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Uniqueness and Indeterminacy: Transitional Dynamics in a Model of Endogenous Growth

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Abstract

In this paper we study the dynamics of the endogenous growth model of Lucas [1988]. We characterize the regions of the parameter space that give rise to unique equilibria as well as the regions that yield a continuum of equilibria with positive growth rates. We find that a continuum of equilibria exists for very plausible parameters.

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1. Introduction

In this paper we study the dynamics of the endogenous growth model of Lucas [1988]. We characterize the regions of the parameter space that give rise to unique equilibria as well as the regions that yield a continuum of equilibria with positive growth rates.[1] Our results are very much in line with those of Caballè and Santos [1992], who find a unique growth equilibrium when there are no human capital externalities. On the other hand, when modest and realistic externalities are introduced, such multiplicities are easily found.

In Lucas [1988] the accumulation of human capital is described by:

$$\dot{h} = \delta h(1-u)$$

where h is human capital, $1-u$ is the fraction of labor allocated to the production of h, and δ is a parameter estimated at 0.05. The utility function is of the standard isoelastic type, given by $U(c) = (1-\sigma)^{-1}(c^{1-\sigma}-1)$. For the same estimates of factor shares and the externality parameter associated with human capital reported by Lucas [1988], we prove that a continuum of equilibria with positive growth rates exists provided that the discount factor is larger than δ and σ is sufficiently low. Conversely, a sufficient condition for local uniqueness is that the discount factor is less than δ (see Proposition 1 and Proposition 2 in section 2.3) and that σ is sufficiently large. However, when we extend the model to allow for labor supply to be endogenous, we can obtain a continuum of equilibria even when the utility of consumption is logarithmic (i.e. $\sigma = 1$), provided the marginal utility of leisure is not too

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[1] For similar results see, among others, Benhabib and Farmer [1991] or Boldrin and Rustichini [1991].

strongly diminishing (see section 3).

What accounts for the continuum of equilibria in growth models with externalities? We offer the following explanation. Starting with an arbitrary equilibrium path, consider constructing another one by increasing the savings rate and accelerating the accumulation of capital. For this to be an equilibrium, the rate of return on capital must increase sufficiently to justify its higher accumulation rate. Speeding up the rate of human capital accumulation by reallocating labor towards the production of human capital might help, provided the externality parameter is strong enough. Then the marginal product of capital may indeed rise sufficiently to justify the higher rate of its accumulation. In a model without externalities, however, the complementarity between physical and human capital in production is simply not strong enough to accommodate the multiplicity of equilibria.

We can compare our multiplicity results with those of Benhabib and Farmer [1991]. Again, starting with an arbitrary equilibrium path consider constructing another path with a higher capital accumulation rate. In Benhabib and Farmer [1991] labor supply must be sufficiently elastic (i.e. the utility of leisure must be close to linear) so that higher capital stocks which increase the marginal product of labor can induce a sufficiently strong labor supply response: this raises the marginal product of capital and justifies the faster accumulation rate while still constituting an equilibrium. In Lucas [1988], to warrant a faster accumulation of physical capital, human capital stocks must also grow faster, and since total labor supply is fixed, consumption must be initially reduced. Such a reduction of consumption is possible in equilibrium only if the curvature of the utility function is not too high (i.e. if σ is low). So, to have a continuum of equilibria, we must have not only externalities, but also a high intertemporal elasticity of substitution of consumption. If labor supply is made endogenous, however, both physical and human capital can be increased without reducing consumption too much by drawing on additional labor. As pointed out above, we can get a continuum of equilibria even with a logarithmic utility of consumption, provided the marginal utility of leisure is not too strongly diminishing.

The implications of the continuum of equilibria are as follows. Two identically endowed economies with identical initial conditions may consume and allocate labor between the production of human and physical capital at completely different rates (see Lucas [1993], who poses the question as to why the growth rates of South Korea and the Philippines differ so drastically, even though their initial wealth and endowment levels were close). Only in the long run will those economies converge to the same growth rate, but not to the same level of output and human and physical capital. Therefore it is possible to view cultural and non–economic factors in such models not as affecting fundamentals like technology or preferences, but simply as selection devices for equilibria which differ on the transition paths.

The rest of the paper is organized as follows: section 2.1 below briefly outlines the model of Lucas [1988]; section 2.2 describes, following Mulligan and Sala–i–Martin [1992], a way of reducing from four to three the dimension of the system of differential equations arising from the first order necessary conditions, thereby greatly simplifying the analysis, and identifies the balance growth path (BGP henceforth); section 2.3 contains the main results concerning the stability of the BGP. In section 3.1 we augment the previous model with endogenous labor supply, and in section 3.2 we show how a continuum of equilibria can arise for different values of the labor supply elasticity. The appendix contains the proofs of the main results.

2.1 The Standard Model

We study the two–sector model introduced by Lucas [1988] and further extended and analyzed by Caballè and Santos [1991] and Mulligan and Sala–i–Martin [1992]. Formally the model is given as:

$$\max_{c(t),u(t)} \int_0^\infty \frac{c^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt \qquad (P)$$

subject to:

$$\begin{aligned}
\dot{k} &= Ak^{\beta}(uh)^{1-\beta}h_a^{\gamma} - c \\
\dot{h} &= \delta h(1-u) \\
k(0) &= k_0 > 0 \\
h(0) &= h_0 > 0
\end{aligned}$$

where c is consumption, k is physical capital, h is human capital, u is the fraction of labor allocated to the production of physical capital, ρ is a positive discount factor, A and δ are positive technology parameters, β is the share of capital, γ is a positive externality parameter in the production of human capital, and σ is the inverse of the intertemporal elasticity of substitution. For notational convenience, we define $\theta \equiv (A, \beta, \gamma, \delta, \rho, \sigma)$, and $\theta \in \Theta$, where $\Theta = R^4_{++} \times R_+ \times (0,1)$.

The problem (P) is solved by defining the current value Hamiltonian:

$$H = \frac{c^{1-\sigma}-1}{1-\sigma} + \lambda_1(Ak^{\beta}h^{1-\beta}h_a^{\gamma}u^{1-\beta} - c) + \lambda_2[\delta h(1-u)]$$

and obtaining the first order necessary conditions for an interior solution:

$$c^{-\sigma} = \lambda_1$$

$$A(1-\beta)\lambda_1 k^{\beta} h^{1-\beta} h_a^{\gamma} u^{-\beta} = \lambda_2 \delta h \tag{1}$$

$$\dot{\lambda}_1 = \rho\lambda_1 - \lambda_1 A\beta k^{\beta-1} h^{1-\beta} h_a^{\gamma} u^{1-\beta} \tag{2}$$

$$\dot{\lambda}_2 = \rho\lambda_2 - (1-\beta)A\lambda_1 k^{\beta} h^{-\beta} h_a^{\gamma} u^{1-\beta} - \lambda_2\delta(1-u) \tag{3}$$

plus the usual two transversality conditions:[2]

$$\lim_{t\to\infty} e^{-\rho t}\lambda_1(t)k(t) = 0 \tag{4}$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda_2(t)h(t) = 0 \tag{5}$$

We note that in problem (P) the representative optimizing agent treats h_a^{γ} as an exogenous function of time: because of the externality, the competitive solution differs from the planner's solution. In equilibrium, of course, h must be set equal to h_a.

A sufficient condition for a solution of the first order conditions to solve (P) is that the Hamiltonian be jointly concave in (k, h) after the controls c and u have been substituted out with their maximizing values.[3] Replacing $\lambda_1^{-\sigma}$ for c and the expression derived from equation 1 for u into the Hamiltonian, we find that:

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[2] To be precise, the transversality condition should be written as:

$$\lim_{t\to\infty}[e^{-\rho t}\lambda_1(t)k(t) + e^{-\rho t}\lambda_2(t)h(t)] = 0$$

Our two conditions are of course sufficient for this single condition to be satisfied.

[3] This is known as the Arrow's condition: see Kamien and Schwartz [1981]. We could have also used Mangasarian's condition, which says that the (not maximized) Hamiltonian must be jointly concave in the states k and h and in the controls c and u, and that $\lambda_i > 0$ for all t. The condition we use turns out to be of simpler application.

$$H^* = B_1 + B_2 A k h_a^{\frac{\gamma}{\beta}} + B_3 h$$

where the B_i are terms not including k and h. Being linear in k and h, H^* is clearly concave.[4]

We want now to express the two multipliers λ_1 and λ_2 in terms of their corresponding control variables c and u in order to end up with an autonomous system of four differential equations in the four variables k, h, c, and u. We now impose the consistency condition that $h_a = h$; since $\lambda_1 = c^{-\sigma}$, we have, from 2:

$$\dot{c} = \frac{A\beta}{\sigma} k^{\beta-1} h^{1-\beta+\gamma} u^{1-\beta} c - \frac{\rho}{\sigma} c \tag{6}$$

Substituting for λ_1 in equation 1 we derive an expression for λ_2:

$$\lambda_2 = \frac{A(1-\beta)}{\delta} c^{-\sigma} k^{\beta} h^{-\beta+\gamma} u^{-\beta}$$

Taking logarithms on both sides and differentiating with respect to time we get:

$$\frac{\dot{\lambda}_2}{\lambda_2} = \beta \frac{\dot{k}}{k} + (\gamma - \beta) \frac{\dot{h}}{h} - \sigma \frac{\dot{c}}{c} - \beta \frac{\dot{u}}{u}$$

Now we can substitute for the rates of growth of k, h, and c, obtaining:

$$\frac{\dot{\lambda}_2}{\lambda_2} = \delta(\gamma - \beta)(1 - u) - \beta \frac{c}{k} + \rho - \beta \frac{\dot{u}}{u} \tag{7}$$

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[4] Note again that the term $h_a^{\frac{\beta+\gamma}{\beta}}$ is treated as an exogenous function of time by the optimizing agents. However, a planner can not ignore this term, so that the relevant Hamiltonian is not concave for the planner and the sufficient conditions are not met in that case.

Next we derive another expression for the rate of growth of λ_2 from equation 3 and set it equal to the expression in equation 7:

$$\rho - \delta = \delta(\gamma - \beta)(1-u) - \beta\frac{c}{k} + \rho - \beta\frac{\dot{u}}{u}$$

From this equation we can thus obtain an expression for $\dot{u}$:

$$\dot{u} = \frac{\delta(\beta-\gamma)}{\beta}u^2 + \frac{\delta(1-\beta+\gamma)}{\beta}u - \frac{c}{k}u$$

Then the four dimensional system in k, h, c, and u is:

$$\begin{aligned}
\dot{k} &= Ak^{\beta}h^{1-\beta+\gamma}u^{1-\beta} - c \\
\dot{h} &= \delta h(1-u) \\
\dot{c} &= \frac{A\beta}{\sigma}k^{\beta-1}h^{1-\beta+\gamma}u^{1-\beta}c - \frac{\rho}{\sigma}c \\
\dot{u} &= \frac{\delta(\beta-\gamma)}{\beta}u^2 + \frac{\delta(1-\beta+\gamma)}{\beta}u - \frac{c}{k}u
\end{aligned} \tag{8}$$

What we want to do next is to reduce the system by one dimension, since the analytical study of a three dimensional system is much easier.

2.2 The Reduced Standard Model

As in Mulligan and Sala–i–Martin [1992] we can reduce the dimension of the system 8 by a change of variables. First, recall that a BGP is defined as a set of functions of time $\{k(t),\ h(t),\ c(t),\ u(t)\}$ that solve the optimal control problem (P), and such that k, h, and c grow at a constant rate, and u is constant. We define these rates of growth as μ_k, μ_h, and μ_c respectively. In this model k and c grow at the same constant rate on the BGP (see Lucas

[1988]), so $\mu_k = \mu_c$. We therefore look for some combination of the original variables that is stationary on the BGP. We can write:

$$\begin{aligned}\frac{\dot{k}}{k} &= Ak^{\beta-1}h^{1-\beta+\gamma}u^{1-\beta} - \frac{c}{k} = \mu_k \\ \frac{\dot{h}}{h} &= \delta(1-u) = \mu_h\end{aligned}$$

Notice that μ_h is always constant when u is constant, and it is positive if $u < 1$; since c and k grow at the same rate, c/k is also constant. Taking logarithms and differentiating with respect to time in the equation for $\dot{k}/k$, we see that, for μ_k to be constant, it must be the case that, on the BGP:

$$\mu_k = \frac{1-\beta+\gamma}{1-\beta}\mu_h$$

We can thus define the following new stationary variables, as suggested by Mulligan and Sala-i-Martin [1992]:

$$\begin{aligned}x &\equiv kh^{\frac{1-\beta+\gamma}{\beta-1}} &(9)\\ q &\equiv \frac{c}{k} &(10)\end{aligned}$$

By taking logarithms in equation 9, differentiating with respect to time, and substituting for μ_k and μ_h, we get an expression or $\dot{x}/x$:

$$\frac{\dot{x}}{x} = Ax^{\beta-1}u^{1-\beta} + \frac{\delta(1-\beta+\gamma)}{\beta-1}(1-u) - q \tag{11}$$

We can do the same with equation 10, and obtain an expression for $\dot{q}/q$:

$$\frac{\dot{q}}{q} = q + A\left(\frac{\beta}{\sigma} - 1\right)x^{\beta-1}u^{1-\beta} - \frac{\rho}{\sigma} \tag{12}$$

We see immediately that k, h, and c do not appear in equations 11 and 12; this means that we end up with a new system in only three dimensions, x, q, and u. To simplify the notation, we define the following new constants:

$$\psi \equiv \frac{\delta(1-\beta+\gamma)}{\beta-1}; \qquad \phi \equiv \frac{\beta}{\sigma} - 1; \qquad \zeta \equiv \frac{\rho}{\sigma} \qquad \eta \equiv \frac{\delta(\beta-\gamma)}{\beta}$$

In this way the three dimensional system is:

$$\begin{aligned} \dot{x} &= Ax^{\beta}u^{1-\beta} + \psi(1-u)x - qx \\ \dot{u} &= \eta u^2 + \frac{\psi(\beta-1)}{\beta}u - qu \\ \dot{q} &= q^2 + A\phi x^{\beta-1}u^{1-\beta}q - \zeta q \end{aligned}$$

In order to find the steady states of this reduced system, we first observe that $x = 0$ can not be a steady state value, since x is raised to a negative power in the $\dot{q}$ equation. This implies that there are no steady states where one of the variables is zero; we can thus consider the system in the rates of growth and find its interior steady states. From the $\dot{u}/u$ equation we find the steady state value of q as a function of the steady state value of u:

$$q^* = \eta u^* + \frac{\delta(1-\beta+\gamma)}{\beta} \tag{13}$$

Next, substitute equation 13 into the equation for $\dot{q}/q$ and obtain an expression for the steady state value of x, again as a function of u^*:

$$x^* = \left[\frac{\beta\zeta - \delta(1-\beta+\gamma) - \delta(\beta-\gamma)u^*}{A\beta\phi}\right]^{\frac{1}{\beta-1}} u^* \tag{14}$$

Finally, substitute equations 13 and 14 into the $\dot{x}/x$ equation to find an expression for u^* as a function of the parameters only:

$$u^* = \frac{\beta\zeta + \beta\phi\psi - \delta(1-\beta+\gamma)(1+\phi)}{\delta(\beta-\gamma)(1+\phi)+\beta\phi\psi} \tag{15}$$

If we substitute for ψ, ϕ, and ζ, we get the following expression:

$$u^* = 1 + \frac{(1-\beta)(\rho-\delta)}{\delta[\sigma(1-\beta+\gamma)-\gamma]} \tag{16}$$

Equations 13, 14, and 16 represent the only steady state of the reduced model, and correspond to the unique family of BGPs of the original model. The problem with equation 16 is that there is no guarantee that u^* lies on the unit interval; hence we turn now our attention to the conditions needed for this to be true; these conditions consist in some restrictions on the "feasible" parameter values.

From equation 16, we see that for u^* to be less than one we need:

$$(\rho-\delta)[\sigma(1-\beta+\gamma)-\gamma)] < 0$$

Also, for u^* to be positive, we must require:

$$\frac{(1-\beta)(\rho-\delta)}{\delta[\sigma(1-\beta+\gamma)-\gamma]} > -1$$

Combining these two inequalities we obtain two cases for which $0 < u^* < 1$:

$$0 < \rho < \delta \qquad \text{and} \qquad \sigma > 1 + \rho/\psi \tag{17}$$

$$\delta < \rho < -\psi \qquad \text{and} \qquad \sigma < 1 + \rho/\psi \tag{18}$$

The additional restriction in 18, $\rho < -\psi$, assures that $1 + \rho/\psi$ is positive; otherwise the case in equation 18 is empty, since we must have $\sigma \geq 0$.

These conditions are necessary for the existence of an interior steady state or BGP, and they define a subset Θ_0 of Θ. From now on we require thus

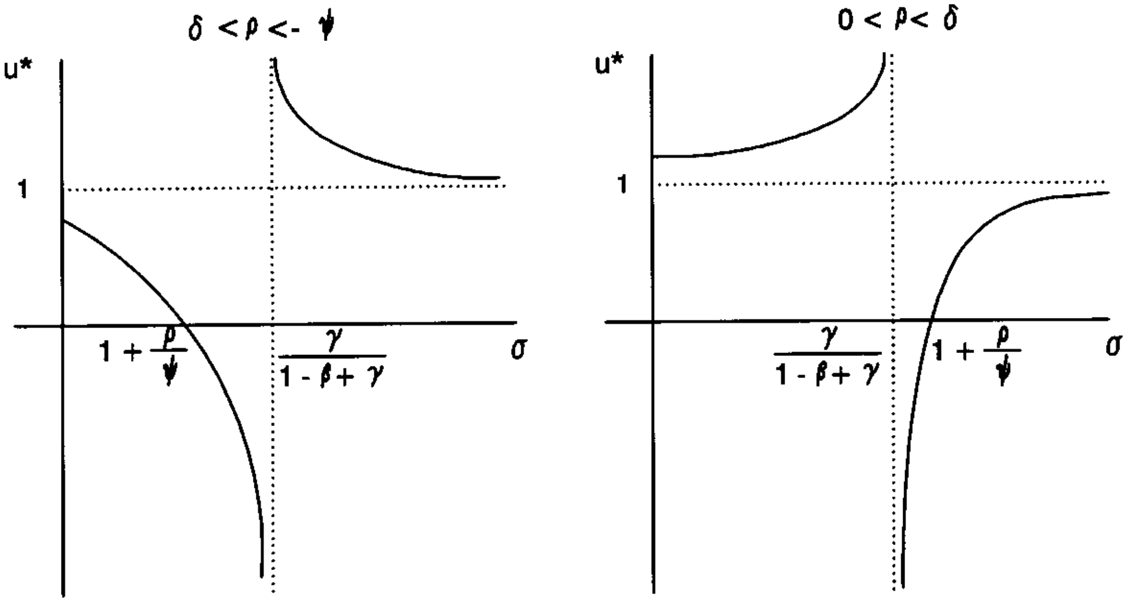

Figure 1: Equilibrium values of u as a function of the parameters.

that $\theta \in \Theta_0$, where $\Theta_0 = \{\theta \in \Theta \mid \sigma < (1+\rho/\psi) \text{ if } \delta < \rho < -\psi \text{ and } \sigma > (1+\rho/\psi) \text{ if } \rho < \delta\}$; the values of u^* as a function of σ are depicted in figure 1.

We have now to check whether this steady state satisfies the transversality conditions. For the first condition, given by equation 4, to be satisfied it must be that:

$$\lim_{t\to\infty} -\rho + \frac{\dot{\lambda}_1}{\lambda_1} + \frac{\dot{k}}{k} < 0$$

Substituting from equation 2 for $\dot{\lambda}_1/\lambda_1$ and from the equation of motion of k for $\dot{k}/k$, the first transversality condition reduces to:

$$-\left[A(\beta-1)x^{*\beta-1}u^{*1-\beta} + q^*\right] < 0 \tag{19}$$

The first term in the expression in square bracket turns out to be the partial derivative of $\dot{x}/x$ evaluated at the equilibrium; in the appendix (see equation A3) we will show that this quantity plus q^* is always positive if u^* lies in the unit interval, and thus the first transversality condition is satisfied. As for the second transversality condition, given by equation 5, we have:

$$\lim_{t\to\infty} -\rho + \frac{\dot{\lambda}_2}{\lambda_2} + \frac{\dot{h}}{h} < 0$$

If we substitute $\rho - \delta$ for $\dot{\lambda}_2/\lambda_2$ (derived from equation 3), and $\delta(1-u^*)$ for $\dot{h}/h$, we obtain:

$$-\delta u^* < 0 \tag{20}$$

Also the second transversality condition is thus always satisfied, provided again that $0 < u^* \leq 1$. Hence, since the Hamiltonian is concave, and both the transversality conditions are satisfied at the steady state, we can be sure that the solution described above is a maximum.

When the parameters of our problem do not satisfy the conditions in 17 or 18 above, we will have corner solutions for u^*. If $u^* = 1$, then it is clear that there can not be sustained growth, since $\dot{h}$ is always zero. On the other hand parameter configurations that yield $u^* = 0$ may result in unbounded utility, and it is easy to check that the transversality conditions fail to hold in this case.

2.3 Stability Properties of the BGP in the Standard Model

In this section we investigate the local stability properties of the BGP found in the previous section to describe the regions in the parameter space which yield unique and indeterminate equilibria. We restrict our analysis to the case of interior solutions for u^*.

The competitive equilibrium solution is locally unique (i.e. the BGP is determinate) if the Jacobian of the reduced system has two eigenvalues with positive real parts and one with negative real part. This is because the initial condition $x(0)$ is given, but $q(0)$ and $u(0)$ are free. The Jacobian evaluated at the BGP (see the Appendix) is given by:

$$J^* = \begin{pmatrix} J_{11}^* & -\frac{x^*}{u^*}(J_{11}^* + \psi u^*) & -x^* \\ 0 & \eta u^* & -u^* \\ J_{11}^* \phi \frac{q^*}{x^*} & -J_{11}^* \phi \frac{q^*}{u^*} & q^* \end{pmatrix}$$

where $J_{11}^* \equiv \left.\frac{\partial \dot{x}}{\partial x}\right|_E = A(\beta - 1)x^{*\beta-1}u^{*1-\beta}$. We can confirm the results of Mulligan and Sala–i–Martin [1992] only under some restrictions on the parameters:

Proposition 1 *Suppose that* $0 < \rho < \delta$ *and* $\sigma > 1 + \rho/\psi$; *then* J^* *has one eigenvalue with negative real part and two eigenvalues with positive real parts: the equilibrium is locally unique.*

Proof: see the Appendix.

Note that Mulligan and Sala–i–Martin [1992], in studying the Lucas [1988] model find saddle–path stability, consistent with Proposition 1 above, for $\rho = 0.065$ and $\delta = 0.1$ (δ and ϕ respectively, in their notation). On the other hand the estimate of δ for the U.S. reported by Lucas [1988] is 0.05, which would make $\rho > \delta$. In that case there can be a continuum of solutions in the neighborhood of the BGP (i.e. an indeterminate BGP).

Proposition 2 *Suppose that* $\delta < \rho < -\psi$ *and* $\sigma < 1 + \rho/\psi$; *then:*
(i) if $\beta < \gamma < 2\beta$, J^* *has two eigenvalues with negative real parts and one eigenvalue with positive real part: there is a continuum of equilibria;*
(ii) A necessary but not sufficient condition for J^* *to have three eigenvalues with positive real parts is that* $\gamma < \beta$.

Proof: see the Appendix.

The restriction $\beta < \gamma < 2\beta$ easily holds for the parameter values estimated by Lucas [1988]: he gives $\beta = 0.25$ and $\gamma = 0.417$. This suggests that there can be an indeterminate BGP in economies where the discount factor is greater than the parameter δ and the intertemporal elasticity of substitution is high enough (or, equivalently, where the degree of risk aversion is low enough, i.e. $\sigma < 1 + \rho/\psi$). [5] Note that if there is no externality ($\gamma = 0$), the parameter restrictions in *(i)* of Proposition 2 cannot be satisfied.

When $\gamma < \beta$, there exists a subset Θ_1 of Θ_0 such that, if $\theta \in \Theta_1$, J^* has three positive roots. Since this occurs when the real parts of two complex eigenvalues changes sign (the eigenvalues are surely complex because the determinant does not become zero), it is clear that the reduced system undergoes a Hopf bifurcation. This indicates the presence of periodic orbits within some range of the parameter space Θ, the stability properties of which remain to be investigated. Examples with three positive roots are (not so) easily developed: for example we have three eigenvalues with positive real parts for $A = 0.1$, $\beta = 0.9$, $\gamma = 0.01$, $\delta = 0.0799$, $\rho = 0.8$, and $\sigma = 0.01$. In this paper, however, we have chosen to stress the indeterminacy results, since they are more relevant for the diverse growth experiences of some countries which initially were at similar levels of wealth and development.

We conclude this section by noting that when there are a continuum of equilibria parametrized by initial conditions, it should be possible to rank them in terms of consumer welfare. This can easily be done in the one–

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[5] For example, it is easy to check that J^* has two complex roots with negative real parts for $\beta = 0.25$, $\gamma = 0.417$, $\rho = 0.065$, $\delta = 0.06$, $A = 1$, and $\sigma = 0.15$. In this case we have $u^* = 0.74$, $\mu_k = \mu_c = 0.024$ and $\mu_h = 0.015$. If $\delta = 0.05$ and other parameters are unchanged, we have two real negative roots but $\mu_k = mu_k = 0.072$ and $\mu_h = 0.046$.

dimensional case of Benhabib and Farmer [1991]. In our three–dimensional case this is harder, and is left for future research.

3.1 A Model with Endogenous Labor

In this section we augment the standard Lucas [1988] model with endogenous labor. In Lucas [1988] the labor supply is fixed and normalized to unity; here we relax this assumption by introducing labor into the utility function as a decision (control) variable. We do this because we expect that indeterminacy may arise for sufficiently high levels of the intertemporal elasticity of substitution of labor, as in Benhabib and Farmer [1991]. We choose a separable functional form for the utility function:

$$\max_{c(t),u(t),L(t)} \int_0^{\infty} \left(\frac{c^{1-\sigma}-1}{1-\sigma} - \frac{L^{1-\epsilon}-1}{1-\epsilon} \right) e^{-\rho t} dt \qquad (PL)$$

subject to:

$$\begin{aligned} \dot{k} &= Ak^{\beta}(uh)^{1-\beta}h_a^{\gamma} - c \\ \dot{h} &= \delta h(L-u) \\ k(0) &= k_0 > 0 \\ h(0) &= h_0 > 0 \end{aligned}$$

For the utility function to be concave, it must be that $\epsilon < 0$. We also modify the equation for the accumulation of human capital: now the maximum quantity of labor that can be devoted to the production of one of the two types of capital is L, instead of one. The Hamiltonian for this system is:

$$H = \frac{c^{1-\sigma} - 1}{1-\sigma} - \frac{L^{1-\epsilon} - 1}{1-\epsilon} + \lambda_1(Ak^{\beta}h^{1-\beta}h_a^{\gamma}u^{1-\beta} - c) + \lambda_2[\delta h(1-u)]$$

and the first order necessary conditions become:

$$c^{-\sigma} = \lambda_1$$

$$L^{-\epsilon} = \lambda_2 \delta h \tag{21}$$

$$A(1-\beta)\lambda_1 k^{\beta}h^{1-\beta}h_a^{\gamma}u^{-\beta} = \lambda_2 \delta h \tag{22}$$

$$\dot{\lambda}_1 = \rho\lambda_1 - \lambda_1 A\beta k^{\beta-1}h^{1-\beta}h_a^{\gamma}u^{1-\beta} \tag{23}$$

$$\dot{\lambda}_2 = \rho\lambda_2 - (1-\beta)A\lambda_1 k^{\beta}h^{-\beta}h_a^{\gamma}u^{L-\beta} - \lambda_2\delta(1-u) \tag{24}$$

From equation 22 we derive the same expression for λ_2 as in the standard model:

$$\lambda_2 = \frac{A(1-\beta)}{\delta} c^{-\sigma}k^{\beta}h^{-\beta+\gamma}u^{-\beta}$$

Replacing this equation into equation 21, we get an expression for the labor supply:

$$L = \left[A(1-\beta)k^{\beta}h^{1-\beta+\gamma}c^{-\sigma}u^{-\beta}\right]^{-\frac{1}{\epsilon}} \tag{25}$$

From equation 23 we derive the same expression for $\dot{c}$ as in equation 6. Following exactly the same procedure as in section 2, we derive two expression for $\dot{\lambda}_2/\lambda_2$, we equate them, and we get the following equation for $\dot{u}$:

$$\dot{u} = \frac{\delta(\beta-\gamma)}{\beta}u^2 + \frac{\delta(1-\beta+\gamma)}{\beta}Lu - \frac{c}{k}u$$

Substituting the expression found in equation 25 for L into this equation and into the equation of motion for h, and putting these equation together with the low of motion of k and c (equation 6), we obtain the following four dimensional system of ordinary differential equations:

$$\begin{aligned}
\dot{k} &= Ak^{\beta}h^{1-\beta+\gamma}u^{1-\beta} - c \\
\dot{h} &= B\delta k^{-\frac{\beta}{\epsilon}}h^{\frac{\epsilon-(1-\beta+\gamma)}{\epsilon}}c^{\frac{\sigma}{\epsilon}}u^{\frac{\beta}{\epsilon}} - \delta hu \\
\dot{c} &= \frac{A\beta}{\sigma}k^{\beta-1}h^{1-\beta+\gamma}u^{1-\beta}c - \frac{\rho}{\sigma}c \qquad (26) \\
\dot{u} &= \eta u^2 + B\psi\frac{\beta-1}{\beta}x^{\frac{1-\beta}{\epsilon}}u^{\frac{\beta+\epsilon}{\epsilon}}q^{\frac{1}{\epsilon}} - uq
\end{aligned}$$

$$(27)$$

where $B \equiv [A(1-\beta)]^{-1/\epsilon}$.

The next step is to check when a BGP exists, i.e. under what conditions μ_k, μ_h and μ_c are constant and μ_u is zero. As in the previous model, we have again $\mu_k = \mu_c$. Suppose u is constant; then from the equation of motion of k we see that, in order for μ_k to be constant, it must be that:

$$k^{\beta-1}h^{1-\beta+\gamma} = \text{const} \qquad (28)$$

On the other hand, from the law of motion of h, we see that μ_h is constant if:

$$k^{-\frac{\beta}{\epsilon}}h^{-\frac{1-\beta+\gamma}{\epsilon}}c^{\frac{\sigma}{\epsilon}} = \text{const} \qquad (29)$$

Taking logarithms and differentiating with respect to time in equations 28 and 29, we see that a BGP exists when the following homogeneuos system of linear equations is satisfied:

$$(\beta-1)\mu_k + (1-\beta+\gamma)\mu_h = 0$$

$$\frac{\sigma - \beta}{\epsilon}\mu_k - \frac{1 - \beta + \gamma}{\epsilon}\mu_h = 0$$

In order for the BGP to be nondegenerate, i.e. in order for k, h, and c to exhibit strictly positive growth rates, it must be that:

$$\frac{(1-\sigma)(1-\beta+\gamma)}{\epsilon} = 0 \tag{30}$$

Equation 30 gives a necessary condition for our model with endogenous labor to exhibit endogenous growth. Notice that there are basically three ways in which condition 30 can be satisfied: either σ, the inverse of the intertemporal elasticity of substitution of consumption, is exactly equal to one, or ϵ, the inverse of the intertemporal elasticity of substitution of labor is infinite, or $1 - \beta + \gamma$ is equal to zero. Since γ is a nonnegative parameter (i.e., if there is an externality in the production of human capital, it is positive), the latter case implies in its turn that $\beta = 1$ and $\gamma = 0$; we have thus a variation of Rebelo's "Ak" model. The case when ϵ is infinite is clearly identical to the Lucas [1988] model: labor supply is completely inelastic and fixed at one; the results of the previous sections then apply. In what follows we thus concentrate on the "new" case where σ is exactly equal to one.

3.2 The Reduced Model with Endogenous Labor

In this subsection we concentrate on the case where $\sigma = 1$, i.e. where the utility of consumption is logarithmic. We can do the same thing as we did in section 2.2, i.e. we define two new variables and obtain a three-dimensional system. The new variables are exactly the same as those defined in section 2.2:

$$x \equiv kh^{\frac{1-\beta+\gamma}{\beta-1}} \qquad\qquad q \equiv \frac{c}{k}$$

Taking logarithms and differentiating with respect to time in these two equations, we obtain the following system in x, u, and q:

$$\begin{aligned} \dot{x} &= Ax^{\beta}u^{1-\beta} - xq - \psi xu + B\psi x^{\frac{1-\beta+\epsilon}{\epsilon}} u^{\frac{\beta}{\epsilon}} q^{\frac{1}{\epsilon}} \\ \dot{u} &= \eta u^2 + B\psi \frac{\beta-1}{\beta} x^{\frac{1-\beta}{\epsilon}} u^{\frac{\beta+\epsilon}{\epsilon}} q^{\frac{1}{\epsilon}} - uq \\ \dot{q} &= q^2 + A(\beta-1)x^{\beta-1}u^{1-\beta}q - \rho q \end{aligned} \tag{31}$$

Unfortunately this system is much more complicated than the system in section 2.2. It is clear, however, that u^*, the steady state value of u does not depend on ϵ, and is equal to ρ/δ.[6] To see this, just notice that $Bx^{(1-\beta+\epsilon)/\epsilon}u^{\beta/\epsilon}q^{1/\epsilon} = L$; substituting this into the above system, we can obtain an expression for q^* as a function of u^* and L^* from the $\dot{u}/u$ equation. From the $\dot{q}/q$ equation we obtain an expression for x^* as a function of the same variables, and finally, substituting all this into the $\dot{x}/x$ equation, we obtain:

$$u^* = \frac{\rho}{\eta\beta + \psi(\beta-1)} = \frac{\rho}{\delta}$$

If we substitute this value for u^* in the expressions for x^*, q^*, and L^*, we obtain three equations in three unknowns, but they are not analytically solvable. We have thus to resort to numerical simulations. Our main interest is to study the sign of the real parts of the eigenvalues of the Jacobian matrix of this system as a function of the intertemporal elasticity of substitution

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[6] This is exactly the same value that we obtain for the Lucas [1988] model when $\sigma = 1$; to see it, just substitute 1 for σ in equation 16.

ϵ	x^*	u^*	q^*	L^*	ω_1	ω_2	ω_3	μ_k	μ_h
-0.01	6.289	0.8	0.2	0.973	– (c)	– (c)	+	0.013	0.009
-0.1	6.463	0.8	0.197	0.951	– (c)	– (c)	+	0.012	0.008
-0.2	7.037	0.8	0.188	0.919	– (c)	– (c)	+	0.009	0.006
-0.28	8.835	0.8	0.164	0.825	– (c)	– (c)	+	0.001	0.0008
-0.3	4.579	0.8	0.243	1.153	+	+	–	0.028	0.018
-1	5.822	0.8	0.209	1.012	+	+	–	0.016	0.011
-10	5.934	0.8	0.207	1.000	+	+	–	0.016	0.010
-500	5.943	0.8	0.207	1.000	+	+	–	0.016	0.010

Table 1: the ω_i are the eigenvalues; a (c) means that a particular eigenvalue is complex.

of labor, $1/\epsilon$. We expect that indeterminacy arises when labor supply is sufficiently elastic. On the other hand, we know that if ϵ is very large in absolute value (i.e. the intertemporal elasticity of substitution of labor is very small), the labor supply is fixed and equal to one. Then we have exactly the Lucas [1988] model studied in the previous section, which, for $\sigma = 1$, is always determinate (at least when we have an interior solution for u). We expect then that a threshold value for ϵ, call it ϵ_0, exists, such that we have indeterminacy whenever $|\epsilon| \leq |\epsilon_0|$ and determinacy if $|\epsilon| > |\epsilon_0|$. Of course, ϵ_0 is in general a function of all other parameter values.

Our simulation is conducted as follows: first we fix all the parameter of the model, except ϵ, at some plausible values; then we compute numerically the steady state of the system for different values of ϵ in the range $[-500, -0.01]$ using the procedure NLSYS in GAUSS386. Once we have the steady state values, we plug them into the Jacobian matrix, and evaluate numerically its eigenvalues (again using GAUSS386). We do all this for a set of parameters values based on the estimates provided by Lucas [1988], which are $\delta = 0.05$, $\beta = 0.25$, and $\gamma = 0.417$. We arbitrarily fix $A = 1$ and put $\rho = 0.04$.

The results are reported in Table 1: we see that ϵ_0 exists, and, for the particular parameter values that we use, it is somewhere between -0.28 and -0.3. This is consistent with the value of -0.25 used by Benhabib and Farmer [1991]; notice also that in Real Business Cycle models, an utility function which is linear in L is often used, and this, in our case, corresponds to a value of ϵ close to zero.[7]

Appendix

In this appendix we prove Propositions 1 and 2. As a preliminary step we compute the Jacobian matrix of the reduced system. We obtain:

$$\begin{aligned}
J_{11} &= \frac{\partial \dot{x}}{\partial x} = A\beta x^{\beta-1}u^{1-\beta} + \psi(1-u) - q \\
J_{12} &= \frac{\partial \dot{x}}{\partial u} = A(1-\beta)x^{\beta}u^{-\beta} + \psi x \\
J_{13} &= \frac{\partial \dot{x}}{\partial q} = -x \\
J_{21} &= \frac{\partial \dot{u}}{\partial x} = 0 \\
J_{22} &= \frac{\partial \dot{u}}{\partial u} = 2\eta u + \frac{\psi(\beta-1)}{\beta} - q \\
J_{23} &= \frac{\partial \dot{u}}{\partial q} = -u \\
J_{31} &= \frac{\partial \dot{q}}{\partial x} = A(\beta-1)\phi x^{\beta-2}u^{1-\beta}q \\
J_{32} &= \frac{\partial \dot{q}}{\partial u} = A(1-\beta)\phi x^{\beta-1}u^{-\beta}q \\
J_{33} &= \frac{\partial \dot{q}}{\partial q} = 2q + A\phi x^{\beta-1}u^{1-\beta} - \zeta
\end{aligned}$$

We are interested in the Jacobian evaluated at the equilibrium; consider

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[7] See, e.g., Hansen [1985].

first the case of an interior solution for u. Instead of substituting the steady state values of x, u, and q into the above matrix, which would result in a huge mess, we can write:

$$J_{11} = \frac{\dot{x}}{x} + A\beta x^{\beta-1}u^{1-\beta} - Ax^{\beta-1}u^{1-\beta} = \frac{\dot{x}}{x} + A(\beta-1)x^{\beta-1}u^{1-\beta}$$

On the BGP we have thus:

$$J_{11}^* \equiv \left.\frac{\partial \dot{x}}{\partial x}\right|_E = A(\beta-1)x^{*\beta-1}u^{*1-\beta}$$

If we express J_{12}^*, J_{31}^*, and J_{32}^* as functions of J_{11}^*, relate J_{22} to $\dot{u}/u$ and J_{33} to $\dot{q}/q$, we see that we can write the Jacobian matrix evaluated at the BGP as:

$$J^* = \begin{pmatrix} J_{11}^* & -\frac{x^*}{u^*}(J_{11}^* + \psi u^*) & -x^* \\ 0 & \eta u^* & -u^* \\ J_{11}^*\phi\frac{q^*}{x^*} & -J_{11}^*\phi\frac{q^*}{u^*} & q^* \end{pmatrix}$$

The eigenvalues of J^* are the solutions of its characteristic equation:

$$-\lambda^3 + \mathrm{Tr}J^*\lambda^2 - \mathrm{B}J^*\lambda + \mathrm{Det}J^* = 0 \qquad (A1)$$

where:

$$\begin{aligned} \mathrm{Tr}J^* &= J_{11}^* + \eta u^* + q^* \\ \mathrm{B}J^* &= \begin{vmatrix} J_{11}* & -\frac{x^*}{u^*}(J_{11}^* + \psi u^*) \\ 0 & \eta u^* \end{vmatrix} + \begin{vmatrix} \eta u^* & -u^* \\ -J_{11}^*\phi\frac{q^*}{u^*} & q^* \end{vmatrix} + \begin{vmatrix} J_{11}^* & -x^* \\ J_{11}^*\phi\frac{q^*}{x^*} & q^* \end{vmatrix} \\ &= J_{11}^*q^* + \eta u^*q^* + \eta J_{11}^*u^* \\ \mathrm{Det}J^* &= J_{11}^*u^*q^*[\phi\psi + \eta(1+\phi)] \end{aligned}$$

At this point we want to determine the sign of the real parts of the roots of equation A1; to do this, we make use of the following Theorem:

Theorem 1 *The number of roots of the polynomial in equation A1 with positive real parts is equal to the number of variations of sign in the scheme:*

$$-1 \qquad \mathrm{Tr}J^* \qquad -\mathrm{B}J^* + \frac{\mathrm{Tr}J^*}{\mathrm{Det}J^*} \qquad \mathrm{Det}J^* \qquad (A2)$$

Proof: This is an application to our special case of a more general Theorem due to Routh (see, e.g., Gantmacher [1960]).

For example, equation A1 has three roots with positive real parts if $\mathrm{Tr}J^* > 0$, $-\mathrm{B}J^* + \mathrm{Tr}J^*/\mathrm{Det}J^* < 0$, and $\mathrm{Det}J^* > 0$.

To apply Theorem 1, we must be able to sign the three key quantities $\mathrm{Tr}J^*$, $\mathrm{B}J^*$, and $\mathrm{Det}J^*$. Let's start with the determinant; we can write:

$$\mathrm{Det}J^* = J^*_{11} q^* u^* \cdot Q$$

where:

$$Q \equiv \phi\psi + \eta(1+\phi) = \frac{\delta[\gamma - \sigma(1-\beta+\gamma)]}{\sigma(\beta-1)}$$

$\mathrm{Det}J^* = 0$ when Q=0, since $J^*_{11} q^* u^*$ is always negative for any acceptable value of u^*. From this it is straightforward to derive the following result:

Result 1 *Let $\theta \in \Theta_0$; then* $\mathrm{Det}J^*$ *is positive when $\sigma < \gamma/(1-\beta+\gamma)$, while* $\mathrm{Det}J^*$ *is negative when $\sigma > \gamma/(1-\beta+\gamma)$.*

Consider next the trace. We first express q^* and J^*_{11} as a function of the parameters in θ, i.e. we substitute equation 16 for u^* into equation 13 and 14. We find that:

$$
\begin{aligned}
q^* &= \frac{(1-\beta+\gamma)[\beta(\delta-\sigma)-\delta\sigma]+\gamma\rho}{\beta[\gamma-\sigma(1-\beta+\gamma)]} \\
J_{11}^* &= \frac{(\beta-1)[\gamma\rho-\delta\sigma(1-\beta+\gamma)]}{\beta[\gamma-\sigma(1-\beta+\gamma)]}
\end{aligned}
$$

We now see that:

$$
J_{11}^* + q^* = \frac{\delta(\sigma-1)(1-\beta+\gamma)+\rho(1-\beta)}{\sigma(1-\beta+\gamma)-\gamma} = \delta u^* \qquad (A3)
$$

and then the trace can be written as:

$$
\mathrm{Tr}J^* = (\delta+\eta)u^* = \frac{\delta(2\beta-\gamma)}{\beta}u^*
$$

Since u^* is always positive, we have the next result:

Result 2 *Let $\theta \in \Theta_0$; then* $\mathrm{Tr}J^*$ *is positive when $\gamma < 2\beta$, and* $\mathrm{Tr}J^*$ *is negative when $\gamma > 2\beta$.*

Finally consider $-\mathrm{B}J^*$; we can write:

$$
\mathrm{B}J^* = J_{11}^*q^* + \eta u^*(J_{11}^*+q^*) = J_{11}^*q^* + \delta\eta u^{*2} = J_{11}^*q^* + \frac{\delta^2(\beta-\gamma)}{\beta}u^{*2}
$$

Since J_{11}^* is always negative if $\theta \in \Theta_0$, we can state the following result:

Result 3 *Let $\theta \in \Theta_0$; then* $\mathrm{B}J^*$ *is negative if $\gamma > \beta$.*

With these Results in hand, we are ready to prove our two propositions.

Proof of Proposition 1: By assumption we have $\theta \in \Theta_0$ with $\rho < \delta$. Result 1 tells us that in this case $\mathrm{Det}J^*$ is negative; this leaves two possibilities open for the eigenvalues of J^*: they can be either three with negative real parts (this case correspond to a dynamically stable steady state), or one with negative real part and two with positive real parts (this case corresponds to a saddle like steady state with an unstable manifold of dimension two, i.e. to a determinate equilibrium). But by virtue of Theorem 1 and of the former Results, we can exclude the first case.

According to Theorem 1, in order for J^* to have three eigenvalues with negative real parts, there must be no changes of sign in the scheme A2; this implies that $\mathrm{Tr}J^*$ and $\mathrm{Det}J^*$ must both be negative. In the case we are considering, i.e. $\rho < \delta$, the determinant is always negative. The trace, on the other hand, is negative if $\gamma > 2\beta$. This implies, by virtue of Result 3, that $\mathrm{B}J^*$ is also negative, and so $-\mathrm{B}J^* + \mathrm{Tr}J^*/\mathrm{Det}J^*$ is always positive. Hence there is always one change of sign in the scheme A2, which completes the proof.

Proof of Proposition 2: By assumption we have $\theta \in \Theta_0$ with $\rho < \delta < -\psi$. Here Result 1 says that $\mathrm{Det}J^*$ is positive; this implies that there can be either three eigenvalues with positive real parts or one eigenvalue with positive real part and two eigenvalues with negative real parts. In order to have three eigenvalues with positive real parts, there must be three changes of sign in the scheme A2; hence $\mathrm{Tr}J^*$ must be positive, which is true, by Result 2, if $\gamma < 2\beta$. On the other hand, Result 3 says that $\mathrm{B}J^*$ is negative if $\gamma > \beta$; it then follows that, if $\beta < \gamma < 2\beta$, there can only be two eigenvalues with negative real parts and one eigenvalue with positive real part, because $-\mathrm{B}J^* + \mathrm{Tr}J^*/\mathrm{Det}J^* > 0$. This is clearly a case of indeterminacy: there are infinitely many equilibria that converge to the single steady state, one for each possible initial conditions.

If, on the other hand, $\gamma < \beta$, it is possible, for θ in some, presumably narrow, subset of Θ_0, that $-\mathrm{B}J^* + \mathrm{Tr}J^*/\mathrm{Det}J^*$ becomes negative, and then three eigenvalues with positive real parts can emerge. When this happens, a Hopf bifurcation has occurred, since the real parts of a pair of eigenvalues cross the imaginary axes in the complex plane. This means that a periodic cycle has emerged, the stability properties of which remain to be investigated.

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