1 A Detailed Algebraic Development Of The Chapman- Kolmogorov Equation And Its Ancillary Equations

Reference: Gardiner: Handbook of Stochastic methods, Springer-Verlay.

Notation will be modified slightly in interests of conformity to previous results.

We begin with specifying joint probability distributions (finite) for a time-dependent stochastic process $\{X(t)\}$:

$$p(x_1, t_1; x_2, t_2; \ldots)$$

We can consider conditional probabilities between $\{X(t)\}$ and $\{Y(t)\}$, where $\{Y(t)\}$ may be a function of $\{X(t)\}$:

$$p(x_1, t_1; x_2, t_2 \ldots \mid y_1, \tau_1, y_2, \tau_2, \ldots)$$

$$= \frac{p(x_1, t_1, x_2, t_2, \ldots; y_1, \tau_1, y_2, \tau_2, \ldots)}{p(y_1, \tau_1, y_2, \tau_2, \ldots)}.$$

The simplest formulation of a joint probability distribution as a model of "stochastic evolution" is in dependence:

$$p(x_1, t_1, x_2, t_2, \ldots) = \prod_i p(x_i, t_i).$$

The next most simple is a Markov process: the future depends on only the most recent past:

$$p(x_1, t_1, x_2, t_2, \ldots \mid y_1, \tau_1, y_2, \tau_2, \ldots) = p(x_1, t_1, x_2, t_2 \ldots \mid y_1, \tau_1)$$

where

$$\tau_1 > \tau_2 > \ldots$$

and

$$t_1 > t_2 > t_3 \ldots > \tau_1 > \tau_2 > \ldots\ .$$

These assumptions enable us to create **chains** of conditional probabilities

$$p(x_1, t_1, x_2, t_2, \ldots x_n, t_n)$$

$$= p(x_1, t_1 \mid x_2, t_2)\, p(x_2, t_2 \mid x_3, t_3) \ldots p(x_{n-1}, t_{n-1} \mid x_n, t_n)$$

for

$$t_1 \geq t_2 \geq t_3 \ldots \geq t_n .$$

We recall:

a)

$$p(x_1, t_1) = \int p(x_1, t_1; x_2, t_2)\, dx_2$$

$$\equiv \int p(x_1, t_1 \mid x_2, t_2)\, p(x_2, t_2)\, dx_2$$

b)

$$p(x_1, t_1 \mid x_3, t_3) \equiv \int p(x_1, t_1; x_2, t_2 \mid x_3, t_3)\, dx_2$$

$$\equiv \int p(x_1, t_1 | x_2, t_2; x_3, t_3)\, p(x_2, t_2 | x_3, t_3)\, dx_2$$

By Markov assumption for $t_1 \geq t_2 \geq t_3$

$$= \int p(x_1, t_1 | x_2, t_2)\, p(x_2, t_2 | x_3, t_3)\, dx_2$$

This is the Chapman-Kolmogorov equation:

$$p(x_1, t_1 | x_3, t_3) \equiv \int dx_2\, p(x_1, t_1 | x_2, t_2) \times p(x_2, t_2 | x_3, t_3) \qquad (1)$$

If the state space is discrete, we have outcomes $\{n_i\}$:

$$\int dx \rightarrow \sum_n$$

Chapman-Kolmogorov is:

$$P(n_1, t_1 | n_3, t_3) = \sum_{n_2} P(n_1, t_1 | n_2, t_2) \times P(n_2, t_2 | n_3, t_3).$$

We are now dealing with matrix multiplication of possibly infinite discussions.

1.1 An Aside On Continuity And Markov Processes In Reality

$X(t)$ may have a continuous range, but not have a continuous sample path. A related question is whether a process is truly Markovian.

Example: Consider a gas within a closed container. $V(t)$ » vector of velocities; $X(t)$ » vector of positions.

The range of $V(t)$ is certainly continuous, but if we are modelling the system in terms of "hard spheres", the sample paths,

$$\ldots v_i \; v_{i+1} \ldots$$

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are clearly not continuous.

The sample paths of $X(t)$ however, might usefully be regarded as continuous.

Is the system Markovian?

For small enough time interval, clearly not, and realistically the path is important for very short times.

However, we usually assume that the time scale on which memory dependence is important is **very** small relative to the time scale of observation.

Reconsider $X(t)$, which we considered having a continuous sample path- but if we sample at large time scales, even this sample path will be discontinuous) this is the Brownian motion analyzed by Einstein.

We are faced with a compromise between "continuity" and Markov.

1.1.1 Definition Of Continuous Markov Processes

For a Markov process, the sample paths are continuous functions of t, with probability 1, if for $\epsilon > 0$,

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x - z| > \epsilon} dx\, p(x, t + \Delta t | z, t) = 0 \qquad (2)$$

uniformly in $z, t, \Delta t$.

This is the hindberg condition.

<u>Two Examples</u>:

Brownian motion:

(i)

$$p(x, t + \Delta t | z, t) = (K \Delta t)^{-\frac{1}{2}} \times \exp\left[-\frac{1}{2} \frac{(x - z)^2}{D \Delta t} \right]$$

D » a constant.

It is easy to show that (i) satisfies (2)- Expand e^{-x} as [expand $e^{x}]^{-1}$) Brownian motion (for <u>**position**</u>) has continuous sample paths.

(ii)

$$p(x, t + \Delta t | z, t) = \left(\frac{\Delta t}{\pi}\right) \Big/ \left[(x - z)^2 + \Delta t^2 \right].$$

This is a Cauchy process and does **not** satisfy (2) $\Rightarrow$ the sample paths are discrete.

See the next illustration.

However, in both cases we have that:

a) Chapman-Kolmogorov is satisfied.

b)

$$\lim_{\Delta t \to 0} p(x, t + \Delta t \mid z, t) = \delta(x - z)$$

1.2 Derivation Of The Differential Chapman-Kolmogorov Equation- The Master Equation

This formal derivation- see Gardiner- chapter 3- **emphasizes** the distinction between the continuous sample path and discrete sample path versions. Our previous analysis left these distinctions to the analysis of specific cases.

We define $\forall \varepsilon > 0$:

i)

$$\lim_{\Delta t \to 0} \frac{p(x, t + \Delta t \mid z, t)}{\Delta t} = W(x \mid z, t) \qquad (3a)$$

uniformly in x, z, t, for

$$|x - z| \geq \varepsilon.$$

Note: Gardiner **restricts** the use of "$W(x \mid z, t)$" to the discontinuous case.

ii)

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x - z| < \varepsilon} dx\,(x_i - z_i)\, p(x, t + \Delta t \,|\, z, t) \qquad (3b)$$

$$= A_i(z, t) + O(\varepsilon)$$

$i = 1, \ldots n$; there being n components in x, z, vectors.

iii)

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x - z| < \varepsilon} dx\,(x_i - z_i)(x_j - z_j)\, p(x, t + \Delta t \,|\, z, t)$$

(3c)

$$= B_{ij}(z, t) + O(\varepsilon):$$

all limits uniform in x, z, t, ε.

Diagram

Defines Derivate Moment -1

$\Rightarrow W(x, z)$ in Van Kampen notation.

is $a_1(z)$, $v = 1$ from notation before

2nd. Derivate Moment

Similarly here

is $b_i(z)$, $v = 2$, from notation before.

All higher order "ϵ bounded" moments are of $O(\epsilon)$; i.e. for all ϵ

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x - z| < \epsilon} dx\, (x_i - z_i)^p (x_j - z_j)^q\, p(x, t + \Delta t \,|\, z, t)$$

$$= O(\epsilon), \qquad \forall (i, j, p, q), \quad p + q > 2$$

These assumptions yield the Fokker-Planck equation to be discussed again later.

Continuity of sample paths occurs when $W(x|z, t)$ vanishes for $x \neq z$; $\{A_i\}$, $\{B_{ij}\}$ describe **continuous** motion; $W(x|z, t)$ describes **discrete** motion.

Proof for 3rd. order:

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x - z| < \epsilon} dx\, (x_i - z_i)(x_j - z_j)(x_k - z_k) \times$$

$$p(\hat{x}, t + \Delta t \,|\, \hat{z}, t)$$

$$\approx C_{ijk}(\hat{z}, t) + O(\varepsilon)$$

Consider

$$\bar{C}(\alpha, \hat{z}, t) = \sum_{ijk} \alpha_1 \alpha_j \alpha_k C_{ijk}(\hat{z}, t)$$

therefore

$$C_{ijk}(\hat{z}, t) = \frac{1}{3!} \frac{\partial^3}{\partial \alpha_1 \partial \alpha_j \partial \alpha_k} \bar{C}(\cdot)$$

$$\bar{C}(\alpha, \hat{z}, t) \le \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x - z| < \varepsilon} dx \cdot |\alpha \cdot (\hat{x} - \hat{z})| \, [\alpha \cdot (\hat{x} - \hat{z})]^2 \, p(x; t + \Delta t | z, t)$$

$$\le |\alpha| \varepsilon \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int [\alpha \cdot (\hat{x} - \hat{z})]^2 \cdot p(x; t + \Delta t | z, t) \, dx + O(\varepsilon)$$

$$= \varepsilon |\alpha| \left[\sum_{ij} \alpha_i \alpha_j B_{ij}(\hat{z}, t) + O(\varepsilon) \right] + O(\varepsilon)$$