ECONOMIC RESEARCH REPORTS

LORENZ DOMINANCE AND THE VARIANCE OF LOGARITHMS

by **Efe A. Ok**
and
James Foster

RR# 97-22 **May 1997**

C.V. STARR CENTER FOR APPLIED ECONOMICS

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, NY 10003-6687

Lorenz Dominance and the Variance of Logarithms*

James E. Foster† and Efe A. Ok‡

April 27, 1996

Abstract

The variance of logarithms is a widely used inequality measure which is well known to disagree with the Lorenz criterion. Up to now, the extent and likelihood of this inconsistency were thought to be vanishingly small. We find that this view is mistaken: The extent of the disagreement can be extremely large; the likelihood is far from negligible. Implications of our findings for empirical work are also discussed.

Journal of Economic Literature Classification Numbers: D63, C43.

Key words: Lorenz curves, transfer principle, variance of logarithms, inequality measures.

*We would like to thank Kin Blackburn, Roland Benabou, Gary Fields, Peter Lambert and Bob Margo for insightful discussions, and Artyom Shneyerov for his superb research assistance. Kathryn Anderson kindly provided us with a sample drawn from 1991 CPS database; we are grateful to her. This paper was written while Foster was a visitor in the Department of Economics at New York University, and he would like to thank the department for its hospitality. Support from the C. V. Starr Center for Applied Economics at New York University is also gratefully acknowledged.

†*Correspondence address*: Department of Economics, Vanderbilt University, Nashville, TN 37235. Fax: 615 356 2653. E-mail: fosterje@ctrvax.vanderbilt.edu.

‡Department of Economics, New York University, 269 Mercer St., New York NY 10003. Telephone: 212 998 8920. Fax: 212 995 4186. E-mail: okefe@fasecon.econ.nyu.edu.

1 Introduction

The variance of logarithms V_L is a widely used measure of dispersion, owing in part to its natural link with wage determination models, its useful decomposition formula and its special relationship with the lognormal distribution. It has figured prominently, for instance, in the recent empirical literature on US wage inequality (e.g., Karoly, 1992, Levy and Murnane, 1992, Juhn, Murphy and Pierce, 1993, and Card and Lemieux, 1994), and is the standard measure used in self-selection models of income distribution (e.g., Heckman and Honore, 1990). At the same time there is a longstanding view that V_L is not a proper measure of inequality, due primarily to its clear violation of Dalton's (1920) "principle of transfers" and its consequent disregard for the Lorenz criterion, a ranking generally accepted as the unambiguous arbiter of inequality comparisons.[1]

Further study of the relevant literature suggests that the case against the variance of logarithms is less convincing. Cowell (1977), for example, has shown that an infinitesimal transfer from one person to a richer person decreases the value of V_L only when both incomes are very large (more than three times the geometric mean). Likewise, Creedy (1977, 1985) reports an extremely low estimate of the likelihood of problematic transfers assuming that incomes are lognormally distributed. The prevailing view among users of V_L seems to be that these problematic cases are extremely unlikely and, in any event, are of negligible importance since they only involve incomes in the extreme upper tail of a distribution.[2]

This paper reexamines the case against the variance of logarithms and offers new insights into the extent and likelihood of its disagreement with the Lorenz criterion. Our conclusions are strikingly different than the prevailing view. We find that disagreements between V_L and the Lorenz ordering are consistent with broad-based changes in incomes up and down the distribution (not just at the extreme upper tail). Moreover, they can involve comparisons in which the Lorenz curves are widely divergent from one another, indicating that the "Lorenz-inconsistency" of V_L is a more serious problem than suggested by the marginal transfer analysis. In fact, one can show that the following "worst case scenerio" is possible: Two distributions with comparable Lorenz curves may be such that the first Lorenz curve approximates the 45% line of equality and the second approximates the L-shaped curve of complete inequality, and yet V_L wrongly indicates that the first is *more* unequal than the second.

We next reevaluate the likelihood that V_L will produce conclusions which go against the Lorenz criterion. Our second series of results show that the widely quoted estimate "less

[1] See, for example, Atkinson (1970) and Sen (1997).

[2] See, for example, Levy and Murnane (1992, p. 1339) who cite Cowell's (1977) conclusions and Heckman and Honore (1990, fn. 6) who refer to Creedy's (1977) results. For a related discussion of this issue in the context of measures of industrial concentration, see Sawyer (1979) and Hart (1980).

tham 1%" of the likelihood of problematic marginal transfers is considerably flawed. In fact, standard statistical distributions fitted to US data (cf. Singh and Maddalla, 1976, and McDonald, 1984), as well as a more recent sample drawn from the CPS, yield likelihood estimates which range roughly from 8% to 12%. Therefore, the Lorenz inconsistency of V_{L} cannot be readily dismissed as a problem which, for all practical purposes, never arises.[3]

2 Definitions

We focus on income distributions that can be represented as a vector $\mathbf{x} = (x_1, ..., x_n) \in \mathbf{R}^n_{++}$, where n is any positive population size, and x_i is the level of income of the ith individual. Associated with $\mathbf{x}$ is a *distribution function* $F_{\mathbf{x}}$, which, for each s, indicates the share $F_{\mathbf{x}}(s)$ of the population receiving an income of s or below. The *Lorenz curve* $L_{\mathbf{x}}$ gives the cumulative share of income $L_{\mathbf{x}}(p)$ held by the poorest cumulative share p of the population. Noting that $F^{-1}_{\mathbf{x}}(p)$ is the income of the person whose position in the distribution is p, we see that

$$L_{\mathbf{x}}(p) \equiv \frac{1}{\mu_{\mathbf{x}}} \int_0^p F^{-1}_{\mathbf{x}}(t)\mathrm{d}t, \quad 0 < p < 1$$

defines the Lorenz curve of the distribution $\mathbf{x}$, where $\mu_{\mathbf{x}}$ is the mean of the distribution.[4] We say that $\mathbf{x}$ *Lorenz dominates* $\mathbf{y}$, denoted by $\mathbf{x}\ L\ \mathbf{y}$, whenever $L_{\mathbf{x}}(p) \geq L_{\mathbf{y}}(p)$ holds for all $p \in (0,1)$, with strict inequality for some p. We shall concentrate exclusively here on the case where $\mathbf{x}$ and $\mathbf{y}$ have the same population size n and the same mean $\mu_{\mathbf{x}} = \mu_{\mathbf{y}}$ in which case Lorenz dominance reduces to the simple partial sum condition

$$\sum_{k=1}^{r} \hat{x}_k \geq \sum_{k=1}^{r} \hat{y}_k$$

for all $r = 1, ..., n$, with a strict inequality holding for some r, where $\hat{\mathbf{x}} = (\hat{x}_1, ..., \hat{x}_n)$ is a permutation of $\mathbf{x}$ such that $\hat{x}_1 \leq \cdots \leq \hat{x}_n$, and similarly for $\hat{\mathbf{y}}$.

An *inequality measure* I is a mapping from the set of all income distributions of arbitrary size to the reals, where $I(\mathbf{x})$ denotes the level of inequality associated with the distribution $\mathbf{x}$. We say that I is *Lorenz consistent* if $\mathbf{x}\ L\ \mathbf{y}$ implies that $I(\mathbf{x}) < I(\mathbf{y})$ for any two distributions $\mathbf{x}$ and $\mathbf{y}$. Two commonly used inequality measures, *Gini coefficient* and the *coefficient of variation,* are defined as

$$G(\mathbf{x}) \equiv \frac{1}{2n^2\mu_{\mathbf{x}}} \sum_{k=1}^{n}\sum_{l=1}^{n} |x_k - x_l| \quad \text{and} \quad CV(\mathbf{x}) \equiv \frac{1}{\mu_{\mathbf{x}}} \sqrt{\sum_{k=1}^{n} (x_k - \mu_{\mathbf{x}})^2},$$

[3] In view of our results, it is not surprising that V_{L} is actually observed to disagree with the Lorenz curve analysis in a number of empirical studies; see, for instance, Creedy and Gemmell (1984).

[4] Formally speaking, $F^{-1}_{\mathbf{x}}(p) \equiv \inf_{s \geq 0}\{F_{\mathbf{x}}(s) \geq p\}$, and the Lorenz curve of $\mathbf{x}$ is the *graph* of the function $L_{\mathbf{x}}$ (Gastwirth, 1971).

respectively. On the other hand, we define *Theil's second measure* as $T_2(\mathbf{x}) \equiv \ln \mu_{\mathbf{x}} - \ln g_x$, where $g_{\mathbf{x}} \equiv \left(\prod_{k=1}^{n} x_k\right)^{1/n}$ is the geometric mean of $\mathbf{x}$. It is readily observed that the inequality measures G, CV and T_2 are Lorenz consistent.

The *variance* of $\mathbf{x}$ is defined as $V(\mathbf{x}) \equiv \frac{1}{n}\sum_{k=1}^{n}\left(x_k - \mu_{\mathbf{x}}\right)^2$, while the *variance of logarithms*, written as V_{L}, is the variance applied to the distribution of log incomes. In other words, where $\ln \mathbf{x} \equiv (\ln x_1, ..., \ln x_n)$, we have $V_{\mathrm{L}}(\mathbf{x}) \equiv V(\ln \mathbf{x}) = \frac{1}{n}\sum_{k=1}^{n}(\ln x_k - \mu_{\ln \mathbf{x}})^2$. It is easy to see that the mean of the distribution of log incomes, $\mu_{\ln \mathbf{x}}$, is the log of the geometric mean. Consequently, the variance of logarithms can be expressed as

$$V_{\mathrm{L}}(\mathbf{x}) = \frac{1}{n}\sum_{k=1}^{n} f(x_k, g_{\mathbf{x}}),$$

where $f(s,g) \equiv (\ln s - \ln g)^2$.

3 The Problem of Lorenz Inconsistency

3.1 Preliminary Analysis

Consider any two distributions $\mathbf{x}$ and $\mathbf{y}$ for which $\mathbf{x}\ L\ \mathbf{y}$. We are interested in cases where $V_{\mathrm{L}}(\mathbf{x}) > V_{\mathrm{L}}(\mathbf{y})$, so that the variance of logarithms is inconsistent with the Lorenz criterion, or equivalently $\triangle V_{\mathrm{L}} \equiv V_{\mathrm{L}}(\mathbf{x}) - V_{\mathrm{L}}(\mathbf{y})) > 0$. In general, the change in V_{L} can be expressed as

$$\triangle V_{\mathrm{L}} = \frac{1}{n}\sum_{k=1}^{n}\left(f(x_k, g_{\mathbf{y}}) - f(y_k, g_{\mathbf{y}})\right) + \frac{1}{n}\sum_{k=1}^{n}\left(f(x_k, g_{\mathbf{x}}) - f(x_k, g_{\mathbf{y}})\right) \tag{1}$$

when the first term is the "direct" effect holding the geometric mean fixed and the second is the "indirect" effect via the change in the geometric mean. In the special case where $\mathbf{x}$ is obtained from $\mathbf{y}$ by a single discrete transfer (of size $\delta > 0$) from a richer individual (say person h for *high*) to a poorer individual (say person ℓ for *low*), this difference reduces to

$$\begin{aligned}\triangle V_{\mathrm{L}} = \frac{1}{n}&\left(\left(f(y_\ell + \delta, g_{\mathbf{y}}) - f(y_\ell, g_{\mathbf{y}})\right) - \left(f(y_h, g_{\mathbf{y}}) - f(y_h - \delta, g_{\mathbf{y}})\right)\right) \\ &+ \frac{1}{n}\sum_{k=1}^{n}\left(f(y_k(\delta), g_{\mathbf{y}(\delta)}) - f(y_k(\delta), g_{\mathbf{y}})\right) \end{aligned}\tag{2}$$

where the post-transfer distribution $\mathbf{y}(\delta) \equiv \mathbf{x}$ and its geometric mean $g_{\mathbf{y}(\delta)}$ depend on δ. Finally, dividing both sides by δ and letting $\delta \to 0$, we obtain the *rate* at which V_{L} changes as a result of a marginal transfer,

$$\lim_{\delta\to 0}\frac{\triangle V_{\mathrm{L}}}{\delta} = \frac{1}{n}\left(f_1(y_\ell, g_{\mathbf{y}}) - f_1(y_h, g_{\mathbf{y}}) + \left(\frac{\partial g_{\mathbf{y}}}{\partial y_h} - \frac{\partial g_{\mathbf{y}}}{\partial y_\ell}\right)\sum_{k=1}^{n} f_2(y_k, g_{\mathbf{y}})\right),$$

where use has been made of l'Hospital's rule.[5] Note that since $\sum_{k=1}^{n} f_2(y_k, g_{\mathbf{y}}) = 0$, the last term of this expression vanishes, and so

$$\lim_{\delta \to 0} \frac{\Delta V_{\mathrm{L}}}{\delta} = \frac{1}{n}\left(f_1(y_\ell, g_{\mathbf{y}}) - f_1(y_h, g_{\mathbf{y}})\right) \tag{3}$$

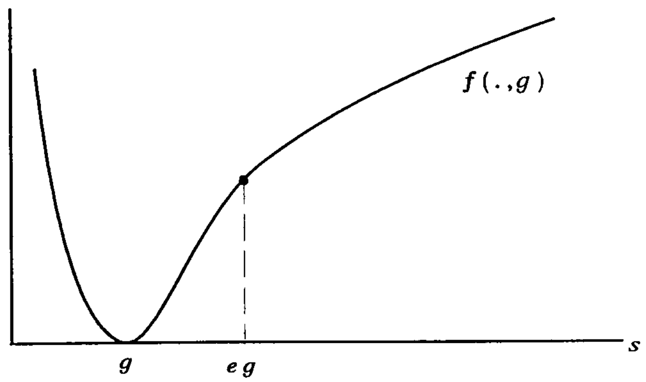
describes the effect of a marginal transfer.

In the present case, where $\mathbf{x}$ and $\mathbf{y}$ have identical population sizes and means, $\mathbf{x}\ L\ \mathbf{y}$ holds if and only if $\mathbf{x}$ can be obtained from $\mathbf{y}$ by a finite number of *progressive* transfers (from richer individuals to poorer ones).[6] Therefore, expression (1) describes the cumulative effect of these transfers on V_{L}. In contrast, expression (2) corresponds to the special case of a single progressive transfer, while expression (3) pertains to the limiting case where the magnitude of the progressive transfer is arbitrarily small.

The final case is clearly the most tractable. Indeed, the sign of (3) depends only on the slope of $f(\cdot, g)$ at y_ℓ and y_h, with $f_1(y_\ell, g_{\mathbf{y}}) < f_1(y_h, g_{\mathbf{y}})$ signalling Lorenz consistency and the reverse inequality indicating a violation. (The graph of $f(\cdot, g)$ is depicted in Figure 1.)

It is easy to see that $f(\cdot, g)$ is decreasing on $(0, g]$ and increasing on (g, ∞) while it is convex on $(0, eg]$ and concave on (eg, ∞).[7] If y_ℓ and y_h are both in $(0, eg]$, then the convexity of f ensures that $f_1(y_\ell, g_{\mathbf{y}}) < f_1(y_h, g_{\mathbf{y}})$; and if y_ℓ is in $(0, g]$ with y_h in (g, ∞), then $f_1(y_\ell, g_{\mathbf{y}})$

FIGURE 1



[5] Alternatively, this rate can be found as $\left.\frac{dV_{\mathrm{L}}(\mathbf{y}(\delta))}{d\delta}\right|_{\delta=0}$ where $V_{\mathrm{L}}(\mathbf{y}(\delta)) = \frac{1}{n}(f(y_\ell + \delta; g_{\mathbf{y}(\delta)}) + f(y_h - \delta; g_{\mathbf{y}(\delta)})) + \frac{1}{n}\sum_{k \neq h,\ell} f(y_k; g_{\mathbf{y}(\delta)})$. See Cowell (1977, p. 163).

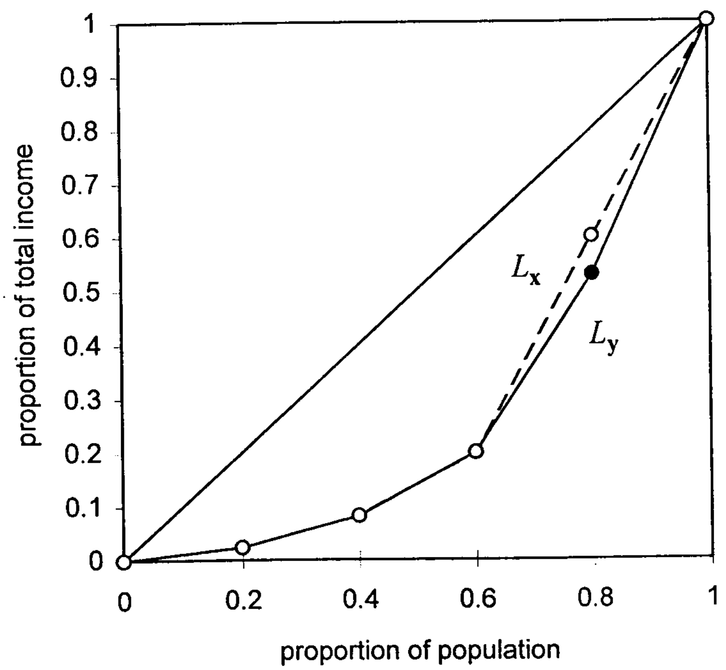
[6] This is a key result in the theory of inequality measurement. See, for example, Kolm (1969), Dasgupta, Sen and Starrett (1973), Fields and Fei (1978) or Foster (1985).

[7] Since $f_1(s,g) = 2(\ln s - \ln g)/s$ and $f_{11}(s,g) = 2(\ln eg - \ln s)/s^2$, we have $f_1(s,g) \gtreqless 0$ iff $s \gtreqless g$ and $f_{11}(s,g) \gtreqless 0$ iff $s \lesseqgtr eg$.

$< 0 < f_1(y_h, g_{\mathbf{y}})$, once again confirming the transfer principle. However, if both incomes lie within (eg, ∞), we obtain $f_1(y_\ell, g_{\mathbf{y}}) > f_1(y_h, g_{\mathbf{y}})$ contrary to both the transfer principle and Lorenz consistency. Finally, the remaining case where y_ℓ is in (g, eg) and y_h is in (eg, ∞) can go either way.

As an illustration, consider the distribution $\mathbf{y} = (2, 5, 10, 28, 40)$. Since $g_{\mathbf{y}} = 10.229 > 10$ and $eg_{\mathbf{y}} = 27.721 < 28$, the only possible violation of the transfer principle due to a marginal transfer involves the final two incomes. Indeed, $f_1(y_4, g_{\mathbf{y}}) = 0.071 > 0.068 = f_1(y_5, g_{\mathbf{y}})$, and so a transfer from person 5 to person 4 increases V_{L}, which is a violation of the transfer principle and hence Lorenz consistency. But how significantly can V_{L} disagree with the Lorenz criterion? The marginal transfer approach is of course not suitable for addressing this question; we should rather consider the discrete transfer case. In the context of the example above, the question is readily answered. One can easily show that to achieve the maximum upward shift in the Lorenz curve of $\mathbf{y}$ compatible with an increase in V_{L} by means of a single transfer, we must equalize the incomes of persons 4 and 5 (i.e. choose $\delta = 6$). The result is that V_{L} rises slightly (from 1.2100 to 1.2124) while the resulting upward shift in the Lorenz curve is modest. (See Figure 2.) Indeed, the associated transfer results in only a 0.028 change in the Gini coefficient (which is a measure of the area contained between the two Lorenz curves). This is typical of examples involving a single transfer.

FIGURE 2



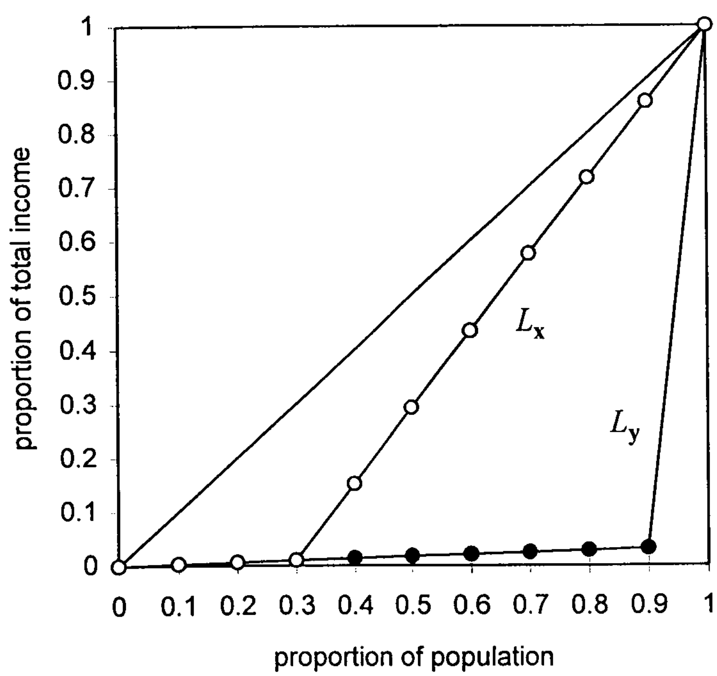
3.2 A Closer Look

In practice, distributional changes are more general in structure, involving *sequences* of transfers of varying sizes to obtain $\mathbf{x}$ from $\mathbf{y}$. This in turn requires the analysis to be conducted in terms of expression (1). Consider, for example, the situation depicted in Figure 3. As before, $\mathbf{x}\ L\ \mathbf{y}$, but this time $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of *six* progressive transfers. Figure 3 makes it clear that the depicted Lorenz dominance is broad-based, as the curves depart for 70% of the population, and quite extensive since the area between them is very large. As noted in Table 1, the Gini coefficient, the coefficient of variation and Theil's second measure strongly support the judgement of the Lorenz ranking. The Gini coefficient, in particular, rises by almost 300% indicating that the area between the line of perfect equality and $L_{\mathbf{y}}$ is three times larger than the area between the line of perfect equality and $L_{\mathbf{x}}$.

TABLE 1: *Example distributions*

Distribution	μ	$G(.)$	$CV(.)$	$T_2(.)$
$\mathbf{x} = (1, 1, 1, 41, 41, 41, 41, 41, 41, 41)$	29	0.29	0.40	0.77
$\mathbf{y} = (1, 1, 1, 1, 1, 1, 1, 1, 1, 281)$	29	0.87	8.39	2.80

FIGURE 3



Yet, the V_{L} declares $\mathbf{x}$ to be *more unequal than* $\mathbf{y}$: $V_{\mathrm{L}}(\mathbf{x}) = 2.90 > 2.86 = V_{\mathrm{L}}(\mathbf{y})$. This is indeed a remarkably powerful example of the Lorenz inconsistency of the variance of logarithms.[8]

We will presently investigate the reason for this surprising case of inconsistency. But, first, we note that its main point can be substantially generalized.

Proposition: *For any* $1 > \epsilon > 0$, *there exist two income distributions* $\mathbf{x}$ *and* $\mathbf{y}$ *satisfying* $G(\mathbf{x}) < \epsilon$ *and* $G(\mathbf{y}) > 1 - \epsilon$ *for which*

$$\mathbf{x}\ L\ \mathbf{y} \quad \text{and} \quad V_{\mathrm{L}}(\mathbf{x}) > V_{\mathrm{L}}(\mathbf{y}).$$

Proof. See Appendix.

This proposition says that the variance of logarithms is not only Lorenz inconsistent, but it can disagree with the Lorenz criterion in such a radical way that it misranks two distributions which are deemed to be essentially perfectly equal and perfectly unequal by the Gini index of inequality (and the Lorenz criterion).[9] In this sense, we must conclude that there is no limit to the disagreement between V_{L} and the Lorenz ordering of income distributions.

Why does V_{L} have these unfortunate characteristics? In other words, what is the driving force behind the above proposition? To see this, let us go back to the example we introduced in the beginning of this subsection (see Table 1). It is easy to see that the geometric mean rises in going from $\mathbf{y}$ to $\mathbf{x}$, which is only natural given the fact that the geometric mean is a Lorenz consistent equality measure for comparing distributions with fixed average income. Now, to understand how important this change in the geometric mean is, let us look back to (1), and note that

$$\begin{aligned} V_{\mathrm{L}}(\mathbf{x}) - V_{\mathrm{L}}(\mathbf{y}) &= \frac{1}{10}\sum_{k=1}^{10}\left(f(x_k, g_{\mathbf{y}}) - f(y_k, g_{\mathbf{y}})\right) + \frac{1}{10}\sum_{k=1}^{10}\left(f(x_k, g_{\mathbf{x}}) - f(x_k, g_{\mathbf{y}})\right) \\ &= (2.90 - 7.01) + (7.01 - 2.86)\,. \end{aligned}$$

[8] It is interesting to note that none of the incomes in $\mathbf{y}$ lie in the problematic region which is determined by the standard marginal transfer approach. This observation underscores how misleading the conclusions of the marginal transfer analysis can be when transfers are large.

[9] This proposition highlights the potentially extreme disagreement between the Lorenz criterion and V_{L} and not just an incompatibility between V_{L} and another single measure (like the Gini). Indeed, we can adapt the proof in the appendix to show the following result for Theil's second measure: For any $\epsilon > 0$, there exist two income distributions $\mathbf{x}$ and $\mathbf{y}$ satisfying $T_2(\mathbf{x}) < \epsilon$ and $T_2(\mathbf{y}) > 1/\epsilon$ for which $\mathbf{x}\ L\ \mathbf{y}$ and $V_{\mathrm{L}}(\mathbf{x}) > V_{\mathrm{L}}(\mathbf{y})$.

This decomposition clearly shows that while the constituent transfers decrease the variance of log incomes evaluated at the original geometric mean, the resultant change in g raises V_{L} back above its original level. In other words, the *direct* effect the transfers have on the income levels of the individuals is here outweighed by their *indirect* effect via the geometric mean. Consequently, this example points to variations in the geometric mean as a main culprit in the massive disagreement between V_{L} and the Lorenz criterion.

An interesting implication of this observation is that such large disagreements might be avoided if the variation in the geometric mean stayed within certain bounds. In the case of the above example, for instance, the effect of progressive transfers would be to decrease V_{L}, for the direct effect of the transfers (which agrees with the Lorenz criterion) would then be stronger than their indirect effect on the geometric mean (which disagrees with the Lorenz criterion). Given the definition of Theil's second measure as $T_2(\mathbf{x}) = \ln \mu_{\mathbf{x}} - \ln g_x$, there may well be grounds for using V_{L} in conjunction with T_2 in practice, a point which we shall revisit in the concluding section.

4 The Likelihood of Lorenz Inconsistency

The above analysis has documented the unusual extent to which V_{L} may disagree with the Lorenz criterion. We now briefly address the related question: *How likely are V_{L}'s Lorenz inconsistencies?* Put differently, the problem is to determine the likelihood of $V_{\mathrm{L}}(\mathbf{x}) > V_{\mathrm{L}}(\mathbf{y})$ given $\mathbf{x}\ L\ \mathbf{y}$, or more precisely, to find the conditional probability of a reversal given that Lorenz dominance holds. This question is, however, difficult to formulate at this level of generality, for it is not readily evident how one can construct the joint probability distribution over all possible income distributions. Rather than attempting to provide an exhaustive solution to the problem at hand, therefore, we shall follow the transfer-based approach of Creedy (1977, 1985).

Suppose that two incomes, s and t, are chosen at random from a given probability distribution which is represented by the distribution function $F(\cdot)$. The question we ask is this: Given that a *small* transfer takes place from s to t, what is the probability, π, that the variance of logarithms will disagree with the principle of transfers? In other words, what is the probability of the two events $[\triangle V_{\mathrm{L}} < 0$ and $s < t]$ and $[\triangle V_{\mathrm{L}} > 0$ and $s > t]$? (We shall henceforth refer to π as the *probability of reversal.*) This approach, of course, ignores the difficulties brought on by multiple transfers including the shift of the geometric mean described above. However, it has the virtue of tractability: we can derive probabilities for this special case.

In view of the discussion presented in the latter half of Subsection 3.1 (recall Figure 1), it is easy to obtain some bounds for π. If both incomes s and t lie above the income level eg, then the marginal transfer will necessarily shift V_{L} in the wrong direction. Hence a *lower bound* for π is $(1-F(eg))^2$. Analogously, we have found earlier that any time one of the incomes was below income level g, a marginal transfer could not shift V_{L} in the wrong direction. Therefore, $\min\{s,t\} > g$ is a necessary condition for V_{L} to violate the transfer principle, and hence, $(1-F(g))^2$ is an *upper bound* for π. To summarize:

$$(1-F(g))^2 \geq \pi \geq (1-F(eg))^2 .$$

The only remaining possibility is that one of the incomes, say s, lies in the range (g, eg) while $t > eg$, and the inequality $f_1(s,g) > f_1(t,g)$ holds (recall (3)). Now $f_1(\cdot, g)$ increases from 0 to $f_1(eg,g)$ on (g,eg), and it decreases from $f_1(eg,g)$ to 0 on (eg,∞). Therefore, for every s in (g,eg) there exists some $h(s)$ in (eg,∞) such that $f_1(s,g) = f_1(h(s),g)$, and hence $f_1(s,g) > f_1(t,g)$ holds precisely when $t > h(s)$. Consequently,

$$\pi = (1-F(eg))^2 + 2\int_g^{eg} (1-F(h(w)))\,\mathrm{d}F(w) \tag{4}$$

provides a straightforward way of computing the probability of reversals associated with F. Of course, a closed form solution may not be possible, in which case well-known numerical approximation methods can be brought to bear.[10]

Table 2 reports the probability of reversal π for the lognormal distribution, the Singh-Maddala and the generalized beta of the second kind (beta-II), all fitted to 1970 U.S. family income data (cf. Singh and Maddala, 1976, and McDonald, 1984), along with the respective lower and upper bounds discussed above.[11] As can be seen from this table, the likelihood of choosing incomes associated with a problematic marginal transfer is higher than previous results would indicate. For example, in the log-normal case, we find an 8.4% probability of reversal as opposed to the widely quoted "less than 1%" estimate of Creedy (1977, 1985).

[10]The results reported below are all obtained using Mathematica® subprograms, the details of which are available from the authors upon request.

[11]The density functions of the distributions of log-normal and beta-II are defined on $\mathbf{R}_{++}$ as

$$f_{LN}(x;\mu,\sigma) \equiv \frac{1}{x\alpha\sqrt{2\pi}}\exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2}\right] \quad \text{and} \quad f_{B2}(x;a,b,p,q) \equiv \frac{ax^{ap-1}}{b^{ap}B(p,q)(1+(y/b)^a)^{p+q}},$$

respectively, where $B(p,q)$ is the beta function. The density of the Singh-Maddala distribution is, on the other hand, defined as $f_{SM}(x;a,b,q) \equiv f_{B2}(x;a,b,1,q)$.

TABLE 2: Findings on the probability of reversal for various distributions approximating the 1970 U.S. family income distribution

Distribution	estimated parameters*	lower bound	π	upper bound
Lognormal	$\mu = 2.1924$, $\sigma = 0.6977$	0.0067	0.0840	0.2661
Singh-Maddala	$a = 1.9652$, $b = 18.7288$ $q = 2.9388$	0.0031	0.0917	0.2897
Beta-II	$a = 5.0573$, $b = 13.5815$ $p = 0.2961$, $q = 0.6708$	0.0031	0.1014	0.3315

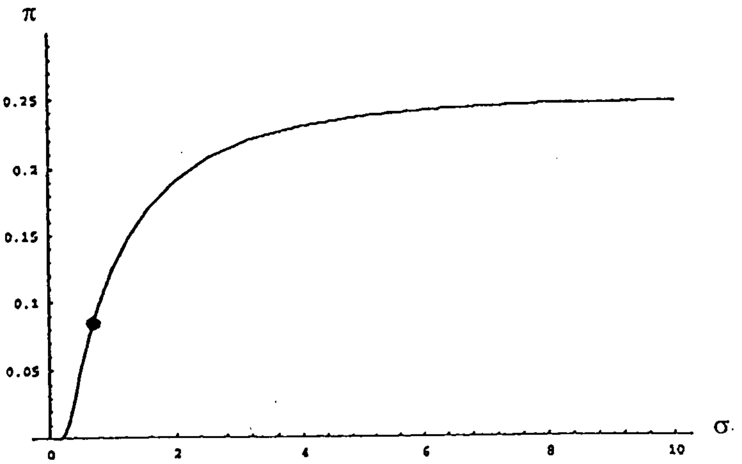
* Source: McDonald, 1984, p. 656, Table III.

The Singh-Maddala distribution (which provides a better fit to the actual 1970 family income distribution than the lognormal) has 9.1% chance of yielding Lorenz inconsistent transfers, while the beta-II distribution (which approximates the actual income distribution even better) yields a π on the order of 10%, certainly a considerable margin of error. These results are also robust to changes in the parameter values. Figure 4, for example, depicts the graph of the probability of reversal as a function of V_{L} (i.e. σ) for the lognormal distribution where μ is fixed at its 1970 level. (The thick point on the graph corresponds to the value of π reported in Table 2.) As can easily be seen from this figure, the probability of reversal exceeds 10% when $V_{\mathrm{L}} > 1$, and monotonically approaches to its upper bound of 26.6% as V_{L} increases.

One remaining question is whether these high values for π arise from the particular functional forms of the probability distributions used above to approximate the actual distribution of incomes. To check this, we applied our method directly to a sample drawn from the 1991 Current Population Survey,[12] and found that the probability of reversal π for the sample is even larger than the previous findings: 12.84%.

[12]The sample distribution consists of 13720 individuals, and excludes self-employed workers, government officials, and certain other groups.

FIGURE 4



In view of these results, we must conclude that, even without considering simultaneous discrete transfers or changes in the geometric mean, the likelihood of the event that variance of logarithms goes counter the Lorenz criterion is rather significant. In the final analysis, it appears extremely difficult to agree with the statement that "the fact that the variance of logarithms *can* violate the principle of transfers need be regarded as no more than a *curiosum*." [Creedy (1977, p. 157).]

5 Conclusion

In this paper, we have evaluated the extent and likelihood of Lorenz reversals associated with the variance of logarithms. Our results show that extreme cases of inconsistency with the Lorenz criterion are indeed possible and, moreover, that such reversals have a fair chance of occuring. The conclusion to be drawn is that, except in certain special contexts,[13] the potential costs of using V_L as a measure of inequality are nontrivial and, inasmuch as there are many well-accepted Lorenz-consistent indicators of inequality available, the use of V_L is hardly justified. Even so, there are a number of beneficial characteristics of the variance of logarithms - such as its decomposition and its ready link with earnings equations.[14] To be sure, many researchers are familiar with the measure and may continue using it for that reason alone. However, if one decides to employ the measure despite its potential drawbacks, we would at least hope that it would not be the only one presented. Rather, it should be

[13] On special case where V_L makes sense is when all distributions under consideration have the lognormal form (e.g., Benabou, 1996, and Glomm and Ravikumar, 1992). In this case V_L is Lorenz consistent, and in fact rises *exacty* when the Lorenz curve falls.

[14] On this, see Foster and Sen (1997) and Foster and Shneyerov (1997).

used in conjunction with other measures, both to evaluate the robustness of conclusions and to help identify potential Lorenz reversals (which can only occur where V_{L} and a Lorenz consistent measure disagrees). In addition, as we noted above, one may get some added insight on likely situations of reversal by monitoring changes in the geometric mean of the distribution. Since the second Theil measure is both a Lorenz consistent measure and one that is based on the geometric mean, it may well be a natural companion to use with V_{L}.

We conclude on a cautionary note. Our treatment up to now seems to suggest that the use of V_{L} is problematic only when it disagrees with the Lorenz criterion, and that if reversals of this sort could be ruled out in a given situation, then its use would be entirely proper. In fact this is *not* the message we are intending to send. For if a measure makes large mistakes in cases where Lorenz curves do not cross, how can we be sure that its judgment over the remaining cases is reasonable? This is a serious problem whether one is concerned only with the ordering induced by V_{L}, or with the cardinal values upon which regression analysis depends. For example, suppose that we find that inequality (as measured by V_{L}) has risen slightly in going from one distribution $\mathbf{x}$ to another $\mathbf{y}$. Even if this change is not itself inconsistent with the Lorenz criterion, it may be the case that its magnitude has been dampened by the Lorenz inconsistency of V_{L} at a nearby (unobserved) pair of distributions, $\mathbf{x}^*$ and $\mathbf{y}^*$, for which $\mathbf{x}^* \ L \ \mathbf{y}^*$ and $V_{\mathrm{L}}(\mathbf{x}^*) > V_{\mathrm{L}}(\mathbf{y}^*)$. Even if the observations themselves contain no instances of reversals, the presence of (unobserved) cases nearby may affect their measured inequality levels. This could clearly affect empirical conclusions or at the very least make their interpretation less clear.

6 Appendix: *Proof of the Proposition*

For any integer $n \geq 3$ and any integer $1 \leq m < n$, define

$$\mathbf{x}^{m,n} \equiv (\underbrace{1, ..., 1}_{m \text{ times}}, \underbrace{\theta, ..., \theta}_{n-m \text{ times}})$$

where $\theta \equiv \dfrac{nB}{n-m} + 1$ for an arbitrary $B > 0$.[15] By direct computation, we find

$$G(\mathbf{x}^{m,n}) = \frac{B}{B+1}\left(\frac{m}{n}\right). \tag{5}$$

and

$$V_{\mathrm{L}}(\mathbf{x}^{m,n}) = \frac{m}{n}\left(1 - \frac{m}{n}\right)\left(\ln\left(\frac{B}{1-(m/n)} + 1\right)\right)^2. \tag{6}$$

Now define

$$\varphi(\alpha, B) \equiv \alpha(1-\alpha)\left(\ln\left(\frac{B}{1-\alpha} + 1\right)\right)^2 \qquad \text{for all } 1 > \alpha > 0 \text{ and } B > 0.$$

Four observations about this function $\varphi(\alpha, B)$ are in order. First, notice that an immediate application of the l'Hospital's rule yields $\varphi(0, B) = \varphi(1, B) = 0$ for all $B > 0$. Second, we have

$$\varphi(\alpha, B) < \varphi(1-\alpha, B) \qquad \text{for all } \frac{1}{2} > \alpha > 0 \text{ and } B > 0 \tag{7}$$

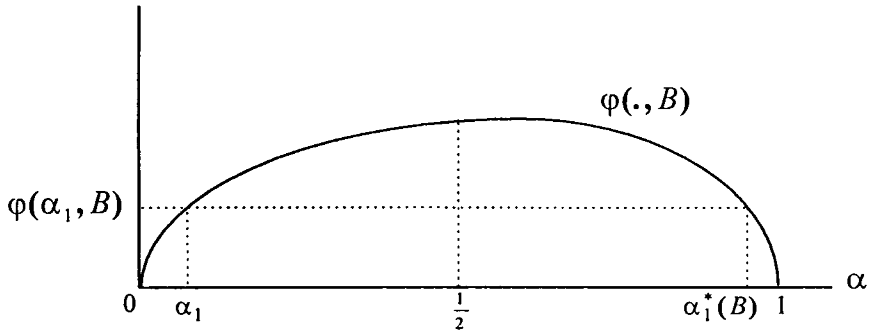
as can be readily verified. Third,

$$\frac{\partial\varphi(\alpha, B)}{\partial\alpha} = \left((1-2\alpha)\ln\left(\frac{B}{1-\alpha} + 1\right) + \frac{2\alpha B}{B + (1-\alpha)}\right)\ln\left(\frac{B}{1-\alpha} + 1\right)$$

so that, given any $B > 0$, φ is a strictly increasing function of α on $[0, 1/2]$. Finally, it is clear that for any $\bar{\alpha} > 1/2$, there exists a $\bar{B} > 0$ large enough to guarantee that $\frac{\partial\varphi(\alpha,B)}{\partial\alpha} < 0$ for all $\alpha \geq \bar{\alpha}$ and $B \geq \bar{B}$. (A generic $\varphi(., B)$ is depicted in Figure 5.)

[15] This construction is also used in the example of Subsection 3.2, and can be interpreted as follows. The initial distribution $\mathbf{x}^{0,n}$ has n persons with $1 + B$ dollars. (Intuitively, each person is receiving a subsistence level of income 1 along with a surplus $B > 0$.) Now suppose that all but one of the rich persons "gang up" on the remaining person, confiscating the surplus and redistributing it evenly among themselves. Then the resulting distribution $\mathbf{x}^{1,n}$ will have one person with income 1, while the remaining $n-1$ persons receive $1 + \frac{nB}{n-1}$. Repeating this process m times, we obtain a distribution $\mathbf{x}^{m,n}$ in which m persons receive 1 and $n - m$ receive $1 + \frac{nB}{n-m}$. Note that each step entails several simultaneous regressive transfers, resulting in unambiguously greater inequality and a downward shifted Lorenz curve.

FIGURE 5



Now pick any $0 < \epsilon < 1$, and choose any $0 < \alpha_1 < \epsilon$. By using the last observation on φ noted above, we may find a $B_1 > 0$ such that

$$\varphi(., B) \text{ is strictly decreasing on } [1-\alpha_1, 1] \text{ for all } B \geq B_1. \tag{8}$$

But then, using the intermediate value theorem, we may conclude that there must exist a $1/2 < \alpha_1^*(B) < 1$ such that

$$\varphi(\alpha_1, B) = \varphi(\alpha_1^*(B), B) \quad \text{for all } B \geq B_1.$$

Moreover, by (7) and (8), we must have $1 - \alpha_1 < \alpha_1^*(B)$ for all $B \geq B_1$. Now take any $\delta < \epsilon$ and define

$$\alpha_2(B) \equiv \max\left\{\alpha_1^*(B) + \frac{1}{2}(1 - \alpha_1^*(B)), (1-\epsilon) + \delta\right\} \quad \text{for all } B > 0.$$

Clearly, $1 > \alpha_2(B) > \alpha_1^*(B) > 1 - \alpha_1$ (for $B \geq B_1$) so that, by (8), we have

$$\varphi(\alpha_2(B), B) < \varphi(\alpha_1^*(B), B) \quad \text{for all } B \geq B_1. \tag{9}$$

In addition, it is clear that there must exist $B_2 > 0$ such that

$$\frac{B}{B+1}\alpha_2(B) \geq \frac{B}{B+1}((1-\epsilon)+\delta) > 1 - \epsilon \quad \text{for all } B \geq B_2. \tag{10}$$

We now let $\hat{B} = \max\{B_1, B_2\}$ and notice that, by choosing n large enough, we may find m_1 and m_2 such that $\frac{m_1}{n} \approx \alpha_1$ and $\frac{m_2}{n} \approx \alpha_2(\hat{B})$. But then letting $\mathbf{x} \equiv \mathbf{x}^{m_1,n}$ and using (5) we obtain

$$G(\mathbf{x}) \approx \frac{\hat{B}}{\hat{B}+1}\alpha_1 < \epsilon$$

and by letting $\mathbf{y} \equiv \mathbf{x}^{m_2,n}$, we find

$$G(\mathbf{y}) \approx \frac{\hat{B}}{\hat{B}+1}\alpha_2(\hat{B}) > 1 - \epsilon$$

by (5) and (10). Moreover, (6) and (9) yield that

$$V_{\mathrm{L}}(\mathbf{y}) \approx \varphi(\alpha_2(\hat{B}), \hat{B}) < \varphi(\alpha_1^*(\hat{B}), \hat{B}) = \varphi(\alpha_1, \hat{B}) \approx V_{\mathrm{L}}(\mathbf{x}).$$

Finally, note that since $\alpha_1 < \alpha_2(\hat{B})$, we must have $m_1 < m_2$, and hence by construction, $\mathbf{x}\ L\ \mathbf{y}$. The proof is completed by a standard continuity argument using the denseness of rationals in the reals. □

References

[1] Atkinson, A. B. (1970), On the measurement of inequality, *Journal of Economic Theory*, **2**, 244-63.

[2] Benabou, R. (1996), Unequal societies, C.V. Starr Center Working Paper 96-17, New York University.

[3] Card, D. and T. Lemieux (1994), Changing wage structure and black-white wage differentials, *American Economic Review*, **84**, 29-33.

[4] Cowell, F. A. *Measuring Inequality*, Wiley, New York, 1977.

[5] Creedy, J. (1977), The principle of transfers and the variance of logarithms, *Oxford Bulletin of Economics and Statistics*, **39**, 152-8.

[6] Creedy, J. (1985), *Dynamics of Income Distribution*, Basil Blackwell, Oxford, 1985.

[7] Creedy, J. and Gemmell, N. (1984), Income redistribution through taxes and transfers in Britain, *Scottish Journal of Political Economy*, **31**, 44-59.

[8] Dalton, H. (1920), The measurement of inequality of incomes, Economic Journal, **30**, 348-361.

[9] Dasgupta, P., Sen, A. K. and Starrett, D. (1973), Notes on the measurement of inequality, *Journal of Economic Theory*, **6**, 180-7.

[10] Fields, G. and J. C. H. Fei (1978), On inequality comparisons, *Econometrica*, **46**, 303-16.

[11] Foster, J. E. (1985), Inequality measurement, in Young, H. P. (ed.), *Fair Allocation*, American Mathematical Society Proceedings in Applied Mathematics, Vol. 33, Providence.

[12] Foster, J. E. and A. Sen (1997), On economic inequality after a quarter century, in Sen, A., *On Economic Inequality,* Clarendon Press, Oxford, 1997.

[13] Foster, J. E. and A. Shneyerov (1997), Path independent inequality measures, mimeo, Department of Economics, Vandertbilt University.

[14] Gastwirth, J. L. (1971), A general definition of the Lorenz curve, *Econometrica*, **100**, 1037-39.

[15] Glomm, G. and B. Ravikumar (1992), Public vs. private investment in human capital: endogeneous growth and income inequality, *Journal of Political Economy*, **39**, 818-834.

[16] Hart, P. (1980), Lognormality and the principle of transfers, *Oxford Bulletin of Economics and Statistics*, **42**, 263-8.

[17] Heckman, J. J. and B. Honore (1990), The empirical content of the Roy model, *Econometrica*, **58**, 1121-50.

[18] Juhn, C., K. M. Murphy and B. Pierce (1993), Wage inequality and the rise in returns to skill, *Journal of Political Economy*, **101**, 410-42.

[19] Karoly, L. A. (1992), Changes in the distribution of individual earnings in the United States: 1967-1986, *Review of Economics and Statistics*, **74**, 107-15.

[20] Kolm, S-Ch. (1969), The optimal production of social justice, in Margolis, J. and H. Guitton (eds.), *Public Economics*, MacMillan, London.

[21] Levy, F. and R. J. Murnane (1992), U.S. earnings levels and earnings inequality: a review of recent trends and proposed explanations, *Journal of Economic Literature*, **30**, 1333-1381.

[22] McDonald J. B. (1984), Some generalized functions for the size distribution of income, *Econometrica*, **52**, 647-64.

[23] Sawyer, M. (1979), The variance of logarithms and industrial concentration, *Oxford Bulletin of Economics and Statistics*, **41**, 165-81.

[24] Sen, A. K. *On Economic Inequality*, Expanded Edition, Clarendon Press, Oxford, 1997.

[25] Singh, S. H. and G. S. Maddala (1976), A function for size distribution of incomes, *Econometrica*, **44**, 963-70.