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A Model of Project Evaluation With Limited Resources

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ABSTRACT

This paper characterizes the optimal policy for a model in which manager may adopt an endogenous number of projects but has only limited resources to devote to their evaluation and maintenance. In any period, the manager may discard any subset of existing projects but may evaluate only one existing or one new project which is then either discarded or restored. Both its current return and the probability with which a project may be restored depends only on the number of periods since its last evaluated. For a manager whose objective is to maximize the sum of discounted returns, the optimal policy takes one of two forms. A "discard" policy specifies that the manager evaluate a new project in each period and discard current projects at some critical age. An "age inspection" policy specifies that the manager evaluate a new project only if all current projects are sufficiently young.

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1. Introduction

The problem to be analyzed in this paper is motivated by a model of product innovation for an entrepreneurial firm developed by Gifford (1989). The essential feature of the model is that it treats the attention of the manager as a limited resource and focuses on the allocation of his attention between maintaining existing projects and developing new ones. Her work complements much of the existing literature on R&D which focuses on the allocation of resources with respect to the development and innovation of a single project (e.g. Lee, (1982a), (1982b), Grossman and Shapiro (1986), and Lucas (1971)) or the optimal timing of adoption of a new technology (e.g. Jensen (1982), (1983)).

In this paper, we derive the optimal policy for a generalized version of Gifford's model. A manager maintains a "menu" of projects each of which is indexed by the number of periods since its last evaluation. In each period, he chooses a subset of projects to discard and at most one project to evaluate. Upon evaluation, a current project is determined to be either functioning or failed. A functioning project may be effectively restored to its initial state. A failed project is permanently discarded. The manager may also choose to forgo the evaluation of any existing project and instead evaluate a new project. With some probability, the manager will determine that the project is viable, and it will be adopted. Otherwise, it is permanently discarded. Both the current observed (expected) return and the probability that a given current project will be restored depend only on the number of periods since it was last evaluated. Otherwise, there are no restrictions on the parameters of the model.

Depending on these parameters, we show that the optimal policy takes one of two forms. If the probability of restoring any existing project upon evaluation is not high enough to compensate for the loss of revenues from keeping the project another period and evaluating a new project instead, then the optimal decision rule is a "discard" policy. In this case, an adopted project is retained for a fixed number of periods and then discarded. A new project is evaluated each period. Otherwise, the optimal decision rule is an "age inspection" policy. In this case, the manager retains each current project until some critical age at which point it is reevaluated. A new project is evaluated only if there is no project at the critical age in the current menu.

The model is related to the machine maintenance models of inspection and repair, beginning with the work of Barlow, Hunter and Proschan (1963, 1965). The basic model consists of a single project which at any time is either functioning or failed. A functioning project may deteriorate in any moment, but a failed project remains failed unless explicitly renewed. At any time, the state of the project may be observed and, if necessary, renewed at a cost. The problem is to determine the age at which a project should be inspected, given the history of inspections since its last renewal. Ross (1971) provides a characterization of the optimal policy when the objective is to minimize the sum of discounted costs. Luss (1976) and Rosenfield (1976) extend the model to allow the project to take on a finite number of states. Wappanapanom and Shaw (1979) further modify the model to allow for inspection to affect the rate of deterioration of the project. Starting with Eckles (1968), a number of authors have analyzed various versions of the model where the current state is imperfectly observed upon inspection (see e.g. Smallwood and Sondik (1973), White (1977), Sondik (1978), Albright (1979), and Lovejoy (1978)). These models have also been extended to incorporate a system of multiple machines in which the failure rate and/or the cost of maintenance of a component may depend on the number of components serviced or their status. (See Cho and Parlar (1991) for a review of this literature).

We depart from this literature by supposing that the resources which may be devoted to project maintenance are limited. This assumption allows us to focus on the allocation of this resource among an endogenous number of competing projects. As a consequence, the "cost" of evaluation is endogenous, depending on the forgone benefits of evaluating alternative projects.

In the next two section, we present the model and characterize the optimal policy. Section 4 concludes with a brief discussion of the results and their applications.

2. The Model

Time is discrete and indexed by the nonnegative integers. Projects are indexed by the number of periods since their last evaluation. A project last evaluated i periods ago is an **age i project** and earns a current return r_i. Future returns are discounted by a factor β per period, $0 < \beta < 1$. Let Q denote the set of sequences $(q_i)_{i=1}^{\infty}$ with a finite sum defined over $\{0,1\}$. An element of Q is called a **menu**, typically denoted by q, where q_i denotes the number of projects of age i. If $q_i = 0$, $i = 1,2,\ldots$, then q is called the **null menu**. For menus q and q′, we write $q' \leq q$ if $q_i' \leq q_i$, $i = 1,2,\ldots$

In each period, the manager possesses some menu q from which he chooses to retain a submenu of projects q′, $q' \leq q$. From menu q′, the manager earns a current return $R(q') \equiv \sum_{i=1}^{\infty} q_i' r_i$. All other projects are permanently discarded from the menu. Simultaneously, either one existing project or one new project is selected for evaluation. If an age i project is evaluated, then with probability p_i it is restored and appears in the next period as an age 1 project. Otherwise, upon yielding its return in the current period, it is permanently discarded. If a new project is evaluated, it is adopted in the next period as an age 1 project with probability p_0 and otherwise discarded. We assume that (r_i) is a bounded sequence and that the objective of the manager is to maximize the discounted sum of returns.

Let e_i denote the menu q for which $q_i = 1$ and $q_j = 0$, $j \neq i$. For $q \in Q$, and $i = 1,2,\ldots$, let q_{-i} denote the menu q' such that $q_i' = 0$ and $q_j' = q_j$, $j \neq i$, and let $q_{-0} \equiv q$. Let $S:Q \to Q$ denote the shift operator defined by $(Sq)_1 = 0$ and $(Sq)_i = q_{i-1}$, $i = 2,3,\ldots$. Then, a manager who retains submenu q' and evaluates project i realizes in the next period menu $Sq'+e_1$ with probability p_i and menu Sq' with probability $(1-p_i)$.

Let I denote the nonnegative integers. For each menu q, let $\Lambda(q) \equiv \{(q',i) \in Q\times I\colon q' \leq q$ and $i > 0$ implies $q_i' = 1\}$ denote the set of 2-tuples consisting of a retained menu and an evaluation choice. A (stationary) **policy** λ is a function defined on Q for which $\lambda(q) \in \Lambda(q)$.[1] Given initial menu q, let $g_\lambda(q)$ denote the expected discounted return from following policy λ and let $g^*(q) = \sup_\lambda g_\lambda(q)$. A policy λ^* is **optimal** if $g_{\lambda^*} = g^*$.

For any real valued function f on Q, let $h(q',i,f) = R(q') + \beta[p_i f(Sq'+e_1) + (1-p_i)f(Sq')]$. If the undiscounted value of realizing menu q in the following period is f(q), then h(q',i,f) denotes the expected total discounted return to retaining menu q' and evaluating project i in the current

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[1] We restrict attention to stationary policies since any optimal stationary policy is also optimal is the space of all history dependent policies (Denardo, Theorem 5).

period. For any policy λ and any function f, define $[H_\lambda(f)](q) = h(\lambda(q),f)$, $q \in Q$. Similarly, for any function f, define the value of the maximization operator A by $[A(f)](q) = \sup_{(q',i)\in A(q)} h(q',i,f)$, $q \in Q$. In the Appendix, we establish that, for any policy λ, g_λ is the unique fixed point of H_λ, and g^* is the unique fixed point of A. The existence of an optimal policy then follows from Denardo (1967, Corollary 2 and Theorem 3).

3. The Main Result

In this section, we derive a characterization the optimal policy. A menu is **attainable** for a policy λ if it can be realized in some future period starting from the null menu when policy λ is applied in every period. We will refer to the subset of attainable menus for a policy as its **attainable set**. We demonstrate that, over its attainable set, a policy takes one of two forms. In some cases, it is a discard policy, in which case a new project is evaluated in every period and, upon reaching a critical age, current projects are discarded without evaluation. Otherwise, it is an age inspection policy in which case each current project is evaluated periodically and a new project is evaluated only if no current project requires evaluation. Which type of policy is adopted depends on whether the benefit of evaluating an existing project at the critical date exceeds the benefit from evaluating a new project instead and retaining the existing project until some later date.

For any nonnegative integer t, let $Q_t = \{q \in Q: q_i = 0,\ i = t+1,t+2,\dots\}$ be the set of menus containing no project older than t. Then, a **discard** policy δ^t is any policy for which, given $q \in Q_{t+1}$, $\delta^t(q) = (q_{-(t+1)},0)$. If the current menu consists of no projects older than t+1, a manager following discard policy δ^t retains any project of age t or less and drops any project of age t+1. It always evaluates a new project. Notice that Q_{t+1} is the attainable set for δ^t so that if a menu lies in Q_{t+1}, any menu realized in the following period must also lie in Q_{t+1}.

For any positive integer t, an **age inspection** policy α^t is any policy for which, given $q \in Q_t$, $\alpha^t(q) = (q,t)$ if $q_t = 0$, and $\alpha^t(q) = (q,0)$ otherwise. If the current menu consists of no projects older than t, a manager following an age inspection policy α^t retains all existing projects and evaluates any age t project. If the current menu contains no age t projects, he evaluates a new project. The attainable set for α^t is Q_t. For any policy δ^t or α^t, we refer to t as its **critical age**. Notice that there are no restrictions on a policy of either type outside its attainable set.

Our analysis proceeds as follows. First, for any discard or age inspection policy we calculate the incremental value of adding an additional project to an existing menu, assuming that the augmented menu is attainable. Noting that these incremental values are independent of the menu to which the projects are added, we then demonstrate that the value of any attainable menu is just the sum of the incremental values of its projects.

Next, using these expressions for the incremental value of a project, we establish an upper bound for the optimal value function g^*. We then demonstrate that the value function for the policy which maximizes the incremental value of an age 1 project over the set of all discard and age inspection policies attains this upper bound on its attainable set. This result provides the desired characterization of an optimal policy.

Consider first the incremental value of acquiring an age k project when the manager adopts the discard policy δ^t. For $k = 1,\ldots,t$, the project is retained until age t and then discarded, resulting in an incremental value of $d_k^t \equiv \sum_{i=1}^t r_k \beta^{i-1}$. If $k = t+1$, the project is discarded in which case its incremental value is $d_{t+1}^t \equiv 0$.

Alternatively, suppose a_k^t denotes the incremental value of acquiring an age k project when the manager adopts the age inspection policy α^t. In this case, he retains the project until period t-k earning a discounted return $\sum_{i=k}^t r_i \beta^{i-k}$, whereupon he evaluates the project. With probability p_t, it passes the evaluation resulting in an age 1 project in the next period and an expected discounted net return of $\beta^{t+1-k} a_1^t$. Otherwise, it is discarded in the following period. However, if the manager had not acquired the age k project in period 0, he would have evaluated a new project in period t-k instead. In this case, with probility p_0, he acquires an age 1 project in the following period, resulting in an expected discounted net return of $\beta^{t+1-k} p_0 a_1^t$. Therefore, the incremental value of acquiring a project of age k must satisfy $a_k^t \equiv \sum_{i=k}^t r_i \beta^{i-k} + \beta^{t+1-k}(p_t - p_0) a_1^t$. Setting $k = 1$, and solving for a_1^t then yields $a_1^t = [\sum_{i=1}^t r_i \beta^{i-1}]/[1-\beta^t(p_t - p_0)]$. Define $a_1^0 = 0$.

Notice that the expressions for d_k^t and a_k^t are independent of the composition of the current menu. Consequently, if the manager follows either type of policy, the addition of projects younger than age t imposes no externalities on the return of the projects already in the menu. This suggests that the total discounted value of a menu should be the sum of the incremental values of its projects plus the value of a null menu. All that remains, therefore, is to determine the value of the null menu. To do this, we exploit the fact that the value of a menu consisting of a single age 1 project is equal to the value of a null menu plus the incremental value of the project.

When a discard policy δ^t is applied to the null menu, the next period realization is either the null menu with probability $(1-p_0)$ or the menu consisting of a single age 1 project with probability p_0. It follows, therefore, that, if d_0^t is the value of the null menu, it must satisfy $d_0^t = \beta[p_0(d_1^t+d_0^t) + (1-p_0)d_0^t]$. Therefore, given policy δ^t, $d_0^t = p_0 d_1^t[\beta/(1-\beta)]$ represents the value of the null menu. A similar argument yields $a_0^t = p_0 a_1^t[\beta/(1-\beta)]$ as the value of the null menu under policy α^t.

LEMMA 1: (a) For $q \in Q_{t+1}$, $g_{\delta^t}(q) = d_0^t + \sum_{i=1}^t q_i d_i^t$, and (b) for $q \in Q_t$, $g_{\alpha^t}(q) = a_0^t + \sum_{i=1}^t q_i a_i^t$.

PROOF: (a) Since g_{δ^t} is the unique fixed point of H_{δ^t}, it follows by definition that, for $q \in Q_{t+1}$, it is must satisfy

$$(1) \qquad g_{\delta^t}(q) = R(q_{-(t+1)}) + \beta[p_0 g_{\delta^t}(e_1+Sq_{-(t+1)}) + (1-p_0)g_{\delta^t}(Sq_{-(t+1)})].$$

Moreover, since, for all $q \in Q_{t+1}$, $e_1+Sq_{-(t+1)}$ and $Sq_{-(t+1)}$ are contained in Q_{t+1}, it follows that (1) uniquely determines g_{δ^t} restricted to Q_{t+1}. To establish part (a), therefore, it is sufficient to verify that $g_{\delta^t}(q) = d_0^t + \sum_{i=1}^t q_i d_i^t$ satisfies (1) for $q \in Q_{t+1}$. Substituting $g_{\delta^t}(q)$ into the right hand side of (1), and recalling that $d_0^t = \beta[d_0^t p_0 + d_1^t]$, we obtain $g_{\delta^t}(q) = R(q_{-(t+1)}) + \beta[d_0^t+p_0 d_1^t] + \beta\sum_{i=1}^t d_{i+1}^t q_i = \sum_{i=1}^t r_i q_i + \beta[d_0^t+p_0 d_1^t] + \sum_{i=1}^t[\sum_{k=i+1}^t r_k \beta^{k-i}]q_i = d_0^t + \sum_{i=1}^t[\sum_{k=i}^t r_k \beta^{k-i}]q_i = d_0^t + \sum_{i=1}^t d_i^t q_i$.

(b) The proof is similar except that, for $q \in Q_t$, g_{α^t} must satisfy

$$(2) \qquad \begin{aligned} g_{\alpha^t}(q) &= R(q) + \beta[p_0 g_{\alpha^t}(e_1+Sq) + (1-p_0)g_{\alpha^t}(Sq)] & \text{if } q_t = 0, \\ g_{\alpha^t}(q) &= R(q) + \beta[p_t g_{\alpha^t}(e_1+Sq_{-t}) + (1-p_t)g_{\alpha^t}(Sq_{-t})] & \text{if } q_t = 1. \end{aligned}$$

Q.E.D.

Our next step is to establish an upper bound for g^*. Let $v_1 \equiv \sup\{\max\{a_1^t, d_1^t\},\ t = 0,1,\ldots\}$ denote supremum of the incremental values of an age 1 project for all discard and age inspection policies. Let $v_0 \equiv p_0 v_1[\beta/(1-\beta)]$ denote the corresponding value it induces for the null menu. For $k = 2,3,\ldots$, let $v_k \equiv \max\{0, \sup\{\sum_{i=k}^{t} r_i\beta^{i-k} + \beta^{t+1-k}\max\{(p_t-p_0)v_1, 0\},\ t = k,k+1,\ldots\}\}$. In words, v_k denotes the maximal incremental value of an age k project when a manager may choose among the following options: (i) discard it immediately, (ii) hold it until some future date and then discard it, or (iii) hold until it reaches some age t and then sell it for $(p_t-p_0)v_1$ (its incremental value to the manager if it evaluates it at age t and follows the best age inspection or discard policy thereafter, assuming that in the absence of the project the manager would evaluate a new project in period t-k.)

LEMMA 2: For $k = 1,2,\ldots$, (a) $v_k \geq 0$, (b) $v_k \geq r_k + \beta(p_k-p_0)v_1$, and (c) $v_k \geq r_k + \beta v_{k+1}$.

PROOF: (a) Obvious. (b) Let $t = k$ in the definition of v_k. (c) There are three cases to consider. (i) $v_{k+1} = 0$, (ii) $v_{k+1} = \sum_{i=k+1}^{t} r_i\beta^{i-k} \geq 0$, or (iii) $v_{k+1} = \sum_{i=k+1}^{t} r_i\beta^{i-k-1} + \beta^{t-k}(p_t-p_0)v_1 \geq 0$. (If the supremum is not attained for any finite t, set $t = \infty$.) In each case, substitution into the definition of v_k establishes the result. Q.E.D.

LEMMA 3: $g^*(q) \leq \hat{g}(q) \equiv v_0 + \sum_{i=1}^{\infty} q_i v_i$, $q \in Q$.

PROOF: Consider any menu q and any $(q',k) \in \Lambda(q)$, and recall that $h(q',k,\hat{g}) = R(q') + \beta[p_k\hat{g}(e_1+Sq'_{-k}) + (1-p_k)\hat{g}(Sq'_{-k})]$. By definition, $v_0 = \beta[v_0 + p_0v_1]$.

For $k = 0$, Lemmma 2c then implies that $h(q',k,\hat{g}) = \sum_{i=1}^{\infty} q_i' r_i + \beta[v_0 + p_0 v_1 + \sum_{i=1}^{\infty} q_i' v_{i+1}] \le v_0 + \sum_{i=1}^{\infty} q_i' v_i = \hat{g}(q')$. Similarly, if $k > 0$, Lemma 2 implies that $h(q',k,\hat{g}) = \sum_{i=1}^{\infty} q_i' r_i + \beta[v_0 + p_k v_1 + \sum_{i=1,i\neq k}^{\infty} q_i' v_{i+1}] \le v_0 + r_k + \beta(p_k - p_0)v_1 + \sum_{i=1,i\neq k}^{\infty} q_i' v_i \le v_0 + \sum_{i=1}^{\infty} q_i' v_i = \hat{g}(q')$. Since $q' \le q$, Lemma 2a implies that $\hat{g}(q') \le \hat{g}(q)$ from which it follows that $\sup_{(q',k)\in\Lambda(q)} h(q',k,\hat{g}) = [A(\hat{g})](q) \le \hat{g}(q)$. The result then follows from Denardo (Theorem 3 and Lemma 2a). Q.E.D.

THEOREM 1: (a) There is an optimal discard policy δ^t if and only if $d_1^t = v_1$. (b) There is an optimal age inspection policy α^t if and only if $a_1^t = v_1$.

PROOF: (a) Given Lemma 3, it is sufficient to prove that $d_1^t = v_1$ if and only if $g_{\delta^t}(q) \ge \hat{g}(q)$, $q \in Q_{t+1}$. By definition, $d_k^t \le v_k$, $k = 1,\ldots,t+1$, $t = 0,1,2,\ldots$. Therefore, for $q \in Q_{t+1}$, the definitions of $\hat{g}$ and g_{δ^t} imply that $g_{\delta^t}(q) \ge \hat{g}(q)$ if and only $d_k^t = v_k$, $k = 1,\ldots,t$. All that remains is to show that $d_1^t = v_1$ implies $d_k^t = v_k$, $k = 1,\ldots,t+1$.

Suppose $d_k^t < v_k$, for some $k \le t+1$. Then there is an integer $s \ge k$ for which at least one of the following three conditions is satisfied: (i) $d_k^t < 0$, (ii) $d_k^t < \sum_{i=k}^{s} r_i \beta^{i-k}$, or (iii) $d_k^t < \sum_{i=k}^{s} r_i \beta^{i-k} + \beta^{s+1-k}(p_s - p_0)v_1$. We will show that in each case, $d_1^t < v_1$.

Recall that $d_k^t = \sum_{i=k}^{t} r_i \beta^{i-k}$. Therefore, in case (i), $d_1^t = \sum_{i=1}^{t} r_i \beta^{i-1} = \sum_{i=1}^{k-1} r_i \beta^{i-1} + \beta^{k-1} d_k^t < \sum_{i=1}^{k-1} r_i \beta^{i-1} = d_1^{k-1} \le v_1$. Similarly, in case (ii), $d_1^t = \sum_{i=1}^{k-1} r_i \beta^{i-1} + \beta^{k-1} d_k^t < \sum_{i=1}^{k-1} r_i \beta^{i-1} + \beta^{k-1} \sum_{i=k}^{s} r_i \beta^{i-k} = \sum_{i=1}^{s} r_i \beta^{i-1} = d_1^s \le v_1$. In case (iii), $d_1^t = \sum_{i=1}^{k-1} r_i \beta^{i-1} + \beta^{k-1} d_k^t < \sum_{i=1}^{k-1} r_i \beta^{i-1} + \beta^{k-1} \sum_{i=k}^{s} r_i \beta^{i-k} + \beta^{s}(p_s - p_0)v_1 = \sum_{i=1}^{s} r_i \beta^{i-1} + \beta^{s}(p_s - p_0)v_1 \le v_1$.

The proof of part (b) is identical. Q.E.D.

For all practical purposes Theorem 1 is a complete characterization of the set of optimal policies. Over its attainable set, an optimal discard or age inspection policy is completely specified. Moreover, if we include the discard policy δ^{∞}, either condition (a) or (b) must be satisfied. It is also apparent from the proof that, except for knife edge cases, any optimal policy must be one of these two types.

The simple form of the optimal policy reflects the fact that the best alternative to evaluating any existing project is to evaluate either a new project or a project of the critical age. Consequently, within any attainable menu, the opportunity cost of evaluating any given project is unaffected by additions or deletions of other projects. For unattainable menus, however, this property is no longer satisfied, in some case resulting in a more complicated form for the optimal policy. If a discard policy is optimal, older projects may be retained and even evaluated for some menus if the (r_t) sequence is not monotonic. If an age inspection policy is optimal, older projects may be retained and evaluated even if both the (r_t) and the (p_t) sequences are monotonically decreasing.

4. Conclusion

Theorem 1 depends on several restrictions embodied in the model. First, there are only two possible consequences to the evaluation of any project. Either the project is restored to some initial state or it is discarded. Second, the transition function for unevaluated projects is deterministic so that the state of a project depends only on the number of periods since its last evaluation. These assumptions guarantee that any feasible menu consists of at most one project of any given age. Third, there is an unlimited homogeneous supply of new projects one of which can always be evaluated instead of an existing project. This assumption provides a constant opportunity cost to the evaluation of existing projects over the attainable set.

Almost any generalization of the model leads to severe complications in characterizing the optimal policy. If the transition function is stochastic or the evaluation process restores projects only partially, then a menu may contain more than one project in a given state. As a consequence, a simple age inspection policy cannot be implemented. Moreover, since the opportunity cost of evaluation will generally depend on the distribution of projects in the menu, a different approach will be required to characterize the optimal policy.

Even with these restrictions, however, the simple form of the optimal policy may provide some useful insights into the decision problem of an entrepreneurial manager. For instance, Gifford (1989) supposes a manager who controls an endogenous number of mutually independent projects each of which is either functioning or failed. A functioning project fails in each period with some fixed probability ϕ, but its status can be determined only if the manager allocates all of his attention in that period to its evaluation. Alternatively, a manager may devote his time to developing a new project which succeeds with probability p_0. If g is the return to a functioning project, b the loss from a failed project, and we assume that a failed project may be restored with probability γ, then, for $t = 1,2,3,\ldots$, $r_t = \phi^t g + (1-\phi^t)b$, $p_t = \phi^t + (1-\phi^t)\gamma$. Gifford then investigates how changes in the parameters of her model affect the (r_t) and (p_t) sequences and hence the optimal policy and steady state size of the firm. For example, she notes that a small increase in p_0 may induce the manager to shift from an age inspection policy α^t to a discard policy $\delta^{t'}$, resulting in a discontinuous increase in the probability of adopting a new project.

Other variations may also be of interest. Consider, for instance, a small firm which specializes in software development. Each project represents a different product whose revenue tends to decline over time as it becomes obsolete due to competition in the market. In any period, the manager may devote his attention to updating an existing product or developing a new product. In this case, (r_t) need not reflect the probability of failure, but rather a deterministic sequence of the revenue generated by the product in each period. If we assume that all products have the same revenue stream and that $p_t = 1$, so that existing products may be completely updated with certainty, this problem can also be cast directly into our framework.

Appendix

Our analysis exploits the results of Denardo (1967) which generalizes may of the results of Blackwell (1965). Unfortunately, Denardo considers only those problems which generate a bounded value function which, for our model, essentially requires that $\sum_{i=1}^{\infty}|r_i| < \infty$. Given our restrictions on the transition map, however, we may extend his analysis to any case where the (r_i) sequence is bounded. We need only demonstrate that there is a complete metric space on which the functional equation satisfies Denardo's contraction and monotonicity properties.

Assume the notation of Section 2. Suppose there is a $K > 0$ such that $|r_i| < K$, $i = 1,2,\ldots$. Recall that Q denotes the set of sequences $(q_i)_{i=1}^{\infty}$ with a finite sum defined over $\{0,1\}$. Choose positive numbers N and ϵ such that $(N+1)\beta < N(1-\epsilon)$. For any $q \in Q$, let $n_q = \max\{N,\{i\colon q_i = 1, i = 1,2,\ldots\}\}$ denote the maximum of N and the age of the oldest project in menu q. Then $|R(q)| \leq Kn_q$.

Let F denote the set of real valued functions for which any element f implies a $K_f > 0$ such that $f(q) \leq K_f n_q$, $q \in Q$. For $f \in F$, define $\|f\| = \sup_q |f(q)|/n_q$. Then, with the metric induced by the norm $\|\cdot\|$, F is a complete metric space. Fix $f \in F$ and choose $K_f > NK/[N-\beta(N+1)]$. Then, $(q',i) \in \Lambda(q)$ implies $|h(q',i,f)| = |R(q') + \beta[p_i f(Sq'+e_1) + (1-p_i)f(Sq')]| \leq Kn_q + \beta K_f(n_q+1) < K_f n_q$. Therefore, for any policy λ, $|[H_\lambda(f)](q)| \leq n_q K_f$ which implies that H_λ maps F into F. Similarly, $|[A(f)](q)| \leq n_q K_f$, $q \in Q$, implies that A maps F into F.

For $q \in Q$, $(q',i) \in \Lambda(q)$ implies $(n_{q'}+1)|h(q',i,f) - h(q',i,g)| \leq (n_{q'}+1)\beta \sup_{q'' \in Q_{n_q+1}}|f(q'') - g(q'')| < n_{q'}(1-\epsilon) \sup_{q'' \in Q_{n_{q'}+1}}|f(q'') - g(q'')|$.

Therefore, $(1/n_{q'}) \sup_{q \in Q_{n_{q'}}} |h(q',i,f) - h(q',i,g)| < (1-\epsilon)[1/(n_{q'}+1)]$ $\sup_{q'' \in Q_{n_{q'}+1}} |f(q") - g(q")| \leq (1-\epsilon)\sup_{q'' \in Q}[1/(n_{q''}+1)]\ |f(q") - g(q")| =$ $(1-\epsilon)\|f-g\|$. Therefore,

(C1) $\|H_\lambda(f) - H_\lambda(g)\| \leq (1-\epsilon)\|f - g\|$.

Following Denardo (Theorem 2), a similar argument establishes that

(C2) $\|A(f) - A(g)\| \leq (1-\epsilon)\|f - g\|$.

Also note that

(M) $f(q) \geq g(q)$, $q \in Q$ implies $h(q',i,f) - h(q',i,g) \geq 0$, $(q',i) \in \Lambda(q)$, $q \in Q$.

Given condition (C1), it follows that g_λ is the unique fixed point of H_λ. (C2) implies that A posseses a unique fixed point. Denardo (Corollary 2 and Theorem 3) then establishes that condition (M) implies that g^* is the fixed point of A and that $g_{\lambda^*} = g^*$ for some policy λ^*.

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