# *THE NUCLEOLUS IS CONTESTED-GARMENT-CONSISTENT: A DIRECT PROOF*

by Jean-Pierre Benoit

**RR# 97-12** **February 1997**

## ABSTRACT

This paper presents a simple direct proof of Aumann and Maschler's result that the nucleolus is the contested-garment solution of a bankruptcy game.

KEY WORDS: Bankruptcy Problem, Talmud, Nucleolus, Consistency, Contested-Garment.
JEL CLASSIFICATION NUMBER: C71

# The Nucleolus is Contested-Garment-Consistent: A Direct Proof.

Jean-Pierre Benoît*
Department of Economics and School of Law, New York University

In a classic paper, Aumann and Maschler(1985) show that a generalization of a Talmudic solution to a bankruptcy problem is the nucleolus of the corresponding game. However, their proof is a subtle indirect one which proceeds via the notion of the kernel. In this paper I provide a simple direct proof.

A bankruptcy problem is denoted $(E; d)$, where $E$ is the size of an estate and $d = d_1, d_2, ...d_n$ are the claims of $n$ agents on the estate, $d_i \geq 0$ for all $i$, and $0 \leq E \leq \sum_{i=1}^{n} d_i \equiv D$. Aumann and Maschler define the CG-consistent (Contested Garment-consistent) solution to the problem (page 199). They show that this solution is as follows, where $d_0 \equiv 0$:

$$\text{For } \sum_{i=1}^{k} \frac{d_i}{2} + (n-k)\frac{d_k}{2} \leq E \leq \sum_{i=1}^{k} \frac{d_i}{2} + (n-k)\frac{d_{k+1}}{2}, 0 \leq k \leq n-1$$

$$\overline{x} : \begin{cases} \overline{x}_i = \frac{d_i}{2}, \ i = 1, ..., k \\ \overline{x}_i = (E - \sum_{j=1}^{k} \frac{d_j}{2})/(n-k), \ i = k+1, ..., n \end{cases}$$

$$\text{For } D - \left( \sum_{i=1}^{k} \frac{d_i}{2} + (n-k)\frac{d_{k+1}}{2} \right) \leq E \leq D - \left( \sum_{i=1}^{k} \frac{d_i}{2} + (n-k)\frac{d_k}{2} \right), n-1 \geq k \geq 0$$

$$\overline{x} : \begin{cases} \overline{x}_i = \frac{d_i}{2}, \ i = 1, ..., k \\ \overline{x}_i = d_i - (D - E - \sum_{j=1}^{k} \frac{d_j}{2})/(n-k), \ i = k+1, ..., n \end{cases}$$

---

*Financial support was provided by the CV Starr Center at NYU.

The *characteristic function game* $(N, v(S))$ corresponding to the bankruptcy problem is determined by:

$$v(S) = (E - \sum_{i \in N/S} d_i)_+$$

For any imputation $x$ and coalition $S$, the coalition's *excess* is defined by:

$$e(x, S) = \sum_{i \in S} x_i - v(S)$$

For any imputation write down the vector of excesses $S \subset 2^N$, ordered from smallest excess to largest. The nucleolus of the game is the imputation with the lexicographically largest such vector.

**Theorem 1** *(Aumann and Maschler) The CG-consistent solution of a bankruptcy problem is the nucleolus of the corresponding characteristic function game.*

**Proof.** Consider a coalition $S$ and an imputation $x$. If $E - \sum_{i \in N/S} d_i \geq 0$ then

$e(x, S) = \sum_{i \in S} x_i - [E - \sum_{i \in N/S} d_i]$.

Since $\sum_{i \in N} x_i = E$, we can write

$e(x, S) = \left[E - \sum_{i \in N/S} x_i\right] - [E - \sum_{i \in N/S} d_i] = \sum_{i \in N/S} (d_i - x_1)$.

Thus,

$$\begin{aligned} \text{if } 0 &\leq E - \sum_{i \in N/S} d_i = v(S) \text{ then} \\ e(x, S) &= \sum_{i \in N/S} (d_i - x_1). \end{aligned} \tag{A}$$

On the other hand:

$$\begin{aligned} \text{if } E - \sum_{i \in N/S} d_i &< 0 \text{ then } v(S) = 0 \text{ and} \\ e(x, S) &= \sum_{i \in S} x_i \end{aligned} \tag{B}$$

*i*) Suppose $\sum_{i=1}^{\hat{k}} \frac{d_i}{2} + \left(n - \hat{k}\right) \frac{d_{\hat{k}}}{2} \leq E \leq \sum_{i=1}^{\hat{k}} \frac{d_i}{2} + \left(n - \hat{k}\right) \frac{d_{\hat{k}+1}}{2}$, for some $\hat{k}$, $0 \leq \hat{k} \leq n-1$

For all $i \leq n-1$

$$\begin{aligned} v(i) &\leq \Big(\sum_{j=1}^{n-1} \frac{d_j}{2} + \frac{d_n}{2} - \sum_{j \neq i} d_j\Big)_+ \\ &\leq \Big(\sum_{j=1}^{n-2} \Big(\frac{d_j}{2} - d_j\Big) + \frac{d_{n-1}}{2} + \frac{d_n}{2} - d_n\Big)_+ = 0 \end{aligned}$$

In addition, for $\hat{k} \leq n-2$,

$$v\left(n\right) \leq \Big(\sum_{i=1}^{n-2} \frac{d_i}{2} + \left(2\right) \frac{d_{n-1}}{2} - \sum_{j=1}^{n-1} d_j\Big)_+ = 0$$

Thus, from $B$, for any imputation $x$ :

$$\begin{aligned} e\left(x, i\right) &= x_i, \text{ for } i = 1, 2, ..., n-1 \qquad (1) \\ \text{and } e\left(x, n\right) &= x_n, \text{ if } \hat{k} \leq n-2 \qquad (2) \end{aligned}$$

For $i = 1, 2, ..., \hat{k}$, $\hat{k} \leq n-1$ :

$$v\left(N/i\right) \geq \Big(\sum_{i=1}^{\hat{k}} \frac{d_i}{2} + \left(n - \hat{k}\right) \frac{d_{\hat{k}}}{2} - d_{\hat{k}}\Big)_+ > 0.$$

Thus, from $A$, for any imputation $x$:

$$e\left(x, N/i\right) = d_i - x_i, \text{ for } i = 1, 2, ..., \hat{k}. \tag{3}$$

Finally, note that for $\overline{x}$ :

$$\overline{x}_i \leq \frac{d_i}{2}, \text{ for } i = 1, 2, ..., n \tag{4}$$

Now let $x'$ be an imputation with $x' \neq \overline{x}$, and let $j, 1 \leq j \leq n-1$ be the lowest index in which these two imputations differ. Consider the set of all coalitions $T_r$ such that $e\left(\overline{x}, T_r\right) < \overline{x}_j$.

If $v\left(T_r\right) = 0$ then $e\left(\overline{x}, T_r\right) = \sum_{i \in T_r} \overline{x}_i \geq \overline{x}_i$ for all $i \in T_r$.
Since $e\left(\overline{x}, T_r\right) < \overline{x}_j$, if $i \geq j$ then $i \notin T_r$. (5)
Therefore, $e\left(\overline{x}, T_r\right) = e\left(x', T_r\right)$.

On the other hand:

If $v(T_r) > 0$ then $e(\overline{x}, T_r) = \sum_{i \in N/T_r} (d_i - \overline{x}_i) \geq \sum_{i \in N/T_r} \frac{d_i}{2}$, from 4.
Since $e(\overline{x}, T_r) < \overline{x}_j \leq \frac{d_j}{2}$, if $i \geq j$ then $i \notin N/T_r$. (6)
Therefore, $e(\overline{x}, T_r) = e(x', T_r)$.

From 5 and 6, for each excess of $\overline{x}$ less than $\overline{x}_j$, there is an identical excess of $x'$. We now show that the converse is not true.

Suppose that $j \leq \hat{k}$. If $x'_j < \overline{x}_j$, from 1:

$$e(x', j) = x'_j < \overline{x}_j = e(\overline{x}, j).$$

If $x'_j > \overline{x}_j$, from 3 and recalling that $\overline{x}_j = \frac{d_j}{2}$:

$$e(x', N/j) = d_j - x'_j < \frac{d_j}{2} = \overline{x}_j = e(\overline{x}, N/j).$$

In either case, compared to $\overline{x}$, $x'$ has an additional excess which is less than $\overline{x}_j$ and so $x'$ is lexicographically smaller than $\overline{x}$.

Suppose that $j \geq \hat{k}+1$, so that $\hat{k} \leq n-2$. Then there is a $\hat{j} \geq \hat{k}+1$ with $x'_{\hat{j}} < \overline{x}_{\hat{j}} = \overline{x}_j$. From 1 and 2

$$e\left(x', \hat{j}\right) = x'_{\hat{j}} < \overline{x}_j = e\left(\overline{x}, \hat{j}\right)$$

Again, $x'$ is lexicographically smaller than $\overline{x}$. Hence, $\overline{x}$ is the nucleolus.

*ii)* Suppose $D - \left(\sum_{i=1}^{\hat{k}} \frac{d_i}{2} + \left(n - \hat{k}\right) \frac{d_{\hat{k}+1}}{2}\right) \leq E \leq D - \left(\sum_{i=1}^{\hat{k}} \frac{d_i}{2} + \left(n - \hat{k}\right) \frac{d_{\hat{k}}}{2}\right)$, for some $n-1 \geq \hat{k} \geq 0$.

Define the game $\left(\hat{E}, d\right) \equiv (D-E; d)$, with characteristic function $\hat{v}(S)$ and excess function $\hat{e}(x, S)$. Notice that each imputation $\hat{x}$ of this game corresponds to an imputation $x = d - \hat{x}$ of $(E, d)$, and vice versa.

As before:

$$\hat{e}(x, S) = \begin{cases} \sum_{i \in S} x_i \text{ if } \hat{v}(S) = 0 \\ \sum_{i \in N/S} (d_i - x_i) \text{ if } \hat{v}(S) > 0 \end{cases}$$

Notice that $\widehat{v}\left(S\right)=\left(\sum_{i=1}^{n}d_i-E-\sum_{i\in N/S}d_i\right)_+=\left(\sum_{i\in S}d_i-E\right)_+$, so

$$\begin{aligned}\widehat{v}\left(S\right) &= 0 \text{ if } \left(\sum_{i\in S}d_i-E\right)\leq 0,\\ \widehat{v}\left(S\right) &> 0 \text{ if } \left(\sum_{i\in S}d_i-E\right)>0.\end{aligned}$$

We can write:

$$\widehat{e}\left(\widehat{x},S\right)=\begin{cases}\sum_{i\in S}\widehat{x}_i \text{ if } \left(\sum_{i\in S}d_i-E\right)\leq 0\\ \sum_{i\in N/S}\left(d_i-\widehat{x}_i\right) \text{ if } \left(\sum_{i\in S}d_i-E\right)>0\end{cases}$$

While for $(E,d)$

$$e\left(d-\widehat{x},N/S\right)=\begin{cases}\sum_{i\in N/S}\left(d_i-\hat{x}_i\right) \text{ if } \left(\sum_{i\in S}d_i-E\right)>0\\ \sum_{i\in S}\widehat{x}_i \text{ if } \left(\sum_{i\in S}d_i-E\right)\leq 0\end{cases}$$

Thus, the vector of excesses associated with an imputation $\widehat{x}$ for $\left(\widehat{E},d\right)$ equals the vector of excesses associated with an imputation $x=d-\widehat{x}$ for $(E,d)$; and the nucleolus $\widehat{\overline{x}}$ of $\left(\widehat{E},d\right)$ gives the nucleolus $\overline{x}=d-\widehat{\overline{x}}$ of $(E,d)$. But $D-\left(\sum_{i=1}^{\hat{k}}\frac{d_i}{2}+\left(n-\hat{k}\right)\frac{d_{\hat{k}+1}}{2}\right)\leq E\leq D-\left(\sum_{i=1}^{\hat{k}}\frac{d_i}{2}+\left(n-\hat{k}\right)\frac{d_{\hat{k}}}{2}\right)$, for $\hat{k}$, $n-1\geq\hat{k}\geq 0$ corresponds to $\sum_{i=1}^{\hat{k}}\frac{d_i}{2}+\left(n-\hat{k}\right)\frac{d_{\hat{k}}}{2}\leq\hat{E}\leq\sum_{i=1}^{\hat{k}}\frac{d_i}{2}+\left(n-\hat{k}\right)\frac{d_{\hat{k}+1}}{2}$, for $\hat{k}$, $0\leq\hat{k}\leq n-1$, so that the results of $i$) apply. ■.

COMMENTS

1) The above proof relies on prior knowledge of the nucleolus. The following heuristic shows that in fact the nucleolus is easy to derive.

Imagine increasing $E$ from 0 to $D$. When $E\leq\min_{i\in N}d_i$, $v(S)=0$ for all $S$ . Hence, from $B$, $e(x,S)=\sum_{i\in S}x_i$. The smallest excess is then given by the smallest $x_i$, and so $\overline{x}_i=\frac{E}{n}$ for all $i$.

As $E$ increases beyond $\min_{i\in N}d_i$, some coalitions now have $v(S)>0$. For these coalitions $e(\overline{x},S)=\sum_{i\in N/S}(d_i-\overline{x}_i)$. The smallest such excess is $(d_1-\overline{x}_1)$. When $E=n\frac{d_1}{2}$, for all $i$, $\overline{x}_i=\frac{d_1}{2}=(d_1-\overline{x}_1)=$ the smallest excess. As $E$ increases beyond $n\frac{d_1}{2}$, increasing $x_1$ would cause $(d_1-\overline{x}_1)$ to fall and hence cause a drop in the smallest excess. Therefore $\overline{x}_1$ is held fixed at $\frac{d_1}{2}$ and the other $\overline{x}_i$ are increased. This continues until $\overline{x}_2$ reaches $\frac{d_2}{2}$. At this point increasing $\overline{x}_2$ would cause a drop in the second smallest excess $\overline{x}_2=\overline{x}_3=...=\overline{x}_n=\frac{d_2}{2}=(d_2-\overline{x}_2)$ etc...[1]

---

[1] Note however, that at $E=\frac{d_1}{2}+\frac{d_2}{2}+(n-2)\frac{d_3}{n}$ the third smallest excess is not necessarily $\frac{d_3}{2}$.

The reader will observe that this heuristic bears some resemblance to Aumann and Maschler's characterization of the Contested-Garment consistent solution on page 200.

2) There is a shorter, albeit less elementary, rigorous proof than my main proof. First, it is easy to establish rigorously that as in 1) above :

(*) for $0 \leq E \leq n\frac{d_1}{2}$, $\overline{x}_i = E/n$ for all $i$, and that as $E$ rises above $n\frac{d_1}{2}$, $\overline{x}_1$ is held fixed at $\frac{d_1}{2}$.

Now consider the "reduced game" in which player $i$ is given $\frac{d_1}{2}$ (see Aumann and Maschler, p. 208). It is easily verified that this corresponds to the bankruptcy problem $\left(E - \frac{d_1}{2}; d_2, d_3, ..., d_n\right)$. From the reduced game property of the nucleolus (see Moulin(1988)), (*) applied to $\left(E - \frac{d_1}{2}; d_2, d_3, ..., d_n\right)$immediately implies that in the region $n\frac{d_1}{2} \leq E \leq \frac{d_1}{2}+(n-1)\,\frac{d_2}{2}$, in the game $(E; d, d_2, d_3, ..., d_n)$ players $2, ..., n$ split $E - \frac{d_1}{2}$ evenly, while 1 gets$\frac{d_1}{2}$. Applying the same reasoning repeatedly to successive reduced games yields the nucleolus.

## REFERENCES

Aumann, R. and Maschler, M. (1985) Game Theoretic Analysis of a Bankruptcy Problem from the Talmud, Journal of Economic Theory 36, 195-213

Moulin, H. (1988) Axioms of Cooperative Decision Making.