ECONOMIC RESEARCH REPORTS

INCOMPLETE MARKETS

BY

Charles Wilson

RR # 91-63 December, 1991

C. V. STARR CENTER FOR APPLIED ECONOMICS

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

Incomplete Markets

by

Charles Wilson
New York University

December 1991

This paper will appear in The New Palgrave for Money and Finance. Support from the C.V. Starr Center for Applied Economics is gratefully acknowledged. I thank Heracles Polomarchakis for several useful discussions on the topic and John Geanokoplos for directing to me to some of the more recent literature.

ABSTRACT

This paper provides a brief summary of some of the essential issues and results in the theory of incomplete markets. JEL Classification Numbers: 021.
Key Words: Incomplete Markets.

Introduction

Markets are complete when every agent is able to exchange every good either directly or indirectly with every other agent. In this case, relatively mild assumptions guarantee the existence, Pareto optimality, and local uniqueness of a competitive equilibrium. When some securities cannot be exchanged with each other, the resulting markets may not be complete. As a consequence, any one of these properties may fail to satisfied. A large literature has evolved which examines the conditions under which different sets of securities generate a complete set of markets and the properties of the equilibrium allocations when they do not. In this essay I will illustrate a few of the main ideas in this literature.

A good starting point is the work of Arrow (1971). He demonstrated that static competitive analysis can be extended to deal with the case of uncertainty, but only by expanding the set of markets to include a separate price for each commodity in each state of the world. With a complete set of markets in state contingent commodities, it follows immediately that, under the usual conditions on preferences, the competitive allocation exists, is *ex ante* Pareto efficient, and locally unique.

Although this approach solves the problem of extending general equilibrium analysis to deal with uncertainty, it strains the credibility of the model by requiring an unrealistic number of goods to be simultaneously exchanged. It was important, therefore, to examine the extent to which the same allocation could be attained with a different market structure requiring a smaller number of instruments. To investigate this question, we require a few definitions.

Consider an economy with S possible realizations of an uncertain state of the world with N goods in each state. A **security** is simply a vector in R^{SN}

which specifies an amount of each state contingent good to be delivered. A "simple" security is a security which promise delivery of one unit of only one state contingent good. A **market** is a set of securities to which we may associate a price vector at which they may be exchanged. An Arrow-Debreu market is a market consisting of the complete set simple securities. A "spot" market for state s is simply a market consisting of simple securities for state s. A spot market is **complete** when it spans the set of all simple securities for that state.

Arrow demonstrated that any equilibrium allocation in an Arrow-Debreu market can also be attained as an equilibrium of an economy with S complete spot markets and one "insurance" market consisting of S simple securities, one for each state. Since spot markets operate only when the actual state is realized, the same allocation requiring SN securities in the Arrow-Debreu economy can be acheived with one "insurance" market consisting of S securities and one spot market consising of N securities.

To understand the logic of Arrow's result, note that with a complete set of spot prices, the only role for an additional market is to transfer purchasing power across states. Let p(s) denote the vector of spot prices in state s and consider some arbitrary security f which promises the delivery of the vector of f_s goods in state s. Then security f transfers $p(s)f_s$ units of purchasing power to state s. I will call $(p(1)f_1,\ldots,p(S)f_S)$ the **income transfer** of security f. To acheive any transfer of purchasing power across states, all that is required is a set of securites whose income transfers span R^S. This requirement is clearly met by the set of simple securities used by Arrow.

Since we rarely see an explicit market in simple securities, it was of interest to determine the conditions under which some other market, e.g. a stock market, might play the same role. In general, the span of transfers of an arbitrary set of securities depends on the prices which prevail in the spot market. For example, Townsend (1978) considers a securities market which includes only "forward" securities, i.e., securities which promise delivery of a unit of some good in every state of the world. He demonstrates that the income transfers of these securities span R^S if and only if there are at least S forward securities (i.e. $N \geq S$) and the set of spot market price vectors are linearly independent. We will see exactly why in the next section.

The Existence of Equilibrium

When the dimension of the span of the transfers of a set of securities depend on the prices in the spot markets, the usual regularity assumptions on preferences no longer guarantee the existence of an equilbrium. Consider the following example based on Hart (1975). There are two agents, a and b, and two states of the world, 1 and 2. In each state there are two goods, labelled X and Y which must be consumed in nonnegative amounts by each agent. The preferences and endowments of the agents are given by the following table where x_i^α and y_i^α are the amounts of goods X and Y respectively consumed by agent α in state i.

Agent	Endowments		Utility
	(X_1,Y_1)	(X_2,Y_2)	
a	(2,2)	(1,1)	$3x_1^a + y_1^a + 3x_2^a + y_2^a$
b	(1,1)	(2,2)	$3x_1^b + y_1^b + 3x_2^b + y_2^b$

Agent a is endowed with two units of each good in state 1 and one unit of each good in state 2. His marginal rate of substitution between X and Y in either state is 3, and his marginal rate of substitution between goods across states is 1. Agent b is endowed with one unit of each good in state 1 and two units of each good in state 2. His preferences are the same as agent a except that the role of X and Y is reversed. Suppose we supplement the spot markets with a "forward" market with two securities one of which promises to deliver one unit of good X in each state and the other a unit of good Y in each state. We will show that a competitive equilibrium does not exist.

Let $p_x(s)$ and $p_y(s)$ denote the equilbrium prices of goods and X and Y in the spot market of state s, and let q_x and q_y denote the equilibrium prices of the two forward securities. Since the spot markets are complete, an exchange of forward security X for forward security Y increases income in state s only if $q_x/q_y < p_x(s)/p_y(s)$. This leads us to consider two cases.

Suppose that the relative spot prices are equal in the two markets. Then, it follows immediately that $q_x/q_y = p_x(1)/p_y(1) = p_x(2)/p_y(2)$. Otherwise, one security dominates the other in the sense that an exchange raises or lowers income in both states. As a result, an exchange of securities transfers no purchasing power at all, and the equilibrium allocation and prices in the spot market are the same as if no securities market existed. Simple calculations reveal that, in both states, agent a obtains all three units of good X and agent b all three units of good Y. However, the spot prices differ, with $p_x(1)/p_y(1) = 2$ and $p_x(2)/p_y(2) = 1/2$. We conclude that the relative spot prices cannot be equal.

If the relative spot prices in the two states are not equal, then one security will dominate the other unless $p_x(s)/p_y(s) < q_x/q_y < p_x(s')/p_y(s')$, for some s,s', $s \neq s'$. In this case, an exchange of securities does imply a non zero transfer of income across states so that the agents are effectively able to exchange any pair of state contingent goods. Consequently, the competitive allocation with the forward securities can be supported as an equilibrium with a full set of Arrow-Debreu securities (with prices $p_x(s)/[p_x(s)q_y - p_y(s)q_x]$ and $p_y(s)/[p_x(s)q_y - p_y(s)q_x]$ in each state s). But since the marginal rate of substitution for good X_1 and X_2 is one for both agents, it follows that the ratio of the spot prices must be the same in both states (in fact, the equilibrium prices of all state contingent goods are equal). We conclude that no equilibrium exists.

An equilibrium fails to exist in this example because the the dimension of the space of net trades in the space of state contingent goods which can be obtained by the exchange of securities in the various markets depends on the prices in those markets. As a consequence, the demand functions for securities may be unbounded. To avoid this problem, Radner (1972) imposes an exogenous lower bound on short sales of securities. Another approach is to assume that the set of securities is sufficiently rich to guarantee that the dimensionality of net trades does not vary with the price as in Geanakoplos and Polemarchakis (1989). In any case, Kreps (1979) notes that the set of transfers for any set of securities has full rank for almost all spot prices and therefore that the existence problem is not generic. A general theorem for the generic existence of an equilibrium is established by Duffie and Shafer (1985).

Pareto Efficiency

When the allocation of goods is determined by the exchange of securities in a single market, all of the standard results on the existence, uniqueness, and welfare properties of equilibrium remain intact. For all practical purposes, the set of securities defines the set of goods with the preference ordering of each agent derived from his preference ordering over the set of state contingent goods. In particular, the first welfare theorem is satisfied. Any equilibrium allocation is Pareto optimal in the space of allocations spanned by the set of securities which define the market. However, when the allocation is determined in two or more segmented markets, this formulation of the first welfare theorem is no longer true.

Suppose there is a complete spot market for each state. Then the set of securities which are traded in at least one market spans the entire state contingent commodity space. Without an additional securities market, however, agents cannot transfer income across states. Therefore, if set of spot market securities defines the feasible set of allocations, the equilibrium allocation is typically not Pareto optimal. Clearly this concept of feasibility is not sufficiently restrictive. It presumes that there is no cost to a central planner in coordinating the allocation of securities in one spot market with the allocation of securities in another. Consequently, it provides no basis for explaining why there is no corresponding market for exchanging simple securities between states.

A more useful definition of feasibility might reflect the ability of a central planner to arbitrarily allocate securities in only one market, the allocation of securities in all other markets being determined by market clearing prices. This approach leads to the following definition suggested by Hart (1975). Suppose there is a complete spot market in each state and one additional market consisting of a set F securities. An allocation of state contingent goods is <u>constrained</u> <u>Pareto</u> <u>efficient</u> if (i) it is attained as an equilibrium in the spot markets for some feasible distribution of securities in F, and (ii) there is no Pareto superior allocation of state contingent goods attained as an equilibrium in the spot markets for some other feasible distribution of securities in F.

We will show that when the number of securities in F is less than S, an equilibrium need not be even constrained Pareto efficient. The reason is that a redistribution of the ownership of securities generally leads to a change in the spot market prices and hence to a change in the vector of income transfers associated with each security. Since the set of transfer vectors generated by the set of securities in F cannot span the space of all income transfers, it follows that subspace spanned by the transfer vectors of the F securities generally depends on the prices in the spot markets. Consequently, the transfer of real income generated by the redistribution of securities following the adjustment of prices in the spot markets typically lies outside the span of the transfers generated by the set of securities at the competitive equilibrium prices. By redistributing existing securities, therefore, it may be possible to increase the welfare of every agent in the economy.

Consider an economy with three agents, a, b, and c, and two states of the world, 1, 2. In each state i there are two goods, labelled X and Y. Suppose the preferences and endowments of the agents are given by the following table where x_i^α and y_i^α is the consumption of goods X and Y respectively by agent α in state i.

Agent	Endowments		Utility
	(X_1,Y_1)	(X_2,Y_2)	
a	(0,2)	(2,0)	$x_1^a + \epsilon \min\{x_2^a,y_2^a\}$
b	(2,0)	(0,2)	$\epsilon \min\{x_1^b,y_1^b\} + x_2^b$
c	(1,1)	(1,1)	$y_1^c + y_2^c$

In this economy, agent a is endowed with two units of good Y in state 1 and two units of good X in state 2. He consumes only good X in state 1 and always consumes an equal amount of both goods in state 2. For each pair of units of the two goods he consumes in state 2 he is willing to give up ϵ units of his consumption of good X in state 1. The endowment and preferences of agent b are the same except that the role of the two states is reversed. Agent c is endowed with one unit of good X in both states but consumes only good Y. His marginal rate of substitution between consumption in the two states is unity.

Suppose there is a single security which promises to deliver one unit of good X in each state. Since there is nothing for which to exchange this security, the equilibrium income and spot prices in each state will be determined solely by the endowments of the agents in that state. It is easy to check that the relative price of the two goods is unity in both states. Agent a consumes two units of good X in state 1 and one unit of each good in state 2. Agent b consumes one unit of each good in state 1 and one unit of good X in state 2. Agent c consumes 2 unit of good Y in both states.

Although the security will never be traded in the market, it can still be used by the government to redistribute purchasing power in the two states and thereby change the spot prices. Suppose, for instance, that agents a and b must each supply agent c with two units of the security. Then the effect is the same as if the endowments were changed as follows.

Agent	Endowments	
	(X_1,Y_1)	(X_2,Y_2)
a	(-2,2)	(0,0)
b	(0,0)	(-2,2)
c	(5,1)	(5,1)

For this economy the equilibrium price of good Y in terms of good X in each state is 5/2. Agent a consumes the three units of good X in state 1 and nothing in state 2. Agent b consumes nothing in state 1 and all three units of good X in state 2. Agent c consumes the three units of good Y in both states.

Now compare the welfare of the two agents in the two economies. Without the transfer payments, agents a and b attain an expected utility of $2+\epsilon$ while agent c attains an expected utility of 4. With the transfer payments, agent a and b both attain an expected utility of 3 while agent c attains a utility of 6. Consequently, for $0 < \epsilon < 1$, the equilibrium with transfer payments Pareto dominates the equilibrium without transfer payments. By transferring purchasing power to agent c in both states, the economy has made the price of the goods demanded by agents a and b cheaper in those states where they value their increased welfare the most.

The possibility that securities can be reallocated to attain a Pareto superior allocation when markets are incomplete was first illustrated by Hart

(1975). He provided an example in which removing securities and hence decreasing the possibilities for trade actually resulted in a Pareto superior allocation. The intuition is similar to that provided in the example above. If markets are not compete, the introduction of a new security may change the spot market prices in such a way that utility of agents decreases unless they can make trades which are not available with the existing set of securities.

Geanokoplos and Polemarchakis (1986) consider a model with two periods and enlarge the commodity space to include consumption before the state of nature is realized. With a complete set of spot markets in the second period and a combined spot and securities market in the first period, they establish that the competitive equilibrium is almost never constrained Pareto optimal whenever the number of securities in F is less than S and there are at least two goods in each state. Geanokoplos, Magill, Quinzii, and Dreze (1990) establish a similar result for a general equilibrium model of the stock market.

Nominal Securities and the Indeterminancy of Equilibrium

Cass (1985) investigates the implications for equilibrium when some of the securities are "nominal". These are securities in which the returns in any state are denominated in some unit of account. When all securities are nominal an equilibrium always exists. Moreover, if the dimension of the span of these securities is less than S, the equilibrium is generally not locally unique. In fact, the dimension of indeterminacy is generally equal to S-1.

This result derives from the fact that the real income actually transferred to any state by a nominal security depends on the price level in that state. Suppose the prices in each spot market s are normalized so that they sum to q_s. Then for each vector, $q = (q_1, \ldots q_S)$, any given nominal security f which promises delivery of f_s units of income in each state s corresponds to a unique "real" security which pays f_s/q_s units of each good in each state s. Let $g_f(q)$ denote the real security $(f_1/q_1, \ldots, f_S/q_S)$, and let F(q) denote the set of all such securities generated by the initial set of nominal securities. For for any security in F(q), the relative prices in each state do not affect the amount of real income which is transferred. Consequently, there will generally be a locally unique equilibrium associated with each vector q.

Suppose that the set of nominal securities does not span R^S. Then, any nonproportional change in q will generically change the span of F(q). Consequently, when we replace the market of nominal securities with a market of real securities F(q), each (normalized) vector q will generally result in a distinct equilibrium allocation. Note, however, that each these allocations can be realized as an equilibrium with the same set of nominal securities. Therefore, since the dimension of normalized vectors q is S-1, it follows that the dimension of equilibrium allocations associated with any incomplete set of nominal securities is generically S-1.

Notice that this argument only works when the set of nominal securities does not span R^S. When the span is complete, the opportunities for the distribution of real income implied by the artificial real securities no longer depends on q and the Arrow-Debreu allocation is attained.

References

Arrow, K (1971), "The Role of Securities in the Optimal Allocation of Risk-Bearing," in Essays in the Theory of Risk-Bearing, Chapter 4, Markham Publishing Company, Chicago.

Cass, D. (1985) "On the 'Number' of Equilibrium Allocations with Incomplete Financial Markets," CARESS Working Paper No. 85-16, University of Pennsylvania.

Duffie, D. and W. Shafer (1985), "Equilibrium with Incomplete Markets, I: A Basic Model of Generic Existence," Journal of Mathematical Economics, 14, pp. 285-299.

Geanokoplos, J., M. Magill, M. Quinzii, and J. Dreze, (1990) "Generic Inefficiency of Stock Market Equilibrium when Markets are Incomplete," Journal of Mathematical Economics, 19, pp. 113-142.

Geanokoplos, J. and H. Polemarchakis (1986)., "Existence, Regularity, and Constrained, Suboptimality of Competitive Allocations when Markets are Incomplete," in W. Heller, R. Starr, and D. Starrett, (eds.) Essays in Honor of Kenneth Arrow, Volume 3. Cambridge: Cambridge University Press.

Geanokoplos, J. and H. Polemarchakis (1990), "Observability and Optimality," Journal of Mathematical Economics, 19, pp. 153-166.

Hart, O. (1975), "On the Optimality of Equilibrium When the Market Structure is Incomplete," Journal of Economic Theory, Vol. 11 (3), pp. 418-443.

Kreps, D., (1979) "Threee Essays on Capital Markets," Technical Report 298. Institute for Mathematical Studies in the Social Sciences, Stanford University.

Radner, R. (1982), "Existence of Equilibrium of Plans, Prices, and Price Expectations," Econometrica, 40, pp. 289-303.

Townsend, R. (1978) "On the Optimality of Forward Markets," American Economic Review, Vol. 68 (1), pp.54-66.