1 Complex Demodulation

We consider the following non-stationary, but deterministic problem.

$$X_t = R_t \cos(\lambda t + \psi_t)$$

R_t is a "slowly changing" amplitude (relative to the period $\frac{1}{\lambda}$)

ψ_t is a slowly changing phase, relative to the period $\frac{1}{\lambda}$.

The idea of complex demodulation is to extract information about R_t and ψ_t.

The analysis is local.

Consider the complex version of this:

$$X_t = R_t \exp\{i(\lambda t + \psi_t)\}$$

If λ were known, then we can "remove" its effect by transformation:

$$\begin{aligned} y_t &= X_t e^{(-i\lambda t)} \\ &= R_t e^{i\psi_t} \end{aligned}$$

and then

$$R_t = |y_t|$$

and

$$e^{i\psi_t} = \frac{y_t}{|y_t|}$$

as

$$\psi_t = \tan^{-1}\left(\frac{v_t}{u_t}\right),$$

where

$$\frac{y_t}{|y_t|} = u_t + i v_t = e^{i\psi_t}$$

and

$$\psi_t = \tan^{-1}\left(\frac{\sin(\psi_t)}{\cos(\psi_t)}\right)$$

Recall

$$e^{i\psi_t} = \cos(\psi_t) + i \sin\{\psi_t\}$$

Note that the fourier transform of y_t is that of X_t shifted to the "left" by the frequency λ; i.e.:

$$F_y(\omega) = F_x(\omega - \lambda).$$

The real form

$$X_t = R_t \cos(\lambda t + \psi_t)$$

can be extracted from the complex version by:

$$X_t = \frac{1}{2}\left[Z_t + \overline{Z}_t\right], \qquad Z_t = R_t e^{i(\lambda t + \psi_t)}$$

$$= \frac{1}{2} R_t \left[\exp i(\lambda t + \psi_t) + \exp\left(-i(\lambda t + \psi_t)\right)\right]$$

therefore

Note the frequency

$$y_t = \underbrace{\frac{1}{2} R_t e^{i\psi_t}}_{\text{what we want}} + \underbrace{\frac{1}{2} R_t \exp\left\{-i\left(\overbrace{2\lambda}\, t + \psi_t\right)\right\}}_{\text{to be removed}}$$

Let us now add noise to the model:

$$X_t = R_t \cos(\theta t + \phi_t) + \epsilon_t$$

ϵ_t iid noise.

$$y_t = X_t e^{-i\theta t}$$

$$= \underbrace{\frac{1}{2} R_t e^{-i\phi_t}}_{\substack{\text{wanted and in ``smooth''} \\ \text{low frequencies have high power.}}} + \underbrace{\frac{1}{2} R_t e^{-i(2\theta t + \phi_t)}}_{\substack{\text{oscillates at a} \\ \text{frequency of } \underline{2\theta}}}$$

$$+ \underbrace{\epsilon_t e^{-i\theta t}}_{\text{high frequency noise}}$$

The component:

$$\frac{1}{2} R_t e^{-i\phi_t}$$

is obtained by smoothing the data using, usually a linear, filter as we discussed earlier.

We want a **low pass** frequency filter with a stop frequency a little below 2θ.

If

$$X_t = R_{1t} \cos(\theta_1 t + \phi_{1t}) + R_{2t} \cos(\theta_2 t + \phi_{2t})$$

$\theta_1 << \theta_2$,

can in principle isolate (R_{1t}, ϕ_{1t}) presumed to be slowly varying with respect to θ_1 by doing the complex demodulation in stages and moving from the lowest to the hghest frequencies.

(Saved under a:,v:CompxDemd)