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Abstract

We axiomatically investigate the problem of rationalizing bargaining solutions by social welfare functions that are linear in every rank-ordered subset of $\mathbf{R}^n_+$. Such functions, the so-called Choquet integrals, have been widely used in the theories of collective and individual choice. We refer to bargaining solutions that can be rationalized by Choquet integrals as *Choquet bargaining solutions.* Our main result is a complete characterization of Choquet bargaining solutions. As a corollary of our main result, we also obtain a characterization of the generalized Gini bargaining solutions introduced by Blackorby et al. (1994).

Keywords: Axiomatic bargaining theory, Choquet integrals, generalized Ginis.
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1. Introduction

In Blackorby, Bossert, and Donaldson (1994) (henceforth B-B-D), the authors present a characterization of a class of bargaining solutions that can be rationalized by the generalized Gini social welfare functions. There are a number of reasons why such solutions, called *the generalized Gini bargaining solutions*, are of interest. First, every generalized Gini solution can be considered as a compromise between the two polar bargaining solutions, namely the utilitarian and the egalitarian solutions. Second, every generalized Gini bargaining solution has a particularly tractable structure: it is linear on every rank-ordered subset of $\mathbf{R}^n_+$. Third, given that the significance of the generalized Gini social welfare functions are well-established in welfare economics (Donaldson and Weymark, 1980, and Weymark, 1981), generalized Gini solutions can be regarded as compelling social choice rules in general.

In this note we advance B-B-D's work by providing a characterization of a class of bargaining solutions that contain generalized Gini solutions as special cases. More precisely, this class consists of all bargaining solutions that can be rationalized by social welfare functions (SWFs) which are strictly increasing and are linear on every rank-ordered subset of $\mathbf{R}^n_+$. Since any such SWF corresponds to a Choquet integral with respect to a monotonic capacity (Schmeidler, 1986), we refer to the members of this class as *Choquet bargaining solutions.* A Choquet bargaining solution differs from a generalized Gini bargaining solution in that it does not need to be anonymous and its representing SWF is not necessarily quasiconcave. Consequently, the class of Choquet bargaining solutions contains a wide range of solutions. For instance, it includes all weighted utilitarian bargaining solutions, and those solutions that can be rationalized by a linear combination of the utilitarian and the max-max SWFs.

Another advantage of the present approach is that it helps one identify the axiomatic basis for the generalized Gini bargaining solutions. This is useful, for B-B-D's approach is not helpful in determining which set of axioms is responsible for what particular aspect of the generalized Gini solutions. By utilizing the elementary Choquet integration theory, we isolate here a weak *linear invariance* property as the main structural assumption behind the Choquet bargaining solutions. This observation, in turn, lets us identify systematically the implications of each of the axioms used in B-B-D in characterizing the generalized Gini bargaining solutions.

Our main result characterizes bargaining solutions that satisfy the aforementioned linear invariance axiom along with three other basic conditions, namely, Pareto optimality, Arrow's choice axiom, and a weak continuity property. Any such bargaining solution can be rationalized by a SWF that is a Choquet integral with respect to a monotonic capacity. When the representing capacity is symmetric and convex, the corresponding Choquet bargaining solution reduces to a generalized Gini solution. Consequently, as a corollary of our main result, we also obtain a complete characterization of the generalized Gini bargaining solutions. This characterization utilizes an independent set of axioms, which are considerably weaker than that used in B-B-D.

2. Choquet and Generalized Gini Bargaining Solutions

As in the standard bargaining literature, we identify an n-player *bargaining problem* by a compact, convex and comprehensive subset S of $\mathbf{R}_+^n$, and treat the origin as the disagreement point (Nash, 1950). As usual, we think of an element of S as a utility profile that would be attained if all players agree on a particular collective action. Alternatively, one may think of S as an arbitrary utility possibility set, and view the present development within the realm of collective choice theory.

Let $\mathbf{\Sigma}^n$ denote the collection of all bargaining problems. A *bargaining solution* on $\mathbf{\Sigma}^n$ is a correspondence $F : \mathbf{\Sigma}^n \rightrightarrows \mathbf{R}_+^n$ such that $\emptyset \neq F(S) \subseteq S$ for all $S \in \Sigma^n$. One may think of $F(S)$ as the set of all resolutions suggested by an impartial arbitrator, or the set of all predictions concerning the outcomes of some underlying strategic bargaining game. We say that a bargaining solution F is (*fully*) *rationalized* by a social welfare function (SWF) $W : \mathbf{R}_+^n \to \mathbf{R}$ if

$$F(S) = \arg\max_{x \in S} W(x) \quad \text{for all } S \in \mathbf{\Sigma}^n.$$

If the social welfare function W is believed to incarnate desirable normative principles, then the bargaining solution F that W rationalizes is seen to implement such principles. Many authors have provided characterizations of bargaining solutions that are rationalized by various SWFs. Among the most notable examples are the Nash bargaining solution (Nash, 1950), the utilitarian and the egalitarian solutions (Myerson, 1981), and the generalized Gini solutions (Blackorby et al., 1994, 1996). In this paper we shall consider a new class of bargaining solutions, which consists of those solutions that can be fully rationalized by the so-called Choquet integrals.

Let us first briefly describe the rudiments of Choquet integrals. A real-valued set function v on $2^{\{1,\ldots,n\}}$ with $v(\emptyset) = 0$ is called a *capacity* on $\{1, \ldots, n\}$. A capacity v is *weakly monotonic* if $v(K) \leq v(L)$ for all $K \subseteq L$, and is *monotonic* if $v(K) < v(L)$ for all $K \subset L$. Given any capacity v, the *Choquet integral* of $x \in \mathbf{R}_+^n$ with respect to v is defined as

$$\int x \, dv \equiv \int_0^\infty v(\{i \in N : x_i \geq t\}) \, dt.$$

A more explicit formula can be given by considering the order statistic of x. For any $x \in \mathbf{R}_+^n$, the *order statistic* of x is defined as $(x_{(1)}, \ldots, x_{(n)}) = x\Pi$, where Π is an $n \times n$ permutation matrix, such that $x_{(1)} \leq \cdots \leq x_{(n)}$. Then the Choquet integral of x with respect to v can be written as

$$\int x \, dv = \sum_{i=1}^{n} \left[v\left(\{(i), \ldots, (n)\}\right) - v\left(\{(i+1), \ldots, (n)\}\right) \right] x_{(i)} \tag{1}$$

where, by convention, $\{(n+1), (n)\} \equiv \emptyset$.

Let v be any monotonic capacity and consider the function $W_v : \mathbf{R}_+^n \to \mathbf{R}_+$ defined as $W_v(x) \equiv \int x \, dv$. We shall refer to this (strongly Paretian) SWF function as a *Choquet social welfare function.* It follows from (1) that W_v is a linear mapping on each of the rank-ordered subspaces of $\mathbf{R}_+^n$ for any given v. Conversely, any increasing function that is linear on each rank-ordered subset of $\mathbf{R}_+^n$

is the Choquet integral with respect to some capacity v. Consequently, Choquet integrals can be considered as interesting generalizations of the utilitarian SWFs, which are, by definition, linear on the entire $\mathbf{R}^n_+$. Given the overwhelming popularity of the utilitarian SWFs, this observation motivates the investigation of the bargaining solutions that are rationalized by Choquet SWFs. We call such a solution F_v a *Choquet bargaining solution*:

$$F_v(S) \equiv \arg\max_{x \in S} W_v(x) \quad \text{for all } S \in \Sigma^n.$$

The normative properties of a Choquet solution F_v are completely embodied in v. Hence, by focusing on certain subclasses of monotonic capacities, we may obtain particularly interesting subclasses of the Choquet bargaining solutions. Here are two important examples.

EXAMPLE 1: (*Inequality Averse Choquet Solutions*) A Choquet integral W_v can be thought of as an "inequality averse" SWF, if it is *quasiconcave* on $\mathbf{R}^n_+$. As will be shown later, this requirement selects capacities v that are *convex*, i.e., those capacities with the property that $v(K \cup L) + v(K \cap L) \geq v(K) + v(L)$ for all $K, L \subseteq \{1, ..., n\}$. We thus define an *inequality averse* Choquet bargaining solution as a bargaining solution that is rationalized by a Choquet integral with respect to a monotonic and convex capacity.

EXAMPLE 2: (*Symmetric Choquet Solutions*) If one wishes to concentrate on "anonymous" SWFs, we would then need to consider a *symmetric* capacity for a Choquet bargaining solution, i.e. a capacity with the property that $v(K) = v_k$ for all $K \subseteq \{1, ..., n\}$ with $|K| = k$. The resulting Choquet solution is then called a *symmetric* Choquet bargaining solution.

We shall demonstrate later that if a Choquet SWF W_v is both symmetric and quasiconcave, then we have

$$W_v(x) = G_\alpha(x) \equiv \sum_{i=1}^{n} \alpha_i x_{(i)} \quad \text{for some } \alpha_1 \geq \cdots \geq \alpha_n > 0,$$

for all $x \in \mathbf{R}^n_+$. In the literature, G_α has been called a *generalized Gini social welfare function* (Weymark, 1981). Put differently, W_v is a generalized Gini SWF if and only if v is monotonic, symmetric, and convex. We have already stressed the importance of Gini SWFs at the very beginning of this paper. Consequently, the bargaining solutions that are fully rationalized by generalized Ginis, called the *generalized Gini bargaining solutions*, emerge as a compeling class of bargaining solutions, which is indeed the main motivation behind the work of B-B-D. The fact that Choquet solutions subsume all generalized Gini solutions thus underscores the significance of the class of Choquet bargaining solutions.

3. The Characterization Theorems

This section aims to examine the axiomatic basis of the Choquet bargaining solutions in general, and the generalized Gini bargaining solutions in particular. In what follows, for each $S \in \Sigma^n$, we denote by PO(S) the set of all strongly Pareto optimal points of S, i.e., PO$(S) \equiv \{x \in S :$ there exists no $y \in S$ and $y \neq x$ with $y_i \geq x_i$ for all $i\}$. Our first two axioms are standard in the cooperative bargaining and social choice literature.

Strong Pareto Optimality (SPO): For all $S \in \Sigma^n$, $F(S) \subseteq \text{PO}(S)$.

Arrow's Choice Axiom (ACA): For all $S, T \in \Sigma^n$, if $T \subseteq S$ and $F(S) \cap T \neq \emptyset$, then $F(T) = F(S) \cap T$.

SPO is self-explanatory. ACA (Arrow, 1959) simply requires that choices be consistent with respect to contractions of the choice set, and it reduces to Nash's *independence of irrelevant alternatives* (IIA) property in the case of a single-valued bargaining solution.

For any pair $x, y \in \mathbf{R}^n$, we say that x and y are *comonotonic* if $(x_i - x_j)(y_i - y_j) \geq 0$ for all i, j, that is, if x and y belong to the same rank-ordered subset of $\mathbf{R}^n_+$. Moreover, $S, T \subset \mathbf{R}^n_+$ are said to be *comonotonic* if x and y are comonotonic for all $x \in S$ and $y \in T$. The key axiom that distinguishes Choquet bargaining solutions from others is the next partial additivity property .

Restricted Invariance (R.INV): For all $(S, x) \in \Sigma^n \times F(S)$, and $y \in \mathbf{R}^n$, if $\text{PO}(S)$ and $\text{PO}(S + \{y\})$ are comonotonic, then $x + y \in F(\text{cch}(S + \{y\}))$.[1]

R.INV is obviously weaker than the standard *linearity* condition that requires invariance for translation of S by *any* vector y (see Myerson, 1981). It can also be interpreted as a "fairness" condition. Provided that utilities are viewed as cardinal unit comparable (or that S is interpreted as a cost allocation problem, say), it is possible to justify rationalizable choice correspondences that satisfy R.INV by using the arguments that are advanced by Weymark (1981) and Ben Porath and Gilboa (1994) for the axioms of *weak independence of income source* and *order-preserving gifts*, respectively. We should also note that analogs of such additivity properties are also used in the theory of cooperative games (Kalai and Samet, 1985).

R.INV is also considerably weaker than the *linear invariance* condition (L.INV) used by B-B-D. First, R.INV requires that the translation of an $x \in F(S)$ by y remain in the choice set if the Pareto boundary $\text{PO}(S)$ after the translation by y remain in the same rank-ordered subset of $\mathbf{R}^n_+$ as $\text{PO}(S)$ is originally in. In contrast, L.INV requires invariance whenever $x + y$ is comonotonic with x. Second, in our model, S is always a subset of $\mathbf{R}^n_+$ and we do not consider its possible extension outside $\mathbf{R}^n_+$ relevant. However, B-B-D use a somewhat "unusual" domain of bargaining problems and L.INV is formulated so that the extension of S outside $\mathbf{R}^n_+$ does matter (to the extent that it plays a role in their proof). It is easy to verify that, in the presence of ACA, R.INV is also weaker than L.INV on this account.

As noted earlier, in this paper our primary aim is to isolate the implication of this separability condition alone; this is the main point of departure of our work from that of B-B-D. The first question we pose here is: what can we say about bargaining solutions that satisfy only SPO, ACA, and R.INV? Our first result provides an answer.

[1] *Notation*: $\text{co}\, S$ and $\text{cch}\, S$ denote the *convex hull* and *convex comprehensive hull* of S, respectively.

THEOREM 1: *If a bargaining solution F satisfies SPO, ACA, and R.INV, then there exists a weakly monotonic capacity v on $\{1, ..., n\}$ such that*

$$F(S) \subseteq \arg\max_{x \in S} \int x\, dv \quad \textit{for all } S \in \Sigma^n.$$

Thus, if a bargaining solution satisfies SPO, ACA, and R.INV, then it is partially rationalized by a weakly increasing Choquet SWF, i.e., it is a "selection" from a weakly monotonic Choquet bargaining solution. While it clarifies the basic structure that R.INV axiom brings to the table, Theorem 1 also shows that a *single-valued* bargaining solution that satisfies SPO, IIA, and R.INV must maximize a Choquet SWF (even though such a solution needs not be fully rationalizable). An interesting example of a single-valued bargaining solution that satisfies the conditions of Theorem 1 is the *leximin* bargaining solution (Imai, 1983).[2] This is another point at which our work differs from B-B-D's. Their result does not apply to single-valued bargaining solutions since it deals with only continuous bargaining solutions, which are necessarily multi-valued with the conjuction of other axioms.

Since we are ultimately interested in fully rationalizable bargaining solutions, we need to include a continuity requirement in our basic list of postulates (Peters and Wakker, 1991); hence the next axiom.

CONTINUITY (CON): For any $(S, x) \in \Sigma^n \times S$, if there exist a sequence $\{S_m\}$ such that (i) $|F(S_m)| = 1$ for all m; (ii) $S_m \to S$ (in Hausdorff topology); (iii) $F(S_m) \to \{x\}$, then $x \in F(S)$.

Note that CON is a much less demanding continuity requirement than those widely used in the literature. It is not only weaker than the standard upper hemicontinuity condition, but it is also weaker than the "weak continuity" axiom of B-B-D.

If a bargaining solution satisfies CON in addition to SPO, ACA and R.INV, then it turns out that it must be fully rationalizable. What is more, these four axioms completely characterize the class of Choquet bargaining solutions.

THEOREM 2: *A bargaining solution F satisfies SPO, ACA, R.INV, and CON if and only if F is a Choquet bargaining solution, i.e, there exists a monotonic capacity v on $\{1, ..., n\}$ such that*

$$F(S) = \arg\max_{x \in S} \int x\, dv \quad \textit{for all } S \in \Sigma^n.$$

Of course, a bargaining solution satisfying the axioms of Theorem 2 may not be considered completely satisfactory from a welfaristic point of view, for such a solution could fail to satisfy certain other "fairness" requirements. For example, it may be desirable that a bargaining solution F be inequality averse, or equivalently, that the Choquet integral W_v which rationalizes F be quasiconcave (recall Example 1). This can be achieved by imposing the next condition.

[2] Define $\sim^l \equiv \{(x, y) \in \mathbf{R}^n_+ : x_{(i)} = y_{(i)} \text{ for all } i\}$ and $\succ^l \equiv \{(x, y) \in \mathbf{R}^n_+ : x_{(i)} = y_{(i)} \text{ for } i = 1, ..., k-1 \text{ and } x_{(k)} > y_{(k)} \text{ for some } k\}$. The *leximin bargaining solution* chooses in S the maximal element of S with respect to $\sim^l \cup \succ^l$.

COMPROMISABILITY (COM): For any $S \in \Sigma^n$, $|F(S)| \neq 2$.

The intuition behind COM (Ok and Zhou, 1997) is fairly straightforward. If $|F(S)| = 2$, then only two "extreme" outcomes are chosen, and no other compromises are allowed by F. The axiom of COM requires that at least one *compromise* between two choices be in the choice set. This axiom is obviously a weakening of the "connectedness" axiom of B-B-D which requires that $F(S)$ be a connected set for all $S \in \Sigma^n$.

COROLLARY 2.1: *A bargaining solution F satisfies SPO, ACA, R.INV, CON, and COM if and only if F is an inequality averse Choquet bargaining solution, i.e, there exists a monotonic and convex capacity v on $\{1, ..., n\}$ such that*

$$F(S) = \arg\max_{x \in S} \int x\, dv \quad \textit{for all } S \in \Sigma^n,$$

and in this case the Choquet integral is concave on $\mathbf{R}^n_+$.

Another fairness condition is that all agents be impartially treated regardless of their labels.

ANONYMITY (A): For all $(S, x) \in \Sigma^n \times F(S)$, $x\Pi \in F(\{y\Pi : y \in S\})$ for any $n \times n$ permutation matrix Π.

COROLLARY 2.2: *A bargaining solution F satisfies SPO, ACA, R.INV, CON, and A if and only if there exist $\alpha_1, \cdots, \alpha_n > 0$ such that*

$$F(S) = \arg\max_{x \in S} \sum_{i=1}^{n} \alpha_i x_{(i)} \quad \textit{for all } S \in \Sigma^n.$$

Combining these two corollaries, we obtain a characterization of the generalized Gini bargaining solutions, which improves B-B-D's main characterization theorem.[3]

THEOREM 3: *A bargaining solution F satisfies SPO, ACA, R.INV, CON, COM, and A if and only if F is a generalized Gini bargaining solution, i.e., there exist $\alpha_1 \geq \cdots \geq \alpha_n > 0$ such that*

$$F(S) = \arg\max_{x \in S} \sum_{i=1}^{n} \alpha_i x_{(i)} \quad \textit{for all } S \in \Sigma^n.$$

All axioms in Theorem 1 are logically independent. In fact, given B-B-D's observations, the only non-trivial part of this claim is the independence of CON with the rest of the axioms. The leximin bargaining solution provides an example of a bargaining solution that violates CON yet satisfies all other axioms.

[3]Put precisely, the *sufficiency* part of Theorem 3 generalizes that of Blackorby et al. (1994). The difference between the domains of bargaining solutions considered here and in Blackorby et al. (1994) are inconsequential with regard to this statement.

In closing, we reiterate that the departure point of the present analysis is the recognition of the generalized Gini bargaining solutions as special cases of the Choquet bargaining solutions. On one hand, this allows us to identify the roles played by the axioms that are used in characterizing the generalized Gini solutions by B-B-D, and hence sheds light on the formal structure of this important class of bargaining solutions. On the other hand, the present approach also brings together the two branches of the theories of individual decision-making and social choice, namely, the theory of expected utility without additive probabilities (cf. Schmeidler, 1989) and the theory of linear social evaluation functions (cf. Mehran, 1976, Weymark, 1981, and Ben Porath and Gilboa, 1994).

4. Proofs

We adopt the following notation in the proofs. For any $K \subseteq \{1, ..., n\} = N$, $\mathbf{1}_K \in \mathbf{R}^n_+$ stands for the vector with i-th component being one if $i \in K$ and zero otherwise.

Since the necessity parts of the results are easy to verify, we shall prove here the sufficiency parts only.

Proof of Theorem 1: Let F be any fixed bargaining solution that satisfies SPO, ACA, and R.INV. We prove Theorem 1 by establishing a series of claims.

Claim 1: *Take any $\beta > \alpha > 0$ and $x \in \mathbf{R}^n_+$. If $\alpha\mathbf{1}_N \in F(\text{cch}\{x, \alpha\mathbf{1}_N\})$, then*

$$F(\text{cch}\{x, \beta\mathbf{1}_N\}) = \{\beta\mathbf{1}_N\}.$$

Proof: By using R.INV, we find

$$\beta\mathbf{1}_N \in F(\text{cch}\,\{\text{cch}\{x, \alpha\mathbf{1}_N\} + \{(\beta - \alpha)\mathbf{1}_N\}\}) = F(\text{cch}\{x + (\beta - \alpha)\mathbf{1}_N, \beta\mathbf{1}_N\}).$$

But $\text{cch}\{x, \beta\mathbf{1}_N\} \subset \text{cch}\{x + (\beta - \alpha)\mathbf{1}_N, \beta\mathbf{1}_N\}$ and $\text{PO}(\text{cch}\{x + (\beta - \alpha)\mathbf{1}_N, \beta\mathbf{1}_N\}) \cap \text{cch}\{x, \beta\mathbf{1}_N\} = \{\beta\mathbf{1}_N\}$. Hence, by SPO and ACA, we have $\{\beta\mathbf{1}_N\} = F(\text{cch}\{x, \beta\mathbf{1}_N\})$. *Q.E.D.*

Claim 2: *Take any $\alpha > 0$ and $x \in \mathbf{R}^n_+$, and let $y = \lambda x + (1 - \lambda)\alpha\mathbf{1}_N$ for some $\lambda \in (0, 1)$. If $y \in F(\text{cch}\{x, \alpha\mathbf{1}_N\})$, then*

$$\alpha\mathbf{1}_N \in F(\text{cch}\{x, \alpha\mathbf{1}_N\}).$$

Proof: First suppose that $\lambda \in (0, 1/2]$. By ACA, $y \in F(\text{cch}\{x, y\})$, and thus by R.INV,

$$\alpha\mathbf{1}_N \in F(\text{cch}\{x + (\alpha\mathbf{1}_N - y), \alpha\mathbf{1}_N\}). \tag{2}$$

But, given that $\lambda \leq 1/2$, it is easily verified that $y \in \text{co}\{x + (\alpha\mathbf{1}_N - y), \alpha\mathbf{1}_N\}$ so that $\text{cch}\{x + (\alpha\mathbf{1}_N - y), \alpha\mathbf{1}_N\} \cap F(\text{cch}\{x, \alpha\mathbf{1}_N\}) \neq \varnothing$. Thus, by (2) and ACA,

$$\alpha\mathbf{1}_N \in F(\text{cch}\{x + (\alpha\mathbf{1}_N - y), \alpha\mathbf{1}_N\}) = \text{cch}\{x + (\alpha\mathbf{1}_N - y), \alpha\mathbf{1}_N\} \cap F(\text{cch}\{x, \alpha\mathbf{1}_N\})$$

and we conclude that $\alpha\mathbf{1}_N \in F(\text{cch}\{x, \alpha\mathbf{1}_N\})$.

Next, suppose that $\lambda \in (1/2, 1)$. Define

$$y^0 \equiv y \quad \text{and} \quad y^k = y^{k-1} + (y^{k-1} - x), \quad k = 1, 2, ...,$$

and notice that

$$y^k = \lambda_k x + (1 - \lambda_k)\alpha \mathbf{1}_N \quad \text{where} \quad \lambda_k \equiv 2^k \lambda - (2^k - 1), \quad k = 1, 2, \ldots .$$

We let $\ell \equiv \min\{k : \lambda_k \in (0, 1/2]\}$ and note that ℓ is well-defined. Now, since $y \in F(\text{cch}\{x, y\})$ by ACA, we may use R.INV successively to conclude that

$$y^k \in F(\text{cch}\{x, y^{k-1}\} + \{y^{k-1} - y^k\}) = F(\text{cch}\{y^{k-1}, y^k\}) \quad \text{for all } k = 1, 2, \ldots .$$

We next claim that

$$y^\ell \in F(\text{cch}\{x, \alpha \mathbf{1}_N\}). \tag{3}$$

To see this, notice that $y^1 \in F(\text{cch}\{y, y^1\})$ and $y \in F(\text{cch}\{x, \alpha\mathbf{1}_N\})$ imply that $\text{cch}\{y, y^1\} \cap F(\text{cch}\{x, \alpha\mathbf{1}_N\}) \neq \emptyset$ so that $y^1 \in F(\text{cch}\{x, \alpha\mathbf{1}_N\})$ by ACA. If $\ell = 1$, this proves (3). If $\ell = 2$, then the same reasoning along with the fact that $y^1 \in F(\text{cch}\{x, \alpha\mathbf{1}_N\})$ yields $y^2 \in F(\text{cch}\{x, \alpha\mathbf{1}_N\})$. A straightforward induction argument, therefore, establishes (3). But then since $\lambda_\ell \in (0, 1/2]$, we may use the first part of the proof to conclude that $\alpha\mathbf{1}_N \in F(\text{cch}\{x, \alpha\mathbf{1}_N\})$. *Q.E.D.*

We now define a functional W on $\mathbf{R}^n_+$ by

$$W(x) \equiv \inf\{\alpha > 0 : \alpha\mathbf{1}_N \in F(\text{cch}\{x, \alpha\mathbf{1}_N\})\}. \tag{4}$$

In light of Claims 1 and 2, we have

$$F(\text{cch}\{x, \alpha\mathbf{1}_N\}) = \begin{cases} \{\alpha\mathbf{1}_N\}, & \text{if } \alpha > W(x) \\ S, & \text{if } \alpha = W(x) \\ \{x\}, & \text{if } \alpha < W(x) \end{cases} \tag{5}$$

for some S with $S \cap \{x, \alpha\mathbf{1}_N\} \neq \emptyset$.

Claim 3: *For any* $x, y \in \mathbf{R}^n_+$, if $x \in F(\text{cch}\{x, y\})$, then $W(x) \geq W(y)$.

Proof: Suppose $W(y) > W(x)$. Choose any an α with $W(y) > \alpha > W(x)$. Consider $z \in F(\text{cch}\{x, y, \alpha\mathbf{1}_N\})$. By SPO,

$$z \in \text{co}\{w, \alpha\mathbf{1}_N\} \quad \text{for some } w \in \text{co}\{x, y\}.$$

By Claim 2, either $w \in F(\text{cch}\{w, \alpha\mathbf{1}_N\})$ or $\alpha\mathbf{1}_N \in F(\text{cch}\{w, \alpha\mathbf{1}_N\})$. Since $F(\text{cch}\{x, y, \alpha\mathbf{1}_N\}) \cap \text{cch}\{w, \alpha\mathbf{1}_N\} \neq \emptyset$ (for z belongs to this intersection), ACA implies that either $w \in F(\text{cch}\{x, y, \alpha\mathbf{1}_N\})$ or $\alpha\mathbf{1}_N \in F(\text{cch}\{x, y, \alpha\mathbf{1}_N\})$.

Assume first that $w \in F(\text{cch}\{x, y, \alpha\mathbf{1}_N\}$. Since in this case $F(\text{cch}\{x, y, \alpha\mathbf{1}_N\}) \cap \text{cch}\{x, y\} \neq \emptyset$, by the hypothesis and ACA, $x \in F(\text{cch}\{x, y\}) = F(\text{cch}\{x, y, \alpha\mathbf{1}_N\}) \cap \text{cch}\{x, y\} \subset F(\text{cch}\{x, y, \alpha\mathbf{1}_N\})$. Then, by ACA,

$$x \in F(\text{cch}\{x, y, \alpha\mathbf{1}_N\}) \cap \text{cch}\{x, \alpha\mathbf{1}_N\} = F(\text{cch}\{x, \alpha\mathbf{1}_N\}). \tag{6}$$

This contradicts (5).

Assume next that $\alpha \mathbf{1}_N \in F(\text{cch}\{x, y, \alpha \mathbf{1}_N\}$. Then, by ACA,

$$\alpha \mathbf{1}_N \in F(\text{cch}\{x, y, \alpha \mathbf{1}_N\} \cap \text{cch}\{y, \alpha \mathbf{1}_N\} = F(\text{cch}\{y, \alpha \mathbf{1}_N\}).$$

This also contradicts (5). *Q.E.D.*

CLAIM 4: *For any $S \in \mathbf{\Sigma}^n$, $F(S) \subseteq \arg\max_{x \in S} W(x)$.*

PROOF: Fix an arbitrary $S \in \mathbf{\Sigma}^n$, pick any $x \in F(S)$ and $y \in S$. Since S is convex, $x \in F(\text{cch}\{x, y\})$ by ACA. Therefore, $W(x) \geq W(y)$ according to Claim 3. *Q.E.D.*

CLAIM 5: (Schmeidler) *If $f : \mathbf{R}^n_+ \to \mathbf{R}_+$ is nondecreasing and comonotonically additive (i.e., $f(x+y) = f(x) + f(y)$ for any x and y that are comonotonic), then there exists a weakly monotonic capacity v on $\{1, ..., n\}$ such that*

$$f(x) = \int x \, dv \quad \textit{for all } x \in \mathbf{R}^n_+.$$

PROOF:[4] Let us first note that f must be linear homogeneous. To see this, take any $x \in \mathbf{R}^n_+$, and notice that comonotonic additivity implies that $f(kx) = kf(x)$ for all $k \in \mathbf{N}$. Thus, $\frac{1}{l} f(x) = f(\frac{1}{l} x)$ for all $l \in \mathbf{N}$, and therefore, $f(\lambda x) = \lambda f(x)$ for all $\lambda \in \mathbf{Q}_{++}$. Then by choosing any rational sequences $a_m, b_m \in \mathbf{Q}_{++}$ such that $a_m \uparrow \lambda$ and $b_m \downarrow \lambda$ as $m \to \infty$, we obtain by monotonicity of f that $a_m f(x) = f(a_m x) \leq f(\lambda x) \leq f(b_m x) = b_m f(x)$ for all $m \geq 1$. Letting $m \to \infty$, we find $f(\lambda x) = \lambda f(x)$.

Now, for any $x \in \mathbf{R}^n_+$, we have $x = \sum_{i=1}^n (x_{(i)} - x_{(i-1)}) \mathbf{1}_{\{(i), ..., (n)\}}$, where $x_{(0)} \equiv 0$ by convention. It is easy to verify that $(x_{(i)} - x_{(i-1)}) \mathbf{1}_{\{(i), ..., (n)\}}$ and $(x_{(j)} - x_{(j-1)}) \mathbf{1}_{\{(j), ..., (n)\}}$ are comonotonic for all i and j. Therefore, letting $v(K) = f(\mathbf{1}_K)$ for all $K \subseteq \{1, ..., n\}$, we find

$$\begin{aligned}
f(x) &= f\left(\sum_{i=1}^n (x_{(i)} - x_{(i-1)}) \mathbf{1}_{\{(i), ..., (n)\}}\right) \\
&= \sum_{i=1}^n f\left((x_{(i)} - x_{(i-1)}) \mathbf{1}_{\{(i), ..., (n)\}}\right) \\
&= \sum_{i=1}^n (x_{(i)} - x_{(i-1)}) f\left(\mathbf{1}_{\{(i), ..., (n)\}}\right) \\
&= \sum_{i=1}^n \left[f\left(\mathbf{1}_{\{(i), ..., (n)\}}\right) - f\left(\mathbf{1}_{\{(i+1), ..., (n)\}}\right)\right] x_{(i)} \\
&= \sum_{i=1}^n \left[v\left(\{(i), ..., (n)\}\right) - v\left(\{(i+1), ..., (n)\}\right)\right] x_{(i)},
\end{aligned}$$

and thus recalling (1), we are done. *Q.E.D.*

[4] Claim 5 is a special case of the representation theorem of Schmeidler (1986). For completeness, we provide a short direct proof here.

CLAIM 6: *There exists a monotonic capacity v on $\{1,...,n\}$ such that $W(x) = \int x\,dv$ for all $x \in \mathbf{R}^n_+$.*

PROOF: We shall show that W is increasing and comonotonically additive. In view of Claim 5, this will complete the proof. First, note that monotonicity of W readily follows from Claim 3. To verify comonotonic additivity, take any comonotonic $x, y \in \mathbf{R}^n_+$ and choose any $\varepsilon > 0$. By (5) and R.INV, we have $F(\mathrm{cch}\{x+y, (W(x)+\varepsilon)\mathbf{1}_N + y\}) = \{(W(x)+\varepsilon)\mathbf{1}_N + y\}$ and

$$F(\mathrm{cch}\{(W(x)+\varepsilon)\mathbf{1}_N + y, (W(x)+\varepsilon)\mathbf{1}_N + (W(y)+\varepsilon)\mathbf{1}_N\}) = \{(W(x)+\varepsilon)\mathbf{1}_N + (W(y)+\varepsilon)\mathbf{1}_N\}.$$

Hence, by Claim 3 and definition of W, we have

$$W(x) + W(y) + 2\varepsilon \geq W((W(x)+\varepsilon)\mathbf{1}_N + y) \geq W(x+y).$$

By letting $\varepsilon \downarrow 0$, we obtain $W(x) + W(y) \geq W(x+y)$. Since we can similarly show that $W(x) + W(y) \leq W(x+y)$, we are done. *Q.E.D.*

Claims 4 and 6 completes the proof of Theorem 1. *Q.E.D.*

PROOF OF THEOREM 2: We only need to modify the proof of Theorem 1 at several points. First, note that it follows from the first line of (5) that $\{W(x)\mathbf{1}_N\} = F(\mathrm{cch}\{x, (W(x)+1/m)\mathbf{1}_N\})$ for all $m \geq 1$ and hence CON implies that $W(x)\mathbf{1}_N \in F(\mathrm{cch}\{x, W(x)\mathbf{1}_N\})$. By an analogous reasoning we can in fact conclude that $F(\mathrm{cch}\{x, W(x)\mathbf{1}_N\}) \supseteq \{x, W(x)\mathbf{1}_N\}$. Then, by a similar argument as that in Claim 2, we can show that $F(\mathrm{cch}\{x, W(x)\mathbf{1}_N\}) = \mathrm{co}\{x, W(x)\mathbf{1}_N\}$. With this strengthening of (5), it is easy to modify the proof of Claim 3 to show that $W(x) = W(y)$ if and only if $\{x, y\} \subseteq F(\mathrm{cch}\{x, y\})$. This also shows that W is increasing (recall that W was only shown to be weakly increasing in Theorem 1). Finally, with this strengthening of Claim 3, we can modify the proof of Claim 4 to conclude that $F(S) = \arg\max_{x \in S} W(x)$ for all S. *Q.E.D.*

PROOF OF COROLLARY 2.1: Let F be any bargaining solution that satisfies all axioms in the corollary. By Theorem 2, F can be rationalized by a Choquet integral W_v with respect to some monotonic capacity v. We now show that W_v is convex. Since F satisfies COM, W_v must be quasiconcave on $\mathbf{R}^n_+$ (Ok and Zhou, 1997).[5] On the other hand, W_v is also linear homogeneous, as

[5]Since this result is still not formally published, we include a proof here for completeness. Take any $z \in \mathbf{R}^n_+$ and consider the upper contour set of z, $U(z) \equiv \{y \in \mathbf{R}^n_+ : W(y) \geq W(z)\}$. Suppose that $U(z)$ is not convex, that is, there exist $x, y \in U(z)$ and a $\lambda \in [0,1]$ such that $\lambda x + (1-\lambda)y \equiv w \notin U(z)$. We define

$$\alpha \equiv \max\{\theta \in [0,1] : \theta w + (1-\theta)x \in U(z)\} \quad \text{and} \quad \beta \equiv \max\{\theta \in [0,1] : \theta w + (1-\theta)y \in U(z)\}.$$

Since any Choquet integral is continuous, both α and β are well-defined, and $0 < \alpha, \beta < 1$. Define

$$x' \equiv \alpha w + (1-\alpha)x \quad \text{and} \quad y' \equiv \beta w + (1-\beta)y.$$

Since w is a convex combination of x and y, it is obvious that $x' \neq y'$. Moreover, $W(x') = W(y')$ since $x', y' \in \partial U(z)$ by the choice of α and β. Therefore, since W is strictly increasing and since $\mathrm{int}\,(\mathrm{co}\,\{x', y'\}) \cap U(z) = \emptyset$, we have $F(\mathrm{cch}\{x', y'\}) = \arg\max_{q \in \mathrm{cch}\,\{x', y'\}} W(q) = \{x', y'\}$ contradicting COM. This proves that W must be quasiconcave.

we have shown in the proof of Claim 5. Hence, W_v is concave because any quasiconcave, linearly homogeneous, and non-negative-valued function on $\mathbf{R}^n_+$ must be concave.

Moreover, W_v is superadditive for $W_v(x+y) = 2W_v(\frac{x+y}{2}) \geq 2\left(W_v(\frac{x}{2}) + W_v(\frac{y}{2})\right) = W_v(x) + W_v(y)$ for all x and y. Now, for any $K, L \subseteq \{1, ..., n\}$, since $\mathbf{1}_{K\cup L} + \mathbf{1}_{K\cap L} = \mathbf{1}_K + \mathbf{1}_L$, and $\mathbf{1}_{K\cup L}$ and $\mathbf{1}_{K\cap L}$ are comonotonic,

$$\begin{aligned} v(K\cup L) + v(K\cap L) &= W_v(\mathbf{1}_{K\cup L}) + W_v(\mathbf{1}_{K\cap L}) \\ &= W_v(\mathbf{1}_{K\cup L} + \mathbf{1}_{K\cap L}) \\ &= W_v(\mathbf{1}_K + \mathbf{1}_L) \\ &\geq W_v(\mathbf{1}_K) + W_v(\mathbf{1}_L) \\ &= v(K) + v(L). \end{aligned}$$

Thus v is convex.[6] *Q.E.D.*

PROOF OF COROLLARY 2.2: If F satisfies A, then by the construction of v in the proof of Theorem 1, v must be *symmetric*, i.e., $v(K)$ is a function of only $|K|$. Thus, by (1),

$$W_v(x) = \sum_{i=1}^{n} \left[v\left(\{(i), ..., (n)\}\right) - v\left(\{(i+1), ..., (n)\}\right)\right] x_{(i)} = \sum_{i=1}^{n} \alpha_i x_{(i)}.$$

Q.E.D.

PROOF OF THEOREM 3: By Corollaries 2.1 and 2.2, the representing capacity v is both convex and symmetric. Hence, for all i, the numbers α_i in Corollary 2.2 satisfy

$$\begin{aligned} \alpha_i &= v\left(\{(i), ..., (n)\}\right) - v\left(\{(i+1), ..., (n)\}\right) \\ &\geq v\left(\{(i), (i+2), ..., (n)\}\right) - v\left(\{(i+2), ..., (n)\}\right) \\ &= v\left(\{(i+1), ..., (n)\}\right) - v\left(\{(i+2), ..., (n)\}\right) \\ &= \alpha_{i+1}. \end{aligned}$$

Q.E.D.

[6] The last paragraph is from Proposition 3 in Schmeidler (1986).

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