1 Spectral Analysis of Bivariate Stationary Process

Definition of a Stationary Bivariate Process:

$$\{X_{1,t}\},\ \{X_{2,t}\},\ t : 0,\ \pm 1,\ \pm 2,\ \dots$$

$\{X_{1,t}, X_{2,t}\}$ is bivariate stationary if and only if:

a) $\{X_{1,t}\}, \{X_{2,t}\}$ are univariate stationary

b) $\mathrm{cov}(X_{1,t}, X_{2,t})$ is a function of $(s - t)$ only.

Let both processes be null mean.

$$R_{11}(s) = E\left[X_{1,t} X^{*}_{1,t+s}\right]$$
$$R_{22}(s) = E\left[X_{2,t} X^{*}_{2,t+s}\right]$$
$$R_{12}(s) = E\left[X_{1,t} X^{*}_{2,t+s}\right]$$
$$R_{21}(s) = E\left[X_{2,t} X^{*}_{1,t+s}\right]$$

so,

$$R_{12}(s) = R^{*}_{21}(-s)$$

The corresponding auto- correlations are:

$$\rho_{11}(s) = \frac{R_{11}(s)}{R_{11}(0)}$$

$$\rho_{22}(s) = \frac{R_{22}(s)}{R_{22}(0)}$$

$$\rho_{12}(s) = \frac{R_{12}(s)}{[R_{11}(0) R_{22}(0)]^{\frac{1}{2}}},$$

similarly for $½_{21}$ (²):

$$\mathrm{R}] (s) = \begin{bmatrix} R_{11}(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{bmatrix}$$

$$½g(s) = \begin{bmatrix} ½_{11}(s) & ½_{12}(s) \\ ½_{21}(s) & ½_{22}(s) \end{bmatrix} :$$

The above are the covariance and correlation matrices; there are a countable infinity of them.

$$¡1 \;·\; ½_{ii}(s) \;·\; 1, \quad \max \text{ at } s = 0$$

$$i = 1,2, \quad ½_{ii}(s), \; \underline{\text{symmetric}} \text{ in } s$$

$$¡1 \;·\; ½_{ij}(s) \;·\; 1$$

<u>but</u> $\max ½_{ij}(s)$ can be anywhere; $½_{ij}(s)$ is **<u>not</u>** symmetric in s.

<u>Diagram</u>

1.1 Auto and Cross Spectra

Provided the auto and cross covariances are absolutely summable;
i.e. provided

$$\sum_{s=-\infty}^{\infty} |R_{ij}(s)| < \infty$$

$$i, j = 1, 2$$

then the non-normalized spectra are:

$$h_{ij}(\omega) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} R_{ij}(s) e^{-is\omega}$$

For $i \neq j$ this is the cross-spectral density function; for $i = j$, we have the auto-spectral density function.

Reinterpreting this result:
By univariate stationarity:

$$X_{1,t} = \int_{-\pi}^{\pi} e^{it\omega} dZ_1(\omega)$$

$$X_{2,t} = \int_{-\pi}^{\pi} e^{it\omega} dZ_2(\omega)$$

$dZ_i(\omega)$ are each orthogonal processes.

Substitute the definitions of $\{X_{1,t}\}$, $\{X_{2,t}\}$ into the definition for $R_{12}(s)$:

$$R_{12}(s) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-it\omega} e^{i(s+t)\omega'} E\{dZ_1(\omega)\, dZ_2^*(\omega')\}$$

L.H.S. is a function of s only by assumed stationarity; therefore, so must R.H.S. be.

L.H.S. $\equiv$ R.H.S. as a function of s if, and only if :

$$E\{dZ_1(\omega)\, dZ_2^*(\omega')\} = 0$$

$$\forall \omega \neq \omega', \text{ i.e. } \Gamma_{z_1,z_2} \text{ is a diagonal matrix}$$

Bivariate stationary processes can be represented by orthogonal random process that that are also mutually orthogonal.

$$h_{ii}(\omega) = E\left\{|dZ_i(\omega)|^2\right\},$$

as before. But also:

$$h_{12}(\omega) = E\{dZ_1(\omega)\, dZ_2^*(\omega)\}$$

$h_{ii}(\omega)$ contains the square of the coefficients of $e^{it\omega}$ in $X_{i,t}$.

$h_{ij}(\omega)$ contains the product of the coefficients of $e^{it\omega}$ in $X_{1,t}$ and $X_{2,t}$.

$h_{ii}(\omega)$ gives the contribution to **total variance** of frequency ω.

$h_{12}(\omega)$ gives the contribution to "t**otal covariance**" of frequency ω.

The spectral density matrix can be related by inverse F.T. to the covariance matrix.

$$h_{ij}(\omega) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} R_{ij}(s) e^{-is\omega}$$

$$i, j = 1, 2$$

Similarly:

$$f_{ij}(\omega) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \rho_{ij}(s) e^{-is\omega}$$

1.2 Various Complex Decompositions of $h_{12}(\omega)$

$h_{12}(\omega)$ is complex in general, <u>even</u> for **<u>real processes</u>**. We rewrite:

$$h_{12}(\omega) = \underbrace{c_{12}(\omega)}_{\text{co-spectrum, real component}} - i \underbrace{q_{12}(\omega)}_{\text{quadrative spectrum, imaginary component}}$$

co-spectrum — real component; quadrative spectrum — imaginary component

By replacing $e^{-is\omega}$ by $\cos(s\omega) - i\sin(s\omega)$ in

$$h_{12}(\omega) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} R_{12}(s) e^{-is\omega}$$

we see that

$$c_{12}(\omega) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \frac{1}{2}\left[R_{12}(s) + R_{12}(-s)\right]\cos(s\omega)$$

$$q_{12}(\omega) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \frac{1}{2}\left[R_{12}(s) - R_{12}(-s)\right]\sin(s\omega)$$

$\frac{1}{2}[R_{12}(s) + R_{12}(-s)]$ is the **even** part of $R_{12}(s)$
$\frac{1}{2}[R_{12}(s) - R_{12}(-s)]$ is the **odd** part of $R_{12}(s)$;
i.e. we have separated the non-symmetric $R_{12}(s)$ into the sum of an **even** part and an **odd** part.

At $\omega = 0$:

$$h_{12}(0) = c_{12}(0)$$
$$q_{12}(0) = 0$$

1.3 Polar Form Expressions

$$h_{12}(\omega) = \alpha_{12}(\omega)e^{i\psi_{12}(\omega)}$$

$$\alpha_{12}(\omega) = |h_{12}(\omega)| = \left[c_{12}^2 + q_{12}^2\right]^{\frac{1}{2}}$$

$$\psi_{12}(\omega) = \tan^{-1}\left(\frac{-q_{12}(\omega)}{c_{12}(\omega)}\right)$$

$\alpha_{12}(\omega)$ ~ cross-amplitude spectrum
$\psi_{12}(\omega)$ ~ phase spectrum

$$h_{12}(\omega) = E\{dZ_1(\omega)\,dZ_2^*(\omega)\}$$

Let

$$dZ_i(\omega) = |dZ_i(\omega)| e^{i\psi_i(\omega)}$$
$$i = 1,2$$

Therefore

$$h_{12}(\omega) = E\left\{|dZ_1(\omega)||dZ_2(\omega)|\right\} * E\left[e^{i[\psi_1(\omega) - \psi_2(\omega)]}\right]$$

Therefore

$$\alpha_{12}(\omega) = E\left\{|dZ_1(\omega)||dZ_2(\omega)|\right\}$$
$$\psi_{12}(\omega) = E\left[e^{i[\psi_1(\omega) - \psi_2(\omega)]}\right]$$

Remember $\psi_{12}(\omega)$ is only defined $\bmod 2\pi$.

Complex coherency is a "standardization" of the complex cross spectrum.

$$\omega_{12}(\omega) = \frac{h_{12}(\omega)}{[h_{11}(\omega)h_{22}(\omega)]^{\frac{1}{2}}}$$

and can be interpreted as the **correlation coefficient** between $\{X_{1t}\}$, $\{X_{2t}\}$ at **frequency ω**

$$0 \leq \omega_{12}(\omega) \leq 1$$

$\omega_{12}(\omega) \sim$ a correlation cofficient in frequency domain

But remember, in general, $\omega_{12}(\omega)$ is **complex**.

$\omega_{12}^2(\omega)$ is **invariant** to linear transformations for **stationary** processes only- but there can be a change in phase relationships.

Let $\{X_{1,t}\}$, $\{X_{2,t}\}$ be any two time series that are bivariate stationary.

Let

$$\tilde{X}_{1,t} = \sum_{u=-\infty}^{\infty} a(u) X_{1,t-u} = \alpha(B) X_{1,t}$$

$$\tilde{X}_{2,t} = \sum_{u=-\infty}^{\infty} b(u) X_{2,t-u} = \beta(B) X_{2,t}$$

Therefore

$$d\tilde{Z}_1(\omega) = \alpha\left(e^{-i\omega}\right) dZ_1(\omega)$$
$$d\tilde{Z}_2(\omega) = \beta\left(e^{-i\omega}\right) dZ_2(\omega)$$

Therefore

$$\tilde{h}_{11}(\omega) = \left|\alpha\left(e^{-i\omega}\right)\right|^2 h_{11}(\omega)$$
$$\tilde{h}_{22}(\omega) = \left|\beta\left(e^{-i\omega}\right)\right|^2 h_{22}(\omega)$$
$$\tilde{h}_{12}(\omega) = \alpha\left(e^{i\omega}\right) \beta\left(e^{-i\omega}\right) h_{12}(\omega)$$

And so,

$$|\tilde{\rho}_{12}(\omega)|^2 = |\rho_{12}(\omega)|^2$$

where

$$\tilde{\rho}_{12}(\omega) = \frac{\tilde{h}_{12}(\omega)}{\left[\tilde{h}_{11}(\omega)\tilde{h}_{22}(\omega)\right]^{\frac{1}{2}}}$$

Therefore,

$$|\tilde{\omega}_{12}(\omega)|^2 = \frac{\left|\alpha\left(e^{i\omega}\right)\right|^2 \left|\beta\left(e^{-i\omega}\right)\right|^2 h_{12}^2(\omega)}{|\alpha(e^{i\omega})|^2\,|\beta(e^{-i\omega})|^2\, h_{11}(\omega)\, h_{22}(\omega)}$$

$$= |\omega_{12}(\omega)|^2$$

$|\omega_{12}(\omega)| = 1$ if and only if $\{X_{1,t}\}$ is a linear transform of $\{X_{2,t}\}$.

Let

$$X_{1,t} = \sum_{j=-\infty}^{\infty} \psi_j X_{2,t-j}, \qquad \sum_{-\infty}^{\infty} |\psi_j| < \infty$$

Therefore

$$X_{1,t} = \int_{-\pi}^{\pi} \left(\sum \psi_j e^{-ij\omega}\right) e^{it\omega}\, dZ_2(\omega)$$

so,

$$dZ_1(\omega) = \sum_j \psi_j e^{-ij\omega}\, dZ_2(\omega)$$

but this is a functional linear relationship in frequency space, so,

$$|\omega_{12}(\omega)|^2 = 1$$

Some Examples:

1) **Uncorrelated** (in time) process:

i.e.

$$
\begin{aligned}
& R_{12}(s) = 0, && \forall s \\
\wedge \quad & h_{12}(\omega) = 0, && \forall \omega \\
\wedge \quad & \alpha_{12}(\omega) = 0 && \\
& c_{12}(\omega) = q_{12}(\omega) = 0, && \forall \omega \\
& \omega_{12}(\omega) = 0 &&
\end{aligned}
$$

2) Linearly (structurally) related:

$$X_{1,t} = aX_{2,t} + \varepsilon_t,$$

compare example above.

ε_t is white noise, uncorrelated with $X_{2,t}$.
therefore

$$R_{12}(s) = aR_{22}(s)$$

and

$$h_{12}(\omega) = ah_{22}(\omega)$$

$$h_{11}(\omega) = a^2 h_{22}(\omega) + h_{\varepsilon}(\omega)$$

so,

$$c_{12}(\omega) = ah_{22}(\omega); \qquad q_{12}(\omega) = 0$$

and

$$\alpha_{12}(\omega) = ah_{22}(\omega); \qquad \psi_{12}(\omega) = 0$$

therefore

$$\omega_{12}(\omega) = \frac{h_{12}(\omega)}{[h_{11}(\omega)h_{22}(\omega)]^{\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{h_{22}(\omega)}} \frac{a h_{22}(\omega)}{\sqrt{a^2 h_{22}(\omega) + h_{\epsilon\epsilon}(\omega)}}$$

$$= \left\{ 1 + \frac{h_{\epsilon\epsilon}(\omega)}{a^2 h_{22}(\omega)} \right\}^{-\frac{1}{2}}$$

If ϵ is white noise, $h_{\epsilon\epsilon}(\omega)$ is a **constant**.

1.4 Regression with Delay- the most interesting case

Let

$$X_{1,t} = a X_{2,t-l} + \epsilon_t$$

therefore

$$R_{12}(s) = a R_{22}(s - l)$$

and as

$$h_{12}(\omega) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} R_{12}(s) e^{-is\omega},$$

we get:

$$h_{12}(\omega) = \frac{a}{2\pi} \sum_{s=-\infty}^{\infty} R_{22}(s - l) e^{-is\omega}$$

$$= a e^{-il\omega} h_{22}(\omega)$$

$$h_{11}(\omega) = a^2 h_{22}(\omega) + h_{\epsilon\epsilon}(\omega)$$

$$c_{12}(\omega) = a\cos\left(\underbrace{\omega\tau}\right) h_{22}(\omega)$$

$\downarrow$

Note the increase
in frequency

$\uparrow$

$$q_{12}(\omega) = a\sin\left(\overbrace{\omega\tau}\right) h_{22}(\omega)$$

$$\alpha_{12}(\omega) = a h_{22}(\omega)$$

$$\psi_{12}(\omega) = -\omega\tau$$

$\uparrow$

Particularly important!

A time delay => phase spectrum is a linear function of frequency and the **slope** is the magnitude of the **delay**.

A generalization of this is to consider for any phase spectrum:

$$\triangle(\omega) = -\frac{d\psi_{12}(\omega)}{d\omega}$$

is known as the "envelope, group delay."

1.5 A More Complex Example with Feedback Loop (Like a cob web cycle)

$$X_{2t} = aX_{1\underbrace{t-1}} + \varepsilon_t, \quad X_{1t} = -bX_{2t} + \eta_t$$

$\nearrow$

Note delay

ε_t, η_t are independent white noise processes with spectral densities:

$$h_{\varepsilon}(\omega) = \frac{\sigma_{\varepsilon}^2}{2\pi} \qquad h_{\eta}(\omega) = \frac{\sigma_{\eta}^2}{2\pi}$$

Let us use the orthogonal components approach.

$$dZ_2(\omega) = a e^{-i\omega} dZ_1(\omega) + dZ_{\varepsilon}(\omega)$$

$$dZ_1(\omega) = -b\, dZ_2(\omega) + dZ_{\eta}(\omega)$$

Solve for dZ_1, dZ_2:

$$dZ_1(\omega) = \frac{dZ_{\eta}(\omega) - b\, dZ_{\varepsilon}(\omega)}{(1 + ab e^{-i\omega})}$$

$$dZ_2(\omega) = \frac{a e^{-i\omega} dZ_{\eta}(\omega) + dZ_{\varepsilon}(\omega)}{(1 + ab e^{-i\omega})}$$

Use the result that:

$$h_{ii}(\omega) = |dZ_i(\omega)|^2$$
$$i = 1,2$$

$$h_{11}(\omega) = \left[\frac{\frac{\sigma_{\eta}^2}{2\pi} + b^2 \frac{\sigma_{\varepsilon}^2}{2\pi}}{|(1 + ab e^{-i\omega})|^2} \right]$$

$$h_{22}(\omega) = \frac{\left\{ \frac{1}{2\pi} \left(a^2 \sigma_{\eta}^2 + \sigma_{\varepsilon}^2 \right) \right\}}{|(1 + ab e^{-i\omega})|^2}$$

$$h_{12}(\omega) = \frac{\left[\frac{1}{2\pi}\left(ae^{-i\omega}\sigma_\varepsilon^2 - b\sigma_\eta^2\right)\right]}{\left|(1 + abe^{-i\omega})\right|^2}$$

$$\left|1 + abe^{-i\omega}\right|^2 = 1 + a^2b^2 + 2ab\cos(\omega)$$

$$h_{11}(\omega) = \frac{\sigma_\varepsilon^2 + b^2\sigma_\eta^2}{2\pi(1 + a^2b^2 + 2ab\cos(\omega))}$$

$$h_{22}(\omega) = \frac{a^2\sigma_\varepsilon^2 + \sigma_\eta^2}{2\pi(1 + a^2b^2 + 2ab\cos(\omega))}$$

$$h_{12}(\omega) = \frac{ae^{-i\omega}\sigma_\varepsilon^2 - b\sigma_\eta^2}{2\pi(1 + a^2b^2 + 2ab\cos(\omega))}$$

$$c_{12}(\omega) = \frac{a\sigma_\varepsilon^2\cos(\omega) - b\sigma_\eta^2}{2\pi(1 + a^2b^2 + 2ab\cos(\omega))}$$

$$q_{12}(\omega) = \frac{a\sigma_\varepsilon^2\sin(\omega)}{2\pi(1 + a^2b^2 + 2ab\cos(\omega))}$$