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ON THE MEASUREMENT OF ECONOMIC POVERTY *

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Abstract

We consider four compelling axioms for poverty indices, and show that a familiar class of poverty measures is characterized by these axioms. An unambiguous (partial) poverty ordering is then defined as the intersection of the (complete) orderings induced by the members of the characterized class; given a poverty line, one income distribution is "*poorer than*" another if, and only if, the former is ranked higher than the latter by *all* poverty indices which satisfy our axioms. Our class of poverty measures is defined in terms of the income distribution, poverty line and an additional parameter, and the paper explores the possibility of making poverty comparisons without the knowledge of this parameter value. We present several easily verifible sufficient conditions under which such comparisons can be made, making our poverty ordering of potential use in empirical applications.

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I. INTRODUCTION

In the twenty years that followed the seminal contribution of Sen (1976), the literature on the theory of axiomatic measurement of economic poverty has considerably expanded; there appears now a consensus on the basic set of axioms any poverty index should satisfy. This development parallels that of the theory of income inequality measurement, a theory in which the axiomatic approach proved quite useful. However, while one is able to make unambiguous inequality comparisons between certain pairs of income distributions by using the celebrated Lorenz dominance criterion, a poverty ordering which would let one unambiguously rank two income distributions on the basis of poverty, given a *fixed* poverty line, is yet to be discovered.[1]

In a recent contribution, Foster and Shorrocks (1991) introduced the property of *subgroup consistency*, a property which appears to be unquestionable in many intuitive aspects, and yet, when combined with a basic set of axioms, yields rather refined classes of poverty indices. We feel that subgroup consistency is perhaps just what the theory of axiomatic measurement of poverty needed. In this paper, we shall first show that this axiom, along with a number of widely used postulates, yields a one-parameter class of poverty indices, and second, illustrate that it plays a major role in constructing a poverty ordering which would let one make unambiguous poverty comparisons once a poverty line is determined. Here is a summary of the present work.

Let us refer to the poverty indices which satisfy the basic axioms proposed in the literature as *well-behaved.* Put precisely, a well-behaved poverty index is a normalized index which satisfies *anonymity, population replication invariance, strict monotonicity, focus* and *restricted continuity,* and which declares zero level of poverty when everyone in the population is rich (see Section 2). Having confined the class at hand to well-behaved poverty indices, we introduce, in Section 2, the fundamental *subgroup consistency* axiom. This postulate is very intuitive in simply asserting that the overall poverty level should fall if a subgroup of the population becomes poorer while the poverty of the rest of the population remains the same. Our next postulate is again a rather standard one: the *transfer axiom.* This axiom is a just requirement which warrants that a transfer from a poor person to a richer person should not decrease the value of the poverty index. Our first result (Theorem 1) provides a characterization of the class of all well-behaved and subgroup consistent poverty indices which satisfy the subgroup consistency and transfer axioms, and thereby identifies the extent of technical structure imposed on any well-behaved poverty index by these two apparently mild restrictions.

The two common structural properties for poverty indices that can be introduced at this stage are homogeneity of degree *zero* and translation invariance. The first

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[1] There are, however, candidates; see Jenkins and Lambert (1993) and Shorrocks (1994).

property paves the way towards *relative* poverty indices, and the latter leads to *absolute* poverty measures. Zheng (1994) shows that one has to make a choice between these two properties, for no distribution-sensitive poverty index can satisfy both the scale and translation invariance properties. (We shall see, however, that these two axioms are fully compatible with respect to the ordinal measurement theory developed below.[2]) Of course, Kolm (1976) has taught us that a similar dilemma is evident in the context of the measurement of income inequality. One way that proved useful in escaping this difficulty in the income inequality framework is to consider *compromise* indices; that is, to combine the translation invariance property with homogeneity of degree *one*. (The basic idea parallels the *centrist* approach also introduced by Kolm (1976).) Surprisingly enough, this approach is not studied in the context of poverty measurement. In the present paper, we do follow this route, and concentrate on well-behaved, compromise and subgroup consistent indices which satisfy the transfer axiom. (see Section 3.a.[3])

One who believes in the axioms specified above will be justified in concluding that, given a poverty line, an income distribution $\mathbf{x}$ is poorer than $\mathbf{y}$ if, and only if, *all* well-behaved, compromise and subgroup consistent poverty indices which satisfy the transfer axiom ranks $\mathbf{x}$ poorer than $\mathbf{y}$. (Of course, this ordering is parametric over the chosen poverty line.) It is this ordering which we refer to as an unambiguous poverty ordering (see Section 4.a).

One of our main results (Theorem 2) shows that a well-behaved, compromise and subgroup consistent poverty index which satisfies the transfer axiom is necessarily of the form

$$Q_\alpha(\mathbf{x}; z) := \left[\frac{1}{n}\sum_{i=1}^{n}(\max\{z - x_i, 0\})^\alpha\right]^{1/\alpha} \quad \text{for all } \mathbf{x} \in \mathbf{R}^n_+,\ z > 0,$$

for some $\alpha \geq 1$, where $z > 0$ is the chosen poverty line. The emerging class is therefore closely linked to the well-known P_α-class of Foster, Greer and Thorbecke (1984).[4] In other words, a well-behaved compromise index which satisfies the axioms of transfer and subgroup consistency must be a generalized α-mean of the levels of individual relative deprivation (see Section 3.b).

This result, in turn, gives a precise description of the unambiguous (fixed poverty line) poverty ordering introduced above. Given $z > 0$, $\mathbf{x} \in \mathbf{R}^n_+$ is *poorer than* $\mathbf{y} \in \mathbf{R}^n_+$

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[2] This observation is also noted in Foster and Shorrocks (1991), and is, in fact, exploited quite nicely to get a characterization of the celebrated P_α-class of poverty indices.

[3] A discussion of pros and cons of following such a compromise approach are briefly discussed in this section, but we stress once again that we give up the relative approach only in a cardinal sense. Indeed, the unambiguous poverty ordering that we are after will be supported by many relative poverty indices as well.

[4] $P_\alpha(\mathbf{x}; z) = \left(\frac{Q_\alpha(\mathbf{x};z)}{z}\right)^\alpha$ for all $\mathbf{x} \in \mathbf{R}^n_+$, $z > 0$ and $\alpha \geq 1$.

if, and only if,

$$\sum_{i=1}^{n} (\max\{z - x_i, 0\})^\alpha \geq \sum_{i=1}^{n} (\max\{z - y_i, 0\})^\alpha \quad \text{for all } \alpha \geq 1.$$

We note that although this partial ordering is obtained via a characterization of a compromise class of poverty indices, it also carries the support of many relative poverty indices. In particular, it is *characterized* by all members of the P_α-class which satisfy the transfer axiom; if one dislikes the compromisation axiom and holds the view that the P_α indices, $\alpha \geq 1$, are more useful than Q_α indices, $\alpha \geq 1$, then (s)he should still believe in our poverty ordering.[5]

The present axiomatization suggests that the above poverty ordering is a very useful device in making ordinal poverty comparisons with a fixed poverty line. The problem with this ordering is, of course, its continuous dependence on the parameter α, which, in turn, limits the empirical applicability of our approach. One is thus led to aim at determining sufficient conditions for our partial ordering to apply. We take on such an inquiry in Section 4, and obtain two useful sub-orderings which are characterized without any reference to a parameter. These findings (summarized in Theorems 3 and 4) let one readily check whether two income distributions can be ranked by our poverty ordering, and hence, are likely to be useful in practice. Nevertheless, the important problem of obtaining a parameter-free characterization of this partial poverty ordering stands at the moment open. The paper concludes with a section containing the proofs of the main results.

II. AXIOMS FOR POVERTY INDICES

We shall take $\bigcup_{n=1}^{\infty} \mathbf{R}_+^n$ and $\mathbf{R}_{++}$ as the sets of all *admissible income distributions* and *poverty lines*, respectively. Consequently, we shall define a poverty index on the following domain:

$$\Omega := \left(\bigcup_{n=1}^{\infty} \mathbf{R}_+^n \right) \times \mathbf{R}_{++}.$$

A pair $(\mathbf{x}; z) \in \Omega$ is thought of as a social situation where the society is composed of $n(\mathbf{x})$ members, the income distribution is given by $\mathbf{x}$, and the poverty line is determined to be z. Given a situation $(\mathbf{x}; z) \in \Omega$, $\{i \in \{1, ..., n(\mathbf{x})\} : x_i \leq z\}$ is interpreted as the set of poor people, and the cardinality of this set is denoted by $q(\mathbf{x}; z)$.[6]

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[5] This observation lets one relate our ordering to another poverty ordering introduced in two important papers by Foster and Shorrocks (1988a, 1988b). See Section 3 for an elaboration of this point.

[6] This is the *strong* definition of the poor in the sense of Donaldson and Weymark (1986). We note, however, that the results of the present paper can be easily modified to accomodate the *weak* definition of the poor as well.

A *poverty index* is any function $P : \Omega \to \mathbf{R}_+$ such that $P(\mathbf{x}; z) = 0$ whenever $q(\mathbf{x}; z) = 0$. We interpret $P(\mathbf{x}; z)$ as the poverty level associated with the social situation $(\mathbf{x}; z) \in \Omega$. If no member of the population is poor, a poverty index, by definition, declares *zero* level of poverty.

To simplify our subsequent discussion, we shall concentrate on *normalized poverty indices*; that is, on poverty indices with the property $P(0; 1) = 1$. The interpretation of this normalization postulate is straightforward.

The most common examples of normalized poverty indices are members of the following class:

$$\mathcal{P}_0 := \bigcup_{\alpha \geq 0} \left\{ P_\alpha : P_\alpha(\mathbf{x}; z) := \frac{1}{n(\mathbf{x})} \sum_{i=1}^{q(\mathbf{x};z)} \left(1 - \frac{x_{\sigma(i)}}{z}\right)^\alpha \quad \forall (\mathbf{x}; z) \in \Omega \right\},$$

where, for any $\mathbf{x} \in \mathbf{R}^{n(\mathbf{x})}$, $\mathbf{x}_\sigma$ denotes the permuted form of $\mathbf{x}$ such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n(\mathbf{x}))}$. $\mathcal{P}_0$ is introduced by Foster, Greer and Thorbecke (1984) and is usually referred to as the *P_α-class* in the literature. This family contains a number of indices which are widely used in empirical practice: the *headcount ratio* (P_0), the *per-capita income gap* (P_1) and the *Foster-Greer-Thorbecke* poverty index (P_2). Another well-known normalized poverty index is, of course, that of Sen (1976):

$$S(\mathbf{x}; z) := P_0(\mathbf{x}; z)P_1(\mathbf{x}; z) + P_0(\mathbf{x}; z)(1 - P_1(\mathbf{x}; z))G(\mathbf{x}; z) \quad \forall (\mathbf{x}; z) \in \Omega$$

where $G(\mathbf{x}; z)$ is the Gini value of the income distribution of the poor.[7]

Definition 1. A *well-behaved poverty index* is a normalized poverty index $P : \Omega \to \mathbf{R}_+$ which satisfies the following properties: For all $(\mathbf{x}; z) \in \Omega$,

(a) (*Anonymity*) $P(\mathbf{x}; z) = P(\mathbf{x}_\sigma; z)$;

(b) (*Population Replication Invariance*) $P(\mathbf{x}; z) = P(\langle \mathbf{x} \rangle_k ; z)$ for any $k \in \mathbf{N}$;[8]

(c) (*Strict Monotonicity*) $P(\mathbf{x}; z) < P(\mathbf{y}; z)$ whenever there exists j such that $y_{\sigma(j)} < \min\{z, x_{\sigma(j)}\}$ and $x_{\sigma(i)} = y_{\sigma(i)}$ for all $i \neq j$;

(d) (*Focus*) $P(\mathbf{x}; z) \neq P(\mathbf{y}; z)$ only if $x_{\sigma(i)} \neq y_{\sigma(i)}$ and $n(\mathbf{x}) = n(\mathbf{y})$ imply $\min\{x_{\sigma(i)}, y_{\sigma(i)}\} \leq z$;

(e) (*Restricted Continuity*) $P(\mathbf{x}; z)$ is a continuous function of x_i on $[0, z]$.

We denote the *set of all well-behaved poverty indices* by $\mathcal{P}$.[9]

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[7] We refer the reader to Foster (1984) and Hagenaars (1987) for careful examinations of other types of aggregative poverty indices.

[8] For any $k \in \mathbf{N}$, the *k-replication* operator is defined on $\bigcup_{n=1}^\infty \mathbf{R}_+^n$ as $\langle . \rangle_k : \mathbf{x} \mapsto (\mathbf{x}, ..., \mathbf{x})$ such that $\langle \mathbf{x} \rangle_k \in \mathbf{R}_+^{kn(\mathbf{x})}$.

[9] Examples of well-behaved poverty indices include all the members of the $\mathcal{P}_0$ class other than the headcount ratio, the index of Sen (1976), the index of Shorrocks (1995) and all the members

By a well-behaved poverty index, therefore, we mean a poverty measure which satisfies the most commonly used axioms in the literature. *Anonymity* is a natural requirement which warrants the impartial treatment of the constituents of the population; *population replication invariance* guarantees that the poverty is measured in per capita terms (and it is, of course, analogous to *Dalton's population principle* familiar from the theory of inequality measurement); *strict monotonicity* ensures the sensitivity of the poverty measure to an increase in the level of poverty of a poor individual; *focus* simply says that the value of a poverty index is independent of the incomes of the rich; and finally, *restricted continuity* requires a poverty index not to record a large change in poverty due to an *infinitesimal* alteration of the level of income of a single poor individual.[10]

In passing, we would like to stress that restricted continuity is considerably less demanding than what could be thought of as its standard counterpart; the continuity of $P(.,z)$ on $\bigcup_{n=1}^{\infty} \mathbf{R}_{+}^{n}$ given any $z > 0$. (We shall call this property the *continuity property* and refer to poverty indices which satisfy it as *continuous poverty indices.*) Indeed, as noted by Donaldson and Weymark (1986) and Foster and Shorrocks (1991) *inter alia*, there is a case that could be made against the latter (more general) continuity requirement: since the poverty line is explicitly used as a demarcation criterion in distinguishing between who is poor and who is not, one might argue that a poverty index which is sensitive to the *number* of the poor (and hence, which fails to be continuous) should not necessarily be deemed irregular. (The Sen index is a case in point.) For this reason, restricted continuity appears to be a more readily acceptable continuity property, for it does allow for discontinuity of $P(\mathbf{x}, z)$ in x_i, $i = 1, ..., n(\mathbf{x})$, at the poverty line.[11]

Since the pioneering contribution of Foster et al. (1984), the axiom of *decomposability* entered the basic list of desirable axioms for poverty indices (cf. Hagenaars (1991)). This property is defined as follows:

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of the two families of poverty indices proposed by Clark, Hemming and Ulph (1981). On the other hand, examples of poverty indices which are not well-behaved in the sense of Definition 1 are the members of the class introduced by Kakwani (1980) along with the index of Thon (1979) (because they do not satisfy anonymity and replication invariance); all the members of the class of absolute poverty indices derived by Blackorby and Donaldson (1980a) (because they do not satisfy replication invariance, (cf. Bossert (1990)); the headcount ratio and the index introduced by Takayama (1979) (because they fail to satisfy strict monotonicity and population replication invariance).

[10] Considering the difficulties in determining the set of poor people one is faced in empirical practice, it seems to us that such a continuity requirement is a rather unexceptionable regularity condition.

[11] Having said this, however, we should stress that we do not view the global continuity of a poverty index as an unacceptable property. We shall, in fact, obtain this property as a consequence of more primitive properties of poverty measures (see Theorem 1).

Definition 2. A poverty index P is said to be *decomposable* if, for any $m \geq 2$, $z > 0$ and any $\mathbf{x}^1, ..., \mathbf{x}^m \in \bigcup_{n=1}^{\infty} \mathbf{R}_+^n$,

$$P((\mathbf{x}^1, ..., \mathbf{x}^m); z) = \sum_{h=1}^{m} \left(\frac{n(\mathbf{x}^h)}{n(\mathbf{x}^1) + \cdots + n(\mathbf{x}^m)} \right) P(\mathbf{x}^h; z).$$

Thus, a decomposable index computes the overall poverty level as a weighted average of the subgroup poverty levels where weights are chosen to be the population shares of the subgroups. As noted by many authors, the decomposability property is very useful in calculating how much each subpopulation contributes to aggregate poverty.

It is evident that a poverty index P is decomposable if, and only if,

$$P(\mathbf{x}; z) = \frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} P(x_i; z) \quad \forall (\mathbf{x}; z) \in \Omega. \tag{1}$$

This observation highlights how demanding the decomposability property is. Indeed, postulating the *additivity* of a poverty index in individual poverty levels seems rather *ad hoc*. There is, however, a weakening of the decomposability axiom which appears to be very compelling.

Axiom SC. (*Subgroup Consistency*) If, for any $(\mathbf{x}; z), (\mathbf{y}; z), (\mathbf{x}'; z), (\mathbf{y}'; z) \in \Omega$ for which $n(\mathbf{x}) = n(\mathbf{x}')$ and $n(\mathbf{y}) = n(\mathbf{y}')$,

$$P(\mathbf{x}; z) > P(\mathbf{x}'; z) \quad \text{and} \quad P(\mathbf{y}; z) = P(\mathbf{y}'; z),$$

holds, then

$$P((\mathbf{x}, \mathbf{y}); z) > P((\mathbf{x}', \mathbf{y}'); z).$$

Axiom SC is a slightly weaker version of the well-known property of *subgroup monotonicity* of poverty indices (cf. Foster (1984), Foster et al. (1984) and Hagenaars (1991)). It simply asserts that the overall poverty level should rise if a subgroup of the population becomes poorer while the poverty of the rest of the population remains the same.[12] The present form of the axiom is first articulated in an influential article by Foster and Shorrocks (1991) where a detailed discussion of the appeal and significance of this property is provided. It is indeed striking how useful subgroup consistency

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[12] Among the subgroup inconsistent poverty indices used in empirical work are the indices of Sen (1976), Takayama (1979), Thon (1979), Kakwani (1980), Chakravarty (1983b), and the first class of Clark et al. (1981).

property turns out to be. It is both intuitive and of practical importance,[13] and in addition, it imposes a rather strong structure on the class $\mathcal{P}$.[14]

Another property of poverty indices that has been studied extensively in the literature is:

Axiom T. (*Transfer Axiom*) For any $\mathbf{x} \in \mathbf{R}_+^{n(\mathbf{x})}$ and $i, j \in \{1, ..., n(\mathbf{x})\}$, $\min\{z, x_j\} > x_i$ implies that

$$P(\mathbf{x}; z) \leq P((x_1, ..., x_i - \delta, ..., x_j + \delta, ..., x_{n(\mathbf{x})}); z) \quad \forall \delta \in (0, x_i].$$

The transfer axiom is due Sen (1976) and simply says that a transfer from a poor person to a richer person should increase the value of the poverty index. Although this axiom appears to us to be quite compelling, we should note that its appeal is questioned by, say, Kundu and Smith (1983).[15] Indeed, a transfer from a poor person to another poor-but-richer person might cause the recipient to pass the poverty line, and therefore, it might decrease the number of poor people. Does the society really become poorer as a result of such a transfer? While Thon (1979) gives a positive answer to this question without reservation, Sen (1981) points out carefully that the answer depends on the aspect of poverty one is interested in measuring. Indeed, it is now well understood that finding a poverty index which incorporates all aspects of the notion of "economic poverty" is a rather hopeless task (cf. Sen (1979, 1981), Kundu and Smith (1983), and Foster (1984)). Since there is a clear sense in which the transfer axiom corresponds to at least one aspect of the concept of "poverty", however, we see merit in exploring the consequences of this postulate in our framework. (The reader is referred to Foster (1984) for an illuminating account of the debate surrounding the desirability of the transfer axiom.)

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[13] The practical significance of subgroup consistency relates to the effectivity of decentralized welfare programs. As Foster and Shorrocks (1991), p. 687, eloquently put it, "... a decentralized strategy typically involves a collection of activities targeted at specific subgroups or regions of the country. If the poverty indicator is not subgroup consistent, we may be faced with a situation in which each local effort achieves its objective of reducing poverty within its targeted group, and yet the level of poverty in the population as a whole increases. Subgroup consistency may therefore be viewed as an essential counterpart to a coherent poverty program."

[14] One can easily show that Axiom SC induces an ordering on $\mathbf{R}_+^n$ which is strictly separable in the sense of Gorman (1968). Consequently, if P is a continuous poverty index, one can use Theorem 1 of Gorman (1968), and the population replication invariance property, to conclude that $P(\mathbf{x}; z) = F(\frac{1}{n}\sum_{i=1}^{n} \phi(x_i; z), z)$ for all $(\mathbf{x}; z) \in \Omega$, where, for any $z > 0$, $F(., z)$ and $\phi(., z)$ are continuous and strictly increasing (see Foster and Shorrocks (1991), pp. 693-696.) Consequently, one must note that Axiom SC introduces a nice additive separability result when combined with the continuity property.

[15] Kundu and Smith (1983) shows that a poverty index, the value of which decreases (increases) as a rich (poor) individual is added to the population, cannot satisfy Axiom T.

We conclude this section by stating a preliminary result which characterizes the class of all well-behaved and subgroup consistent poverty indices which satisfy the transfer axiom.

Theorem 1. *A well-behaved poverty index* $P \in \mathcal{P}$ *satisfies Axioms SC and T if, and only if, there exist functions* $\phi : \mathbf{R}_+ \times \mathbf{R}_{++} \to \mathbf{R}_+$ *and* $F : \bigcup_{z>0} (\phi(\mathbf{R}_+, z) \times \{z\}) \to \mathbf{R}_+$ *such that*

$$P(\mathbf{x}; z) = F\left(\frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} \phi(x_i, z),\ z \right) \quad \forall (\mathbf{x}; z) \in \Omega,$$

where $\phi(t, z) = 0$ *for all* $t \in [z, \infty)$; $\phi(., z)$ *is continuous and decreasing on* $\mathbf{R}_+$ *for any given* $z > 0$; *and* $F(., z)$ *is continuous and strictly increasing on* $\phi(\mathbf{R}_+, z)$.

This characterization result is closely linked to Proposition 1 of Foster and Shorrocks (1991). Indeed, Theorem 1 becomes identical to the Foster-Shorrocks characterization if one replaces the transfer axiom with the continuity property. Since it appears that the transfer axiom is a more satisfactory postulate for a poverty index than the continuity property, we feel that Theorem 1 complements the Foster-Shorrocks characterization in a useful way.

III. A CLASS OF COMPROMISE POVERTY INDICES

III.a Compromise poverty indices

When studying poverty indices with a variable poverty line, it is useful to specify the behavior of the indices with regard to *simultaneous* changes in the individual incomes and the poverty line. To this effect, one usually postulates an invariance property for the index with respect to certain types of such simultaneous modifications. The following definition is standard (cf. Blackorby and Donaldson (1980a), Foster and Shorrocks (1991) and Zheng (1994)).

Definition 3. A poverty index P is said to be *relative* if

$$P(\mathbf{x}; z) = P(\lambda(\mathbf{x}; z)) \quad \forall (\mathbf{x}; z) \in \Omega,\ \lambda > 0;$$

and it is said to be *absolute* if

$$P(\mathbf{x}; z) = P((\mathbf{x}; z) + \theta \mathbf{1}_{n(\mathbf{x})+1}) \quad \forall (\mathbf{x}; z) \in \Omega,$$

for all $\theta \in \mathbf{R}$ such that $x_{\sigma(1)} + \theta \geq 0$, where $\mathbf{1}_m = (1, ..., 1) \in \mathbf{R}^m$, $m \geq 1$.

The first question to ask is if a well-defined poverty index can be both relative and absolute. This question is answered in the negative by Zheng (1994) where it is shown that a $P \in \mathcal{P}$ which is both relative and absolute is necessarily of the form

$$P(\mathbf{x}; z) = f(q(\mathbf{x}; z), n(\mathbf{x})) \quad \forall(\mathbf{x}; z) \in \Omega,$$

for some $f : \mathbf{Q} \times \mathbf{N} \rightarrow \mathbf{R}$. In other words, a distribution-sensitive poverty index cannot be both relative and absolute.

An analogue of Definition 3 is formulated in the body of income inequality measurement literature as well.[16] In fact, the counterpart of Zheng's observation is more acute in this context: a continuous inequality index which is both relative and absolute must be a constant function (cf. Eichhorn and Gehrig (1982)). Nevertheless, an escape from this impossibility result can be devised by requiring an absolute inequality index to satisfy linear homogeneity (rather than homogeneity of degree zero). This route is indeed well pursued. Among others, the important contributions of Kolm (1976), Blackorby and Donaldson (1980b) and Ebert (1984, 1988) make it clear that many useful absolute inequality indices do satisfy the property of linear homogeneity. Such indices are typically referred to as compromise inequality indices.

By analogy with the notion of a compromise inequality index, we define a compromise poverty index as any poverty index which satisfies the following axiom:

Axiom C. (*Compromisation*) For all $(\mathbf{x}; z) \in \Omega$ and for all $\theta \in \mathbf{R}$ such that $x_{\sigma(1)} + \theta \geq 0$,

$$P(\mathbf{x}; z) = P((\mathbf{x}; z) + \theta \mathbf{1}_{n(\mathbf{x})+1}),$$

and

$$\lambda P(\mathbf{x}; z) = P(\lambda(\mathbf{x}; z)) \quad \forall(\mathbf{x}; z) \in \Omega,\ \lambda > 0.$$

Clearly, the property of compromisation is not a property specific to poverty measurement, but rather it is a regularity requirement for indices in general. In the present context, it envisages a consistent type of behavior for a poverty index with respect to certain types of simultaneous changes in the individual incomes and the poverty line. Compromisation demands, on one hand, a translation invariance requirement which is a straightforward reflection of Kolm's leftist inequality criterion, and on the other, a consistency property warranting that the response of a poverty index to an equiproportional change in all incomes and the poverty line is always the same, namely, proportional.[17] So, the main use of such a property is admittedly to

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[16] Introduction of these concepts to the related literature is due Kolm (1976). See also Blackorby and Donaldson (1978, 1980b), and Eichhorn and Gehrig (1982).

[17] One should, however, be careful in interpreting this property. Since the value of a poverty index is explicitly a function of the poverty line, it does not make sense to compare the poverty levels of

provide an analytic structure for a poverty index, rather than highlighting a particular aspect of the notion of "poverty".[18]

Here is a quick illustration of the use of the compromisation axiom. Suppose P is a compromise poverty index. By Axiom C, for any $t \geq 0$ and $z > 0$,

$$\begin{aligned} P(t;z) &= \begin{cases} P(0;z-t), & t \leq z \\ 0, & t > z \end{cases} = \begin{cases} P((z-t)(0;1)), & t \leq z \\ 0, & t > z \end{cases} \\ &= P(0;1)\begin{cases} z-t, & t \leq z \\ 0, & t > z \end{cases}. \end{aligned} \tag{2}$$

This observation completely characterizes the class of compromise poverty indices which are decomposable. It turns out that this class is a singleton; if one wishes to use a normalized compromise poverty index which is decomposable, then (s)he may use only one index. Indeed, by virtue of (1) and (2), we have the following elementary result:

Proposition 1. *A normalized poverty index, P, is decomposable and compromise if, and only if,*

$$P(\mathbf{x};z) = \frac{1}{n(\mathbf{x})} \sum_{i=1}^{q(\mathbf{x};z)} \left(z - x_{\sigma(i)}\right) \qquad \forall (\mathbf{x};z) \in \Omega.$$

III.b The Characterization Theorem

As noted in the previous subsection, confining attention to compromise indices turned out to be quite fruitful in the measurement of income inequality. It is thus surprising that the consequences of a similar treatment in the realm of poverty measurement has not yet been fully developed. We have taken on precisely this task in this paper, and found that the compromisation property refines the class of poverty measures obtained in Theorem 1 to a one-parameter class of poverty indices:

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two income distributions with different poverty lines. Therefore, the equation $\lambda P(\mathbf{x};z) = P(\lambda(\mathbf{x};z))$ certainly does not mean that the social situation $\lambda(\mathbf{x};z)$ is λ times poorer than the social situation $(\mathbf{x};z)$. Such a conclusion would, of course, be unacceptable.

[18] For the reader who is uneasy with the choice of compromisation in favor of absolute rather than relative measures, we should point out that our ultimate aim is to derive a partial poverty ordering, and once we introduce our ordering it will be immediately clear that the type of compromisation is inconsequential, for our ordering could also be *characterized* by a canonical class of relative poverty indices. See, in particular, the remark that immediately follows Corollary 1.

Theorem 2. *A well-behaved poverty index $P \in \mathcal{P}$ satisfies Axioms SC, T and C if, and only if, there exists $\alpha \geq 1$ such that*

$$P(\mathbf{x}; z) = \left[\frac{1}{n(\mathbf{x})} \sum_{i=1}^{q(\mathbf{x};z)} \left(z - x_{\sigma(i)} \right)^{\alpha} \right]^{1/\alpha} \qquad \forall (\mathbf{x}; z) \in \Omega.$$

By virtue of this theorem, the class

$$\mathcal{Q}_1 := \bigcup_{\alpha \geq 1} \left\{ Q_\alpha : Q_\alpha(\mathbf{x}; z) := \left[\frac{1}{n(\mathbf{x})} \sum_{i=1}^{q(\mathbf{x};z)} \left(z - x_{\sigma(i)} \right)^{\alpha} \right]^{1/\alpha} \quad \forall (\mathbf{x}; z) \in \Omega \right\}$$

is determined to be precisely the family of all well-behaved compromise poverty indices which are subgroup consistent and satisfy the transfer axiom. Naturally enough, we propose the name Q_α*-class* for this family of poverty indices.

Referring to Runciman (1966), Foster et al. (1984), p. 762, states that "where the "desired situation" is to receive enough income to be able to "meet the accepted conventions of minimum needs" (Sen (1979), p. 29) and the "existing situation" is given by the poor household's income, the magnitude of relative deprivation is precisely the income shortfall of that household." Given this perspective, Theorem 2 clarifies that a well-behaved compromise poverty index which satisfies Axioms SC and T must be an *α-mean* of the *individual magnitudes of relative deprivation, $\alpha \geq 1$.* In particular, the only decomposable member of $\mathcal{Q}_1$ is simply the average of the individual magnitudes of relative deprivation. (Recall Proposition 1.)

We conclude this section by demonstrating that the axioms used in Theorem 2 constitute an independent set of postulates. In other words, the stated characterization of the Q_α-class is tight.

Proposition 2. *The well-behavedness, subgroup consistency, transfer and compromisation axioms are mutually independent.*

Proof: The poverty index

$$P^1(\mathbf{x}; z) = \sqrt{\sum_{i=1}^{q(\mathbf{x};z)} (z - x_{\sigma(i)})^2} \quad \forall (\mathbf{x}; z) \in \Omega$$

is an ill-behaved but a compromise and subgroup consistent index which satisfies axiom T. On the other hand,

$$P^2(\mathbf{x};z) = \frac{P^1(\mathbf{x}, z)}{n(\mathbf{x}) z} \quad \forall (\mathbf{x}; z) \in \Omega$$

defines a well-behaved non-compromise poverty index which satisfies Axioms SC and T, and

$$P^3(\mathbf{x}; z) = \begin{cases} \frac{P^1(\mathbf{x}; z)}{q(\mathbf{x}; z)}, & \text{if } q(\mathbf{x}; z) > 0 \\ 0, & \text{if } q(\mathbf{x}; z) = 0 \end{cases} \qquad \forall (\mathbf{x}; z) \in \Omega$$

yields a well-behaved compromise poverty index which satisfies Axiom T but not Axiom SC.[19] Finally,

$$P^4(\mathbf{x}; z) = \left[\sum_{i=1}^{q(\mathbf{x}; z)} \sqrt{z - x_{\sigma(i)}} \right]^2 \qquad \forall (\mathbf{x}; z) \in \Omega$$

is a well-behaved, compromise and subgroup consistent poverty index which fails to satisfy Axiom T. ■

IV. A PARTIAL ORDERING APPROACH

Theorem 2 is of obvious interest to the theory of *cardinal* poverty measurement. In this section, we shall show that it can be utilized in the context of *ordinal* measurement of poverty as well.

IV.a A Partial Poverty Ordering

For any given $z > 0$, we define the partial ordering $\succcurlyeq_P^{(z)} \subset \left(\bigcup_{n=1}^{\infty} \mathbf{R}_+^n\right) \times \left(\bigcup_{n=1}^{\infty} \mathbf{R}_+^n\right)$ as

$$\mathbf{x} \succcurlyeq_P^{(z)} \mathbf{y} \quad \text{if and only if} \quad [\forall P \in \mathcal{Q}_1 : P(\mathbf{x}; z) \geq P(\mathbf{y}; z)] .$$

$\succ_P^{(z)}$ is defined as the asymmetric factor of $\succcurlyeq_P^{(z)}$ as usual.

By virtue of Theorem 2, it follows that once one fixes the poverty line at $z > 0$, and accepts the axioms of well-behavedness, compromisation, subgroup consistency and transfer, $\succcurlyeq_P^{(z)}$ lends itself as a very useful ordering for *unambiguously* ranking income distributions on the basis of poverty. Indeed, $\succcurlyeq_P^{(z)}$ is nothing but the intersection of all the complete orderings induced by the members of $\mathcal{Q}_1$.[20]

It is important to note that a poverty index need not be compromise, or in fact, not even belong to $\mathcal{Q}_1$ to agree with the poverty ordering $\succcurlyeq_P^{(z)}$ everywhere. As illustrated

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[19] That P^3 is not subgroup consistent is evident from the following example: $P^3(9; 10) = 1 > 0 = P^3(11; 10)$ and $P^3((9,9); 10) = \sqrt{2}/2 < 1 = P^3((11,9); 10)$.

[20] The motivation behind this ordering is analogous to that of the celebrated Lorenz ordering. Indeed, it is well-known that one income distribution Lorenz dominates another if, and only if, *all* relative inequality measures which satisfy Dalton's principle of equalizing transfers agree that the former distribution is less unequal than the other (see, for instance, Fields and Fei (1978)).

below, it is, in fact, possible to *characterize* $\succcurlyeq_P^{(z)}$ by means of *relative* poverty indices. This fact, of course, amplifies the appeal of $\succcurlyeq_P^{(z)}$ as a device for making ordinal poverty comparisons between income distributions with a fixed poverty line.

Corollary 1. *Let* $z > 0$. *For any* $\mathbf{x}, \mathbf{y} \in \bigcup_{n=1}^{\infty} \mathbf{R}_+^n$, $\mathbf{x} \succcurlyeq_P^{(z)} \mathbf{y}$ *if, and only if,*

$$H(Q_\alpha(\mathbf{x}; z), \alpha) \geq H(Q_\alpha(\mathbf{y}; z), \alpha) \quad \forall \alpha \geq 1$$

for all $H : \mathbf{R}_+ \times [1, \infty) \to \mathbf{R}$ *which is strictly increasing in its first component.*

To illustrate the significance of this observation, we note that, defining $H(t, \alpha) = (t/z)^\alpha$, $t \geq 0$, $\alpha \geq 1$, for a fixed $z > 0$, Corollary 1 reveals that, for any $\mathbf{x}, \mathbf{y} \in \bigcup_{n=1}^{\infty} \mathbf{R}_+^n$,

$$\mathbf{x} \succcurlyeq_P^{(z)} \mathbf{y} \quad \text{if and only if} \quad [\forall P \in \mathcal{P}_1 : P(\mathbf{x}; z) \geq P(\mathbf{y}; z)],$$

where

$$\mathcal{P}_1 := \bigcup_{\alpha \geq 1} \left\{ P_\alpha : P_\alpha(\mathbf{x}; z) := \frac{1}{n(\mathbf{x})} \sum_{i=1}^{q(\mathbf{x}; z)} \left(1 - \frac{x_{\sigma(i)}}{z}\right)^\alpha \quad \forall (\mathbf{x}; z) \in \Omega \right\}.$$

In other words, $\succcurlyeq_P^{(z)}$ is characterized by all the members of the P_α-class which satisfy Axiom T; even for those who dislike the compromisation axiom and who hold the view that $\mathcal{P}_1$ is a family of poverty indices which is more useful than $\mathcal{Q}_1$, the poverty ordering $\succcurlyeq_P^{(z)}$ still presents itself as an *unambiguous* poverty ordering.

We conclude this section by comparing $\succcurlyeq_P^{(z)}$ with another poverty ordering on $\bigcup_{n=1}^{\infty} \mathbf{R}_+^n$ introduced by Foster and Shorrocks (1988a, 1988b):

$$\mathbf{x} \succeq_{FS} \mathbf{y} \quad \text{if and only if} \quad [\forall z > 0 : P_1(\mathbf{x}; z) \geq P_1(\mathbf{y}; z)].$$

By the observation that follows Proposition 1 of Foster and Shorrocks (1988a), this ordering is, in fact, quite strong:

$$\mathbf{x} \succeq_{FS} \mathbf{y} \quad \text{if and only if} \quad [\forall z > 0, \alpha \geq 1 : P_\alpha(\mathbf{x}; z) \geq P_\alpha(\mathbf{y}; z)].^{21}$$

Therefore, $\succeq_{FS} \subset \succcurlyeq_P^{(z)}$, for any $z > 0$, and $\succcurlyeq_P^{(z)}$ can, in fact, be thought of as an *extension* of the Foster-Shorrocks poverty ordering.

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[21] Since deciding on the 'right' level of the poverty line is a problematic issue which can only be resolved by a more or less arbitrary procedure at times (cf. Atkinson (1987) and Hagenaars (1991)), there is a strong justification behind the Foster-Shorrocks ordering. $\succeq_{FS}$ ranks $\mathbf{x}$ as 'poorer' than $\mathbf{y}$ only when all indices in $\mathcal{P}_1$ agrees that $\mathbf{x}$ is 'poorer' than $\mathbf{y}$ with respect to *all* possible levels of the poverty line. It is in this sense $\succeq_{FS}$ is an unambiguous ordering of poverty.

When $\mathbf{x} \succsim_{FS} \mathbf{y}$, there is ample reason to declare $\mathbf{x}$ as the poorer distribution. On the other hand, when $\succsim_{FS}$ fails to apply, it appears that a natural way of measuring poverty would be to pick a certain poverty line (perhaps a finite multiplicity of them) and then use a P_α-index evaluated at this chosen poverty level. (Indeed, this certainly appears to be the practice.) But after fixing a certain poverty level, the choice of α becomes of importance again. It is at this point, $\succcurlyeq_P^{(z)}$ presents itself once again as an unambiguous (with respect to the choice of α) poverty ordering.

IV.b Sufficiency Theorems

Provided that a poverty line $z > 0$ is somehow fixed, we have observed above that the partial ordering $\succcurlyeq_P^{(z)}$ is a compelling poverty ordering. However, due to the continuous dependence of $\succcurlyeq_P^{(z)}$ on the parameter α, our development so far does not let us check whether or not two income distributions can be ranked by $\succcurlyeq_P^{(z)}$ except in trivial cases (like when nobody is poor in one of the distibutions). This situation is in contrast to the position of the Lorenz ordering in the theory of income inequality measurement. Indeed, while the Lorenz ordering is defined by the intersection of all the complete orderings induced by inequality measures which satisfy a number of appealing axioms (very much like $\succcurlyeq_P^{(z)}$), it also enjoys a parameter-free characterization which allows one to readily check if it can be used to rank two income distributions (very much unlike $\succcurlyeq_P^{(z)}$). Our task now is therefore to determine a majorization-type characterization of $\succcurlyeq_P^{(z)}$. Unfortunately, this turns out to be a quite difficult problem in the mathematical theory of *inequalities*. Here we shall only be able to give a number of conditions which will characterize two subrelations of $\succcurlyeq_P^{(z)}$. The characterizations of these subrelations will be parameter free (like the Lorenz ordering) so that they may be easily used in empirical practice.[22] The problem of completely characterizing $\succcurlyeq_P^{(z)}$ without giving any reference to a parameter stands at the moment open.

Our first proposition is an immediate consequence of Tomić's well-known *submajorization* theorem[23] which states that, for any $n \in \mathbf{N}$ and $\mathbf{r}, \mathbf{s} \in \mathbf{R}_+^n$,

$$\sum_{i=1}^{n} G(r_i) \geq \sum_{i=1}^{n} G(s_i)$$

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[22] We refer the reader to Mitra and Ok (1995) for several related results proved in the context of the measurement of income mobility.

[23] See, for instance, Marshall and Olkin (1979), Proposition 4.B.2, p. 109.

for all continuous, increasing and convex functions if, and only if

$$s_{\rho(1)} - r_{\rho(1)} \leq 0,$$

$$\left(s_{\rho(1)} - r_{\rho(1)}\right) + \left(s_{\rho(2)} - r_{\rho(2)}\right) \leq 0,$$

....

$$\sum\nolimits_{i=1}^{n} (s_i - r_i) \leq 0$$

where, for any $\mathbf{x} \in \mathbf{R}^{n(\mathbf{x})}$, $\mathbf{x}_\rho$ denotes the permuted form of $\mathbf{x}$ such that $x_{\rho(1)} \geq \cdots \geq x_{\rho(n(\mathbf{x}))}$. Defining, for any $(\mathbf{x}; z) \in \Omega$,

$$\sigma^{(\mathbf{x};z)} := \left(\max\{z - x_{\sigma(1)}, 0\}, ..., \max\{z - x_{\sigma(n(\mathbf{x}))}, 0\}\right),$$ [24]

and observing that $t \mapsto t^\alpha$ is a continuous, increasing and convex mapping for all $\alpha \geq 1$, we therefore have the following sufficiency theorem:

Theorem 3. *Let $z > 0$. For any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_+$, if*

$$\left(\sigma_1^{(\mathbf{y};z)} - \sigma_1^{(\mathbf{x};z)}\right) \leq 0,$$

$$\left(\sigma_1^{(\mathbf{y};z)} - \sigma_2^{(\mathbf{x};z)}\right) + \left(\sigma_1^{(\mathbf{y};z)} - \sigma_2^{(\mathbf{x};z)}\right) \leq 0,$$

....

$$\sum\nolimits_{i=1}^{n} \left(\sigma_i^{(\mathbf{y};z)} - \sigma_i^{(\mathbf{x};z)}\right) \leq 0,$$

then $\mathbf{x} \succcurlyeq_P^{(z)} \mathbf{y}$ must hold.

Although Theorem 3 is stated with regard to the poverty comparison of two income distributions of the same size, we can trivially extend its coverage to the case of income distributions of different sizes by making use of the population replication invariance. To illustrate, let $z = 10$, $\mathbf{x} = (9, 3, 11)$ and $\mathbf{y} = (5, 12)$. Which distribution is poorer? By definition of $\succcurlyeq_P^{(10)}$, we have $\mathbf{x} \succ_P^{(10)} \mathbf{y}$ if and only if $\langle \mathbf{x} \rangle_2 \succ_P^{(10)} \langle \mathbf{y} \rangle_3$, and since the vector $\sigma^{(\mathbf{x};z)} = (7, 7, 1, 1, 0, 0)$ submajorizes $\sigma^{(\mathbf{y};z)} = (5, 5, 5, 0, 0, 0)$ (i.e. the

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[24] In other words, $\sigma^{(\mathbf{x};z)}$ is a permuted form of the vector $\left(\max\{z - x_1, 0\}, \ ..., \ \max\{z - x_{n(\mathbf{x})}, 0\}\right)$ such that $\sigma_1^{(\mathbf{x};z)} \geq \cdots \geq \sigma_{n(\mathbf{x})}^{(\mathbf{x};z)}$. For instance, if $(\mathbf{x}; z) = ((9, 3, 11); 10)$, then $\sigma^{(\mathbf{x};z)} = (7, 1, 0)$.

hypotheses of Theorem 2 hold), we indeed have $\langle \mathbf{x}\rangle_2 \succ_P^{(10)} \langle \mathbf{y}\rangle_3$. [25]

Theorem 3 makes it clear that the submajorization ordering could be very useful in determining whether or not two income distributions can be ranked by $\succcurlyeq_P^{(z)}$, given $z > 0$. In what follows, our aim is to improve upon this observation, and thus, to prove that $\succcurlyeq_P^{(z)}$ is "more complete" than the submajorization ordering.

To simplify our subsequent discussion, let us take $z > 0$ as (arbitrarily) fixed, and write $\sigma^{\mathbf{y}}$ for $\sigma^{(\mathbf{y};z)}$ for any $\mathbf{y} \in \bigcup_{n=0}^{\infty} \mathbf{R}_+^n$. For any $n \in \mathbf{N}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{R}_+^n$, we shall denote the cardinality of the set $\{i \in \{1, ..., n\} : y_i < z\}$ by $q^-(\mathbf{y})$ and define

$$\mathbf{K}_h := \begin{cases} \left(\sum_{i=1}^{h} (\sigma_i^{\mathbf{y}} - \sigma_i^{\mathbf{x}})\right)\left(\frac{\sigma_1^{\mathbf{y}}}{\sigma_{h+1}^{\mathbf{y}}}\right), & \text{if } \sum_{i=1}^{h} (\sigma_i^{\mathbf{y}} - \sigma_i^{\mathbf{x}}) > 0 \\ \sum_{i=1}^{h} (\sigma_i^{\mathbf{y}} - \sigma_i^{\mathbf{x}}), & \text{otherwise,} \end{cases}$$

for all $h = 1, ..., q^-(\mathbf{y}) - 1$. (Of course, $\mathbf{K}_h$ is, in fact, a function of $\mathbf{x}, \mathbf{y}$ and z, but for brevity, we do not use a notation which makes this explicit.)

The main result of this section reads as follows:

Theorem 4. *Let $z > 0$. For any $\mathbf{x}, \mathbf{y} \in \mathbf{R}_+^n$, if*

$$\mathbf{K}_1 (\sigma_1^{\mathbf{y}} - \sigma_2^{\mathbf{y}}) \leq 0,$$

$$\mathbf{K}_1 (\sigma_1^{\mathbf{y}} - \sigma_2^{\mathbf{y}}) + \mathbf{K}_2 (\sigma_2^{\mathbf{y}} - \sigma_3^{\mathbf{y}}) \leq 0,$$

$$\dots\dots\dots\dots$$

$$\sum_{i=1}^{q^-(\mathbf{y})-1} \mathbf{K}_i (\sigma_i^{\mathbf{y}} - \sigma_{i+1}^{\mathbf{y}}) \leq 0,$$

and

$$\sum_{i=1}^{q^-(\mathbf{y})} (\sigma_i^{\mathbf{y}} - \sigma_i^{\mathbf{x}}) \leq 0,$$

then $\mathbf{x} \succcurlyeq_P^{(z)} \mathbf{y}$ must hold.

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[25] To see the non-triviality of this example, we note that $\mathbf{x} = (9, 3, 11)$ and $\mathbf{y}' = (4, 12)$ cannot be ranked by the ordering $\succcurlyeq_P^{(10)}$. Indeed, while $Q_1(\mathbf{x}; z) < Q_1(\mathbf{y}; z)$, it is obvious that there exists an $\alpha_0 > 1$ such that $Q_\alpha(\mathbf{x}; z) > Q_\alpha(\mathbf{y}; z)$ for all $\alpha > \alpha_0$; someone who views the per capita income gap as a "good" poverty measure would conclude that $\mathbf{y}$ is a poorer distribution than $\mathbf{x}$, whereas a sufficiently poverty averse reader would like to arrive at the opposite conclusion. There appears to be an ambiguity in making a poverty comparison between $\mathbf{x}$ and $\mathbf{y}$.

Theorem 4 is a technical result the formulation of which is not so friendly. It is, however, useful in at least two regards. First, the hypotheses of the theorem are stated without giving any reference to a parameter. Given any two social situations $(\mathbf{x}; z)$ and $(\mathbf{y}; z)$ in Ω, one can easily check whether or not the conditions stated in the theorem are satisfied (using, if necessary, the population replication invariance as well).

Second, one may easily verify that Theorem 4 is considerably stronger than the submajorization result noted in Theorem 3. Indeed, if the hypotheses of Theorem 3 hold true, then by definition of $\mathbf{K}_h$, we have $\mathbf{K}_h \leq 0$, $h = 1, ..., q^-(\mathbf{y}) - 1$, so that all of the hypotheses of Theorem 4 are trivially satisfied. Let us now observe that the converse implication is not true. Let $z = 20$, $\mathbf{x} = (17, 6, 16.5, 22)$ and $\mathbf{y} = (14, 8, 17.75, 30)$. We have $\sigma^{\mathbf{x}} = (14, 3.5, 3, 0)$ and $\sigma^{\mathbf{y}} = (12, 6, 2.25, 0)$ so that the antecedents of Theorem 3 are not satisfied; $\sigma^{\mathbf{x}}$ does not submajorize $\sigma^{\mathbf{y}}$, and conversely. However, $\mathbf{K}_1 = -2$ and $\mathbf{K}_2 = (0.5)(12/2.25) = 8/3$ so that

$$\mathbf{K}_1 \left(\sigma_1^{\mathbf{y}} - \sigma_2^{\mathbf{y}}\right) = (-2)6 = -12 < 0$$

$$\mathbf{K}_1 \left(\sigma_1^{\mathbf{y}} - \sigma_2^{\mathbf{y}}\right) + \mathbf{K}_2 \left(\sigma_2^{\mathbf{y}} - \sigma_3^{\mathbf{y}}\right) = -12 + \tfrac{8}{3}(3.75) = -2 < 0$$

and

$$\sum_{i=1}^{3} \left(\sigma_i^{\mathbf{y}} - \sigma_i^{\mathbf{x}}\right) = -0.25 < 0;$$

all of the hypotheses of Theorem 4 are satisfied. We may then conclude that

$$\mathbf{x} = (17, 6, 16.5, 22) \succ_P^{(20)} (14, 8, 17.75, 30) = \mathbf{y};$$

that is, all well-defined poverty indices which satisfy Axioms SC, T and C (or, all P_α indices with $\alpha \geq 1$) would agree that $\mathbf{x}$ is poorer than $\mathbf{y}$. As indicated above, this result cannot be obtained from Theorem 3; a stronger majorization result like Theorem 4 is needed to reach this conclusion.

Theorem 4, therefore, provides us with an easy method of implementing the partial ordering $\succcurlyeq_P^{(z)}$. In view of the axiomatic support behind $\succcurlyeq_P^{(z)}$, we believe that majorization-type results like Theorem 4 are likely to prove useful in empirical practice. The shortcoming of Theorem 4 is, of course, that it does not characterize $\succcurlyeq_P^{(z)}$; the hypotheses of the proposition are not necessary for $\succcurlyeq_P^{(z)}$ to rank two income distributions.[26] The next step of the associated research agenda is then to provide a parameter-free characterization of our poverty ordering.

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[26] Let $z = 20$, $\mathbf{x} = (0, 5, 5)$ and $\mathbf{y} = (2, 2, 7)$. We have $\sigma^{\mathbf{x}} = (10, 5, 5)$, $\sigma^{\mathbf{y}} = (8, 8, 3)$ and $\mathbf{K}_2 = 8/3$ so that $\mathbf{K}_1 \left(\sigma_1^{\mathbf{y}} - \sigma_2^{\mathbf{y}}\right) + \mathbf{K}_2 \left(\sigma_2^{\mathbf{y}} - \sigma_3^{\mathbf{y}}\right) = 40/3 > 0$ while $\sum_{i=1}^{3} \left(\sigma_i^{\mathbf{y}} - \sigma_i^{\mathbf{x}}\right) = -1 < 0$; the hypotheses of Theorem 4 are not satisfied. Yet one can show that $\mathbf{x} \succ_P^{(20)} \mathbf{y}$ hold true. (The proof of this claim is somewhat lengthy, and hence, is omitted here.)

The present paper concludes with the sketches of the proofs of Theorems 1, 2 and 4.

V. PROOFS

V.a Proof of Theorem 1

Since the sufficiency can readily be verified, we shall only study the necessity part of the theorem. The basic idea of the proof is to exploit Proposition 3 of Foster and Shorrocks (1991), p. 699. According to this theorem, for a well-behaved index $P \in \mathcal{P}$ which satisfies Axiom SC, one of the following cases must hold true:

(a) There exist functions $\phi : \mathbf{R}_+ \times \mathbf{R}_{++} \to \mathbf{R}_+$ and $F : \bigcup_{z>0} (\phi(\mathbf{R}_+, z) \times \{z\}) \to \mathbf{R}_+$ such that

$$P(\mathbf{x}, z) = F\left(\frac{1}{n(\mathbf{x})}\sum_{i=1}^{n(\mathbf{x})} \phi(x_i, z),\ z\right) \quad \forall(\mathbf{x}; z) \in \Omega,$$

where $\phi(t, z) = 0$ for all $t \in [z, \infty)$; $\phi(., z)$ is continuous and decreasing on $\mathbf{R}_+$ for any given $z > 0$; and $F(., z)$ is continuous and strictly increasing on $\phi(\mathbf{R}_+, z)$.

(b) There exist functions $\phi : \mathbf{R}_+ \times \mathbf{R}_{++} \to \mathbf{R}_+$, $F : \bigcup_{z>0} (\phi(\mathbf{R}_+, z) \times \{z\}) \to \mathbf{R}_+$ and $\lambda : \mathbf{R}_{++} \to (0, 1)$ such that

$$P(\mathbf{x}, z) = F\left(\lambda(z)\frac{q(\mathbf{x}; z)}{n(\mathbf{x})} + (1-\lambda(z))\frac{1}{n(\mathbf{x})}\sum_{i=1}^{n(\mathbf{x})} \phi(x_i, z),\ z\right) \quad \forall(\mathbf{x}; z) \in \Omega,$$

where $\phi(t, z) = 0$ for all $t \in [z, \infty)$; $\phi(., z)$ is continuous and decreasing on $\mathbf{R}_+$ for any given $z > 0$; and $F(., z)$ is continuous on $\mathbf{R}_{++}$ and strictly increasing on $\phi(\mathbf{R}_+, z)$.

(c) For any $(\mathbf{x}; z), (\mathbf{y}; z) \in \Omega$, $P(\mathbf{x}; z) \geq P(\mathbf{y}; z)$ if, and only if,

$$\frac{q(\mathbf{x}; z)}{n(\mathbf{x})} > \frac{q(\mathbf{y}; z)}{n(\mathbf{y})} \quad \text{or} \quad \left[\frac{q(\mathbf{x}; z)}{n(\mathbf{x})} = \frac{q(\mathbf{y}; z)}{n(\mathbf{y})} \text{ and } \frac{1}{n(\mathbf{x})}\sum_{i=1}^{n(\mathbf{x})} \phi(x_i, z) \geq \frac{1}{n(\mathbf{y})}\sum_{i=1}^{n(\mathbf{y})} \phi(y_i, z)\right]$$

where $\phi(t, z) = 0$ for all $t \in [z, \infty)$; $\phi(., z)$ is continuous and decreasing on $\mathbf{R}_+$ for any given $z > 0$.

It is immediately clear that (c) cannot hold in our case, for no such index satisfies the transfer axiom. (For instance, Axiom T dictates that $P((2, 4); 5) \leq P((0, 6); 5)$ while (c) entails the opposite conclusion.[27]) We now proceed to show that (b) is incompatible with Axiom T as well.

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[27] It is easy to see that, in case (b), $\frac{q(\mathbf{x};z)}{n(\mathbf{x})} > \frac{q(\mathbf{y};z)}{n(\mathbf{y})}$ implies $P(\mathbf{x}; z) > P(\mathbf{y}; z)$ for any $(\mathbf{x}; z), (\mathbf{y}; z) \in \Omega$.

Let $z = 1$ and define

$$\mathbf{x}^m = \left(1 - \frac{1}{2m}, 1 - \frac{1}{4m}\right) \quad \text{and} \quad \mathbf{y}^m = \left(1 - \frac{1}{m}, 1 + \frac{1}{4m}\right)$$

for all $m \geq 1$. Notice that $\mathbf{y}^m$ is obtained from $\mathbf{x}^m$ by transferring an income of $(1/2m)$ from the poorest individual to the richer individual. By Axiom T, therefore, $P(\mathbf{y}^m; 1) \geq P(\mathbf{x}^m; 1)$. But if P was of form (b), the fact that F is strictly increasing in its first argument and that $n(\mathbf{x}^m) = n(\mathbf{y}^m) = 2$ would then imply that

$$\lambda q(\mathbf{y}^m; 1) + (1-\lambda)\phi\left(1 - \frac{1}{m}, 1\right) \geq \lambda q(\mathbf{x}^m; 1) + (1-\lambda)\left[\phi\left(1 - \frac{1}{2m}, 1\right) + \phi\left(1 - \frac{1}{4m}, 1\right)\right]$$

where $\lambda(1) \equiv \lambda$. Since $q(\mathbf{x}^m; 1) = 2$ and $q(\mathbf{y}^m; 1) = 1$ for all $m \geq 1$, we obtain

$$\lambda \leq (1-\lambda)\phi\left(1 - \frac{1}{m}, 1\right) - (1-\lambda)\left[\phi\left(1 - \frac{1}{2m}, 1\right) + \phi\left(1 - \frac{1}{4m}, 1\right)\right].$$

Letting $m \to \infty$, we obtain

$$\lambda \leq \lim_{m \to \infty} \left\{(1-\lambda)\phi\left(1 - \frac{1}{m}, 1\right) - (1-\lambda)\left[\phi\left(1 - \frac{1}{2m}, 1\right) + \phi\left(1 - \frac{1}{4m}, 1\right)\right]\right\} = 0$$

where the last step follows from the continuity of ϕ and the fact that $\phi(1, 1) = 0$. This contradicts the fact that $\lambda \equiv \lambda(1) > 0$.

Consequently, the only possible representation of a poverty index satisfying all the hypotheses of Theorem 1 is as stated in case (a), and the proof is complete. ■

V.b Proof of Theorem 2

We start with the following observation:

Lemma 1.[28] *Let $P \in \mathcal{P}$. If P satisfies Axioms SC, T and C, then there must exist a continuous $\psi : \mathbf{R} \to \mathbf{R}_+$ which is strictly increasing on $\mathbf{R}_+$ such that*

$$P(\mathbf{x}; z) = \psi^{-1}\left(\frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} \psi(z - x_i)\right) \quad \forall (\mathbf{x}; z) \in \Omega,$$

where $\psi(t) = 0$ for all $t \leq 0$.

Proof of Lemma 1. Since we have established in Theorem 1 that a poverty index $P \in \mathcal{P}$ which satisfies Axioms SC and T must be continuous on $\mathbf{R}_+^n$ for any given

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[28] This lemma too borrows heavily from the development of Foster and Shorrocks (1991).

$n \in \mathbf{N}$ and $z > 0$, and since a compromise index is absolute, we can use Proposition 5 of Foster and Shorrocks (1991), p. 703, to conclude that there exist $\psi : \mathbf{R} \to \mathbf{R}$ and $F : \psi(\mathbf{R}) \to \mathbf{R}_+$ such that

$$P(\mathbf{x}; z) = F\left(\frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} \psi\left(z - x_i\right)\right) \quad \forall(\mathbf{x}; z) \in \Omega,$$

where F is continuous and strictly increasing; ψ is continuous and increasing; and $\psi(t) = 0$ for all $t \leq 0$. By using (2) and the normalization axiom, on the other hand, we obtain $t = F(\psi(t))$ for all $t \geq 0$. Therefore, $F = \psi^{-1}$ and ψ is strictly increasing on $\mathbf{R}_+$. ■

Now define $\Psi_n : \mathbf{R}_+^n \to \mathbf{R}_+$ as

$$\Psi_n(t_1, ..., t_n) := \psi^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} \psi\left(t_i\right)\right), \quad n = 1, 2, ... \, .$$

Given the properties of ψ noted in Lemma 1, each Ψ_n is a *quasilinear mean.* Moreover, by linear homogeneity of P, each Ψ_n is homogeneous of degree 1: for all $t_1, ..., t_n \geq 0$, $n \in \mathbf{N}$, and all $\lambda > 0$,

$$\begin{aligned}
\Psi_n(\lambda t_1, ..., \lambda t_n) &= \psi^{-1}\left(\tfrac{1}{n} \textstyle\sum_{i=1}^{n} \psi\left(\lambda t_i\right)\right) \\
&= P(\lambda((z - t_1), ..., (z - t_n)); \lambda z) \;\; \forall z > \max\{t_1, ..., t_n\} \\
&= \lambda P(((z - t_1), ..., (z - t_n)); z) \quad \forall z > \max\{t_1, ..., t_n\} \\
&= \lambda \psi^{-1}\left(\tfrac{1}{n} \textstyle\sum_{i=1}^{n} \psi\left(t_i\right)\right) \\
&= \lambda \Psi_n(t_1, ..., t_n).
\end{aligned}$$

Consequently, by Theorem 2.2.1 of Eichhorn (1978), p. 32, for a given $n \in \mathbf{N}$, either

$$\Psi_n(t_1, ..., t_n) = A(n) \prod_{i=1}^{n} t_i^{\alpha_i(n)} \quad \forall t_1, ..., t_n \geq 0, \tag{3}$$

for some $A(n) > 0$ and $\alpha_i(n) \in \mathbf{R} \backslash \{0\}$, $i = 1, ..., n$, such that $\sum_{i=1}^{n} \alpha_i(n) = 1$, or

$$\Psi_n(t_1, ..., t_n) = \left(\sum_{i=1}^{n} \beta_i(n) t_i^{\alpha(n)}\right)^{1/\alpha(n)} \qquad \forall t_1, ..., t_n \geq 0, \tag{4}$$

for some $\beta_i(n) > 0$, $i = 1, ..., n$ and $\alpha(n) \in \mathbf{R} \backslash \{0\}$. But (3) cannot hold for any $n \in \mathbf{N}$; if it did, we would have

$$\Psi_n(0, t_2, ..., t_n) = 0 \quad \forall t_2, ..., t_n \geq 0$$

which violates the strict monotonicity of P. Thus, we can conclude that (4) must hold. Now, in (4), $\alpha(n) < 0$ cannot hold true for any $n \in \mathbf{N}$, for otherwise we could not have Ψ_n continuous at the origin. Moreover, by the symmetry of Ψ_n, we must have $\beta_i(n) = \beta_j(n)$ for all $i, j = 1, ..., n$ and all $n \in \mathbf{N}$. Therefore, for any $n \in \mathbf{N}$, we have

$$\Psi_n(t_1, ..., t_n) = \left(\sum_{i=1}^{n} \beta(n) t_i^{\alpha(n)} \right)^{1/\alpha(n)} \quad \forall t_1, ..., t_n \geq 0 \tag{5}$$

for some $\beta(n) > 0$ and $\alpha(n) > 0$.

Now pick an arbitrary $a > 0$. Then, (5) implies that, for any $n \in \mathbf{N}$,

$$\Psi_n(\langle a \rangle_n) = (\beta(n)n)^{1/\alpha(n)} a. \tag{6}$$

But since P is population replication invariant, for any $n, m \in \mathbf{N}$,

$$\Psi_n(\langle a \rangle_n) = P(\langle a \rangle_n ; 2a) = P(\langle a \rangle_m ; 2a) = \Psi_m(\langle a \rangle_m). \tag{7}$$

Combining (6) and (7), we have

$$(\beta(n)n)^{1/\alpha(n)} = (\beta(m)m)^{1/\alpha(m)} \quad \forall n, m = 1, 2, ...$$

so that there must exist a constant $\xi > 0$ such that

$$(\beta(n)n)^{1/\alpha(n)} = \xi \quad \forall n = 1, 2, ... \tag{8}$$

But the definition of Ψ_1 and (6) give $\Psi_1(a) = a = (\beta(1))^{1/\alpha(1)} a$ so that $(\beta(1))^{1/\alpha(1)} = 1$. Thus, by (8), $\xi = 1$; that is $\beta(n) = 1/n$ for all $n \in \mathbf{N}$. Consequently, for all $n \in \mathbf{N}$,

$$\Psi_n(t_1, ..., t_n) = \left(\frac{1}{n} \sum_{i=1}^{n} t_i^{\alpha(n)} \right)^{1/\alpha(n)} \quad \forall t_1, ..., t_n \geq 0 \tag{9}$$

for some $\alpha(n) > 0$.

Notice next that, for any $a, b > 0$, (9) yields

$$\Psi_{2n}\left(\langle (a, b) \rangle_n\right) = \left(\frac{a^{\alpha(2n)} + b^{\alpha(2n)}}{2} \right)^{1/\alpha(2n)} \quad \forall n = 1, 2, ...$$

while, by the population replication invariance, $\Psi_{2n}(\langle (a, b) \rangle_n) = \Psi_2(a, b)$ so that

$$\left(\frac{a^{\alpha(2n)} + b^{\alpha(2n)}}{2} \right)^{1/\alpha(2n)} = \left(\frac{a^{\alpha(2)} + b^{\alpha(2)}}{2} \right)^{1/\alpha(2)} \quad \forall n = 1, 2, ...$$

But this equation can hold for all $a, b > 0$ if, and only if,

$$\alpha(2n) = \alpha(2) \quad \forall n = 1, 2, ... \tag{10}$$

Now let ℓ be any odd integer greater than 1. Using (9) and applying the population replication axiom to $(a, b, 0, ..., 0) \in \mathbf{R}^\ell_+$ for any $a, b > 0$, we have

$$\left(\frac{a^{\alpha(\ell)} + b^{\alpha(\ell)}}{2}\right)^{1/\alpha(\ell)} = \left(\frac{a^{\alpha(2\ell)} + b^{\alpha(2\ell)}}{2}\right)^{1/\alpha(2\ell)}.$$

This equation too can clearly hold for all $a, b > 0$ if, and only if, $\alpha(\ell) = \alpha(2\ell)$. Therefore, by (10), we may conclude that there exists an $\alpha > 0$ such that $\alpha(n) = \alpha$ for all $n \in \mathbf{N}$. Consequently, for any $n \in \mathbf{N}$,

$$\psi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\psi(t_i)\right) = \Psi_n(t_1, ..., t_n) = \left(\frac{1}{n}\sum_{i=1}^{n} t_i^\alpha\right)^{1/\alpha} \qquad \forall t_1, ..., t_n \geq 0.$$

Combining this observation with Lemma 1, we may then conclude that there exists an $\alpha > 0$ such that

$$P(\mathbf{x}; z) := \left(\frac{1}{n(\mathbf{x})}\sum_{i=1}^{q(\mathbf{x};z)}\left(z - x_{\sigma(i)}\right)^\alpha\right)^{1/\alpha} \qquad \forall(\mathbf{x}; z) \in \Omega. \tag{11}$$

The final stage of our proof follows easily from the next lemma. (The proof of this result is routine and thus omitted.)

Lemma 2. *Let* $P_n = P|_{\mathbf{R}^n_+\times\mathbf{R}_{++}}$, $n \in \mathbf{N}$. *A well-behaved poverty index* $P \in \mathcal{P}$ *satisfies Axiom T if, and only if, for any* $n \in \mathbf{N}$,

$$\frac{\partial^2}{\partial x^2_{\sigma(1)}} P_n(\mathbf{x}; z) \geq 0 \qquad \forall(\mathbf{x}; z) \in \mathbf{R}^n_+ \times \mathbf{R}_{++} \text{ such that } x_{\sigma(i)} \in [0, z).^{29}$$

Now pick an arbitrary $n \in \mathbf{N}$. By (11), we have, for any $\mathbf{x} \in \mathbf{R}^n_+$,

$$P_n(\mathbf{x}; z) = \left(\frac{1}{n}(z - x_{\sigma(1)})^\alpha + \frac{1}{n}\sum_{i=2}^{q(\mathbf{x};z)} w_i\right)^{1/\alpha}$$

where $w_i := (z - x_{\sigma(i)})^\alpha$, $i = 2, ..., q(\mathbf{x}; z)$, for some $\alpha > 0$, so that

$$\begin{aligned}
\tfrac{\partial}{\partial x_{\sigma(1)}} P_n(\mathbf{x}; z) &= -\tfrac{1}{n}\left(\tfrac{1}{n}(z - x_{\sigma(1)})^\alpha + \tfrac{1}{n}\textstyle\sum_{i=2}^{q(\mathbf{x};z)} w_i\right)^{\frac{1-\alpha}{\alpha}} (z - x_{\sigma(1)})^{\alpha-1} \\
&= -\tfrac{1}{n}\left(\tfrac{1}{n}(z - x_{\sigma(1)})^\alpha + \tfrac{1}{n}\textstyle\sum_{i=2}^{q(\mathbf{x};z)} w_i\right)^{\frac{1-\alpha}{\alpha}} \left((z - x_{\sigma(1)})^{-\alpha}\right)^{\frac{1-\alpha}{\alpha}} \\
&= -\tfrac{1}{n}\left(\tfrac{1}{n} + \tfrac{1}{n}\textstyle\sum_{i=2}^{q(\mathbf{x};z)} w_i(z - x_{\sigma(1)})^{-\alpha}\right)^{\frac{1-\alpha}{\alpha}}
\end{aligned}$$

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[29] (Recall that $\mathbf{x}_\sigma$ is a permuted form of $\mathbf{x}$ such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}$.) Given the anonymity of P, this derivative condition, of course, guarantees the positivity of all the second partials of P_n on the relevant subdomains.

which, in turn, yields

$$\frac{\partial^2}{\partial x_{\sigma(1)}^2} P_n(\mathbf{x}; z) = \frac{(\alpha-1)}{n^2} \left(\sum_{i=2}^{q(\mathbf{x};z)} w_i \right) \left(\frac{1}{n} + \frac{1}{n} \sum_{i=2}^{q(\mathbf{x};z)} w_i (z - x_{\sigma(1)})^{-\alpha} \right)^{\frac{1-2\alpha}{\alpha}} (z - x_{\sigma(1)})^{-\alpha-1}.$$

In view of Lemma 2, we have $\alpha \geq 1$, and Theorem 2 follows. ■

V.c Proof of Theorem 4

Let us define, for any $\mathbf{r}, \mathbf{s} \in \mathbf{R}_{++}^m$, $m \in \mathbf{N}$, such that $r_1 \geq \cdots \geq r_m > 0$ and $s_1 \geq \cdots \geq s_m > 0$,

$$A_h := \sum_{i=1}^{h} (s_i - r_i)$$

and

$$B_h := \begin{cases} A_h \left(\frac{s_1}{s_{h+1}} \right), & A_h > 0 \\ A_h, & A_h \leq 0 \end{cases}$$

for all $h = 1, ..., m$. (Obviously, $B_h \geq A_h$ for all h.) Theorem 4 is an immediate consequence of the following result:

Lemma 3. *For any given $\mathbf{r}, \mathbf{s} \in \mathbf{R}_{++}^m$, $m \in \mathbf{N}$, such that $r_1 \geq \cdots \geq r_m > 0$ and $s_1 \geq \cdots \geq s_m > 0$, if $A_m \leq 0$ and*

$$B_1(s_1 - s_2) \leq 0,$$

$$B_1(s_1 - s_2) + B_2(s_2 - s_3) \leq 0,$$

$$\cdots\cdots\cdots\cdot$$

$$\sum_{i=1}^{m-1} B_i(s_i - s_{i+1}) \leq 0,$$

then we have

$$\sum_{i=1}^{m} (s_i^\alpha - r_i^\alpha) \leq 0 \quad \forall \alpha \geq 1.$$

Proof of Lemma 3. We shall make use of the following preliminary result throughout the proof.

Lemma 4. *Let $\alpha \geq 1$. For any given $\mathbf{r}, \mathbf{s} \in \mathbf{R}^m_{++}$, $m \in \mathbf{N}$, such that $r_1 \geq \cdots \geq r_m > 0$ and $s_1 \geq \cdots \geq s_m > 0$, if*

$$\sum_{i=1}^{m-1} A_i(s_i^{\alpha-1} - s_{i+1}^{\alpha-1}) + A_m s_m^{\alpha-1} \leq 0, \tag{12}$$

then we have

$$\sum_{i=1}^{m}(s_i^\alpha - r_i^\alpha) \leq 0.$$

Proof of Lemma 4. By Abel's partial summation formula, for all $\alpha \geq 1$,

$$\sum_{i=1}^{m} s_i^{\alpha-1}(s_i - r_i) = \sum_{i=1}^{m-1} A_i(s_i^{\alpha-1} - s_{i+1}^{\alpha-1}) + A_m s_m^{\alpha-1}. \tag{13}$$

But since, for all $\alpha \geq 1$, $t \mapsto t^\alpha$ is a convex function on $\mathbf{R}_+$, we have $s_i^\alpha - r_i^\alpha \leq \alpha s_i^{\alpha-1}(s_i - r_i)$. Therefore, by summing over i and combining the outcome with (13), we have

$$\sum_{i=1}^{m}(s_i^\alpha - a_i^\alpha) \leq \alpha\left(\sum_{i=1}^{m-1} A_i(s_i^{\alpha-1} - s_{i+1}^{\alpha-1}) + A_m s_m^{\alpha-1}\right)$$

and hence Lemma 4. ■

We now wish to show that under the hypotheses of Lemma 3, the inequality given in (12) must hold for all $\alpha \geq 1$. Since Lemma 3 is obvious for $\alpha = 1$, we will be concerned henceforth with values of $\alpha > 1$. Define

$$V := \{i \in \{1, ..., m-1\} : A_i > 0\} \quad \text{and} \quad W := \{i \in \{1, ..., m-1\} : A_i \leq 0\}.$$

Let us first consider the case where $\alpha \in (1, 2)$. Then $t \mapsto t^{\alpha-1}$ is a concave function on $\mathbf{R}_+$ so that

$$s_i^{\alpha-1} - s_{i+1}^{\alpha-1} \leq (\alpha-1)s_{i+1}^{\alpha-2}(s_i - s_{i+1}) = (\alpha-1)\left(\frac{s_1}{s_{i+1}}\right)^{2-\alpha} s_1^{\alpha-2}(s_i - s_{i+1}),$$

and since $s_1 \geq s_{i+1}$ for all $i = 1, ..., m-1$ and $\alpha \in (1, 2)$, we conclude that

$$s_i^{\alpha-1} - s_{i+1}^{\alpha-1} \leq (\alpha-1)\left(\frac{s_1}{s_{i+1}}\right) s_1^{\alpha-2}(s_i - s_{i+1}) \qquad \forall i \in V. \tag{14}$$

In addition, by concavity of the mapping $t \mapsto t^{\alpha-1}$, we have

$$s_i^{\alpha-1} - s_{i+1}^{\alpha-1} \geq (\alpha-1)s_i^{\alpha-2}(s_i - s_{i+1}) \qquad \forall i \in W. \tag{15}$$

Since $\left(\frac{s_i}{s_1}\right)^{\alpha-2} \geq 1$ for all i, (14) and (15) together imply that

$$\frac{1}{\alpha-1}\left(\frac{1}{s_1}\right)^{\alpha-2}\left(\sum_{i=1}^{m-1} A_i(s_i^{\alpha-1}-s_{i+1}^{\alpha-1})\right)$$
$$\leq \sum_{i\in W} A_i\left(\frac{s_i}{s_1}\right)^{\alpha-2}(s_i-s_{i+1}) + \sum_{i\in V} A_i\left(\frac{s_1}{s_{i+1}}\right)(s_i-s_{i+1}).$$

But since $\left(\frac{s_i}{s_1}\right)^{\alpha-2} \geq 1$ for all $i \in W$, we have $\sum_{i\in W} A_i\left(\frac{s_i}{s_1}\right)^{\alpha-2}(s_i-s_{i+1}) \leq \sum_{i\in W} B_i(s_i-s_{i+1})$ so that

$$\sum_{i\in W} A_i\left(\frac{s_i}{s_1}\right)^{\alpha-2}(s_i-s_{i+1}) + \sum_{i\in V} A_i\left(\frac{s_1}{s_{i+1}}\right)(s_i-s_{i+1}) \leq \sum_{i=1}^{m-1} B_i(s_i-s_{i+1}).$$

Combining this with the previous inequality, we obtain

$$\frac{1}{\alpha-1}\left(\frac{1}{s_1}\right)^{\alpha-2}\left(\sum_{i=1}^{m-1} A_i(s_i^{\alpha-1}-s_{i+1}^{\alpha-1})\right) \leq \sum_{i=1}^{m-1} B_i(s_i-s_{i+1}).$$

Since $A_m \leq 0$ and $\sum_{i=1}^{m-1} B_i(s_i-s_{i+1}) \leq 0$ by hypothesis, this inequality yields (12), and by using Lemma 5, we may then conclude that

$$\sum_{i=1}^{m}(s_i^\alpha - r_i^\alpha) \leq 0 \quad \forall \alpha \in (1,2). \tag{16}$$

Let us now consider the case $\alpha \geq 2$. In this case, $t \mapsto t^{\alpha-1}$ is convex on $\mathbf{R}_+$, and we thus have

$$s_i^{\alpha-1} - s_{i+1}^{\alpha-1} \leq (\alpha-1)s_i^{\alpha-2}(s_i-s_{i+1}) \qquad \forall i \in V$$

and

$$s_i^{\alpha-1} - s_{i+1}^{\alpha-1} \geq (\alpha-1)s_{i+1}^{\alpha-2}(s_i-s_{i+1}) \qquad \forall i \in W.$$

Therefore, defining

$$\omega_i := \begin{cases} s_i^{\alpha-2}, & i \in V \\ s_{i+1}^{\alpha-2}, & i \in W \end{cases},$$

we have

$$\begin{aligned}\frac{1}{\alpha-1}\sum_{i=1}^{m-1} A_i(s_i^{\alpha-1}-s_{i+1}^{\alpha-1}) &\leq \sum_{i\in W} A_i s_{i+1}^{\alpha-2}(s_i-s_{i+1}) + \sum_{i\in V} A_i s_i^{\alpha-2}(s_i-s_{i+1}) \\ &= \sum_{i=1}^{m-1} \omega_i A_i(s_i-s_{i+1}).\end{aligned}$$

But since $B_i \geq A_i$ for all $i = 1, ..., m-1$, this inequality yields

$$\frac{1}{\alpha-1}\sum_{i=1}^{m-1} A_i(s_i^{\alpha-1}-s_{i+1}^{\alpha-1}) \leq \sum_{i=1}^{m-1} \omega_i B_i(s_i-s_{i+1}). \tag{17}$$

To conclude the proof, we shall need

Lemma 5.[30] (Abel's Inequality) *For any real numbers* $u_1, ..., u_\ell$ *and* $v_1, ..., v_\ell$ *such that* $v_1 \geq \cdots \geq v_\ell \geq 0$,

$$\sum_{i=1}^{\ell} u_i v_i \leq \left(\max_{h \in \{1,...,\ell\}} \sum_{i=1}^{h} u_i \right) v_1.$$

Now one can easily verify that $\omega_1 \geq \cdots \geq \omega_{m-1} > 0$.We can thus apply Lemma 5, and conclude that

$$\sum_{i=1}^{m-1} \omega_i B_i(s_i - s_{i+1}) \leq \left(\max_{h \in \{1,...,m-1\}} \sum_{i=1}^{h} B_i(s_i - s_{i+1}) \right) \omega_1.$$

But by hypothesis, $\sum_{i=1}^{h} B_i(s_i - s_{i+1}) \leq 0$ for all $h = 1, ..., m-1$ so that the above inequality yields that

$$\sum_{i=1}^{m-1} \omega_i B_i(s_i - s_{i+1}) \leq 0.$$

Combining this inequality with (17), and recalling that $A_m \leq 0$ by hypothesis, we conclude that (12) holds which, in view of Lemma 4, entails

$$\sum_{i=1}^{m} (s_i^\alpha - r_i^\alpha) \leq 0 \quad \forall \alpha \geq 2.$$

This completes the proof of Lemma 3. ■

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[30] See Mitrinović (1970), p. 32, Theorem 2.2.1.

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