,q_1,q_2) in the unit simplex in R^4 which are Harsanyi consistent, necessarily satisfy (2.7) and therefore lie in a three dimensional subset of R^4.

3. The General Problem. Now suppose there are N players, i=1,...,N. Each player can be one of a finite number of types, $c_{ik} \in C_i$, for k = 1,..., $\#C_i$, where $\#C_i$ is the cardinality of of the set C_i. Let C = $X^N_{i=1}C_i$ denote set of all possible vectors of types, one for each player, and let $C_{-i} = X_{j\neq i}C_j$ denote the set of vectors of types of all players other than the i-th player. Given any finite integer T let $\Delta(T)$ denote the (T-1)-dimensional unit simplex in R^T, and let $\Delta(T)^+$ denote the strictly positive subset of $\Delta(T)$:

$$\Delta(T) = \{x = (x_1, \ldots, x_T) \in R^T \mid \Sigma^T_{t=1} x_t = 1 \text{ and } x_t \geq 0 \text{ for all } t\}$$

and $$\Delta(T)^+ = \{x=(x_1, \ldots, x_T) \in \Delta(T) \mid x_t > 0 \text{ for all } t\} \qquad (3.1)$$

Associated with the k-th type of the i-th player, c_{ik}, is the belief that player has about the types of the other players. This will be a probability measure, p_{ik}, on C_{-i}; in particular, $p_{ik} \in \Delta(\#C_{-i})$. A **belief system** for the players is a specification of such a vector p_{ik} for each player i, and for each type of player i, c_{ik}. We let B denote the set of all belief systems and let B^+ be the subset of belief systems in B which are strictly positive: I.e.,

$$B = X^N_{i=1} X_{c_{ik} \in C_{ik}} \Delta(\#C_{-i}) \quad \text{and} \quad B^+ = X^N_{i=1} X_{c_{ik} \in C_{ik}} \Delta(\#C_{-i})^+ \qquad (3.2)$$

Recall that $C = X^N_{i=1} C_i$ is the set of all player types. Let G be the set of all probability measures over C and let G^+ be the subset of C which are strictly positive:

$$G = \Delta(\#C) \qquad \text{and} \qquad G^+ = \Delta(\#C)^+ \qquad (3.3)$$

For a belief system of players, $p \in B$, to be consistent in a Harsanyi (1968) sense, and hence to obey the "Harsanyi Doctrine", there must exist a probability measure $\pi \in G$ on the set of all possible vectors of types of players, C, from which the beliefs of each player of each type is obtained by conditioning on their own types; i.e., $p_{ik} = \pi(.|c_{ik})$ for each player i and type c_{ik}, where $\pi(.|c_{ik})$ denotes the conditional probability of π conditional on c_{ik}. Since the set of types are finite the conditional probabilities can be obtained using Bayes' rule.

Indeed, fix any belief system $p \epsilon B$ and probability measure over types, $\pi \epsilon G$. Let $p_{ik}(c_{-i})$ and $\pi(c)$ denote the probability assigned to $c_{-i} \epsilon C_{-i}$ and $c \in C$ by p_{ik} and π respectively. Let $\{c_{-i}, c_{ik}\}$ denote the element of C whose i-th coordinate is c_{ik} and whose other coordinates are the vector c_{-i}; and let $\pi(c_{ik})$ be the probability assigned by π to the event that the i-th player's type is c_{ik}.

For a belief system $p \in B$ to be consistent we require the existence of a probability $\pi \in G$ which obeys Bayes' rule: For each i=1,...,N, for each $c_{ik} \in C_i$ such that $\pi(c_{ik}) > 0$, and for each $c_{-i} \in C_{-i}$,

$$p_{ik}(c_{-i}) = \pi(\{c_{-i}, c_{ik}\})/\pi(c_{ik}) \quad (3.4)$$

Now, each vector in $\Delta(\#C_{-i})$ is a vector in a $(\#C_{-i}-1)$-dimensional space. Hence from (3.2), the set B is a set of vectors of dimension

$$\dim B = \Sigma^N_{i=1}(\#C_i)(\#C_{-i}-1) = NX^N_{i=1}(\#C_i) - \Sigma^N_{i=1}(\#C_i) \quad (3.5)$$

Since we are primarily concerned with dimensionality calculations we may restrict our attention to strictly positive belief systems, i.e., those in B^+ and H^+, and strictly positive joint probability vectors (in G^+). Now each probability $\pi \in G^+$

induces a unique belief system $b(\pi)$ in H^+ via Bayes' rule, (3.4). It is easy to check that this mapping $b:G^+ \to H^+$ is continuous on G^+. From the definition of H^+, for each $p \in H^+$, there exists a $\pi \in G$ such that $b(\pi) = p$; it is easy to check that if p is strictly positive, i.e. $p \in H^+$, then so is π so that the mapping $b:G^+ \to H^+$ is ONTO.

Next, if $p \in H^+$, then p is an indecomposable belief system in the sense of Harsanyi [1968, Theorem III, p.488] and therefore from that theorem there can only be one probability $\pi \in G$ which induces it (i.e., such that $b(\pi) = p$). In particular the mapping $b:G^+ \to H^+$ is ONE-TO-ONE. It is also easy to check that the mapping $b:G^+ \to H^+$ is differentiable. Hence we may identify the set G^+ with H^+ in the dimensionality calculations and in particular we may set

$$\dim H = \dim G = \#C - 1 = [X^N_{i=1}(\#C_i)]-1 \tag{3.6}$$

Of course, without loss of generality we may suppose that there are at least two players and that each player has at least two possible types, so that $\#C_i \geq 2$ for all i. With (3.5) and (3.6) we may now very easily state and prove our main result (which follows immediately from lemma A.1 in the appendix).

Proposition. (a) Fix the number of players $N \geq 2$. Then $\dim H < \dim B$ and in particular the subset H of Harsanyi consistent belief systems has Lebesgue zero in the set B of all belief systems.
(b) As the number of players N tends to infinity the ratio

[dim H/dim B] tends to zero and the difference [dim B - dim H] tends to infinity. Indeed,

[dim H/dim B] $\leq$ 1/(N-1) and [dim B - dim H] $\geq$ $(N-1)2^N + 1$ (3.7)

Appendix.

Lemma A.1. Fix any integer $N \geq 2$ and let $\{z_i\}^N_{i=1}$ be any sequence of real numbers such that $z_i \geq 2$ for each i= 1,...,N. If

$$W_N = N(X^N_{i=1}z_i) - \Sigma^N_{i=1}z_i \qquad \text{and} \qquad U_N = X^N_{i=1}z_i - 1, \text{ then}$$

$$W_N > U_N, \qquad W_N - U_N \geq (N-1)2^N + 1 \quad \text{and} \quad U_N/W_N \leq 1/(N-1) \qquad (A.1)$$

Proof: First we will show that for each $N \geq 2$,

$$\Sigma^N_{i=1}z_i \leq X^N_{i=1}z_i \qquad (A.2)$$

Fix any integer $N \geq 2$. Order the N numbers $\{z_i\}^N_{i=1}$ and relabel if necessary so that $z_1 \geq z_2 \geq \ldots \geq z_N$. Define $S_k = \Sigma^k_{i=1}z_i$ and $P_k = X^k_{i=1}z_i$. Then $S_1 = z_1 = P_1$ and if $S_k \leq P_k$, then,

$$S_{k+1} = S_k + z_{k+1} \leq S_k + z_k \leq 2S_k \leq z_{k+1}.S_k \leq z_{k+1}.P_k = P_{k+1}$$

Hence by induction, (A.2) holds for each N. This in turn implies that for $N \geq 2$, $W_N - U_N = (N-1)(X^N_{i=1}z_i) - \Sigma^N_{i=1}z_i + 1 \geq (N-2)(X^N_{i=1}z_i) + 1 \geq (N-1)2^N + 1$ from which the first two inequalities of (A.1) follow. Further, $U_N/W_N = [P_N - 1]/[NP_N - S_N] \leq P_N/[NP_N - P_N] = 1/[N-1]$. //

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