Abstract

1 Cross Spectral Analysis

References. Brockwell & Davis
Bloomfield
Granger + Hatanaka

Recall the spectral/A.C.F. relationship for a single r.v.:

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\omega} \gamma(h)$$

$$-\pi \le \omega \le \pi$$

$\gamma(\cdot)$ is the autocorrelation function $\gamma(h) = \frac{R(h)}{R(0)}$
$f(\cdot)$ is the spectral density function and

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\omega} f(\omega) d\omega$$

for a discrete parameter process.

And for the time series itself, $\{X_t\}$, $\{X_t\}$ is stationary:

$$X_t = \int_{-\pi}^{\pi} e^{it\omega} dZ(\omega)$$

where $dZ(\omega)$, $-\pi < \omega \le \pi$, is an orthogonal increment process.

One way to write this is:

$$\mathbf{R}^{!_2}_{!_1} f(!)\,d! = E\ jZ(!_2) ¡ Z(!_1)j^2$$

$$¡¼ < !_1 \cdot !_2 \cdot ¼ \qquad "$$

contribution to variance for frequencies in range $(!_1; !_2)$

We immediately generalize to:

$$X_t = f(X_{1t}, X_{2t})g$$

is a stationary, null mean, bivariate process.

Let

$$¡(²) = °_{ij}(h)$$

$$= E\ fX_{it+h}, X_{jt}g \qquad i, j = 1, 2, \ldots$$

$$¡1 < h < 1$$

and we assume

$$¡(h) = \mathbf{X}^{1}_{h=¡1} °_{ij}(h) < 1, \qquad 8i, j$$

$¡(h)$ is the cross-correlation matrix.

The cross-spectral density function:

$$f_{12}(!) = \frac{1}{2¼} \mathbf{X}^{1}_{¡1} e^{¡ih!} °_{12}(h)$$

The $2 £ 2$ matrix, called the spectral density matrix of X_t is:

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\omega}\,\Gamma(h)$$

$$-\pi \le \omega \le \pi$$

$f_{ii}(\omega)$ are the own spectral density functions.

The spectral representations are:

$$\gamma_{ij}(\omega) = \int_{-\pi}^{\pi} e^{ih\omega} f_{ij}(\omega)\,d\omega$$

$$i, j = 1, 2$$

γ_{ij}, $i \neq j$, is **not** in general symmetric about zero. $f_{ij}(\cdot)$ is typically complex.

$$f_{ii}(\omega) = E\,|dZ_i(\omega)|^2$$

$$i = 1, 2$$

$$f_{ij}(\omega) = E\left[dZ_i(\omega)\,dZ_j(\omega)^*\right]$$

$dZ_j(\omega)^*$ is the complex conjugate of $dZ_j(\omega)$.

The spectral covariance over the frequency range (ω_1, ω_2) is:

$$\int_{\omega_1}^{\omega_2} f_{ij}(\omega)\,d\omega = E\{[(Z_i(\omega_2) - Z_i(\omega_1)) \ast$$

$$(Z_j(\omega_2) - Z_j(\omega_1))^*]\}$$

for $-\pi \le \omega_1 \le \omega_2 \le \pi$

Note that:

$$E\{dZ_i(\omega)dZ_j(\omega')\} = 0$$

$\omega \neq \omega'$

From the last expression see that:

$$f_{21}(\omega) = f_{12}(\omega)^*,$$

i.e. $f(\omega)$, a 2×2 matrix is Hermitian

$$f(\omega) = f(\omega)^+,$$

$f(\omega)^+$ is the complex conjugate transpose of $f(\omega)$.

Let $g(\cdot)$, $h(\cdot)$ denote any square integrable function with respect to $f_{ij}(\cdot)$:

$$E\left[\int_{-\pi}^{\pi} g(\omega)dZ_i(\omega)d\omega\ \overline{\int_{-\pi}^{\pi} h(\omega)dZ_j(\omega)d\omega}\right]$$

$$= \int_{-\pi}^{\pi} g(\omega)\overline{h(\omega)}f_{ij}(\omega)d\omega$$

$\overline{h(\omega)}$ is the complex conjugate of $h(\omega)$.

Above is a simple generalization of a previous result for the univariate case:

$$E\left\{\int_{-\pi}^{\pi} f(\omega)dZ(\omega)\ \overline{\int_{-\pi}^{\pi} g(\omega)dZ(\omega)}\right\}$$

$$= E\left\{\int f(\omega)\overline{g(\omega)}dZ^2(\omega)\right\}$$

$$= \int f(\omega)\overline{g(\omega)}\,dH(\omega)$$

Let us represent each component of $X_t = (X_{1t}, X_{2t})'$ by:

$$X_{it} = \int_{-\pi}^{\pi} e^{it\omega}\,dZ_i(\omega)$$

The covariance between dZ_i, dZ_j is

$$f_{ij}(\omega) = E\left\{ dZ_i(\omega)\overline{dZ_j(\omega)} \right\}$$

and the squared correlation between $dZ_i(\omega)\,dZ_j(\omega)$ is $|f_{12}(\omega)|^2$.

$$\frac{|f_{12}(\omega)|^2}{f_{11}(\omega)\,f_{22}(\omega)} = K_{12}^2(\omega)$$

is called the squared coherency function.

Cauchy-Schwarz inequality =>

$$0 \le K_{12}^2(\omega) \le 1$$

$$-\pi \le \omega \le \pi.$$

$$f_{12}(\omega) = c_{12}(\omega) - i q_{12}(\omega)$$

$$c_{12}(\omega) = Re(f_{12})$$

$$q_{12}(\omega) = -Im(f_{12})$$

$c_{12}(\omega)$ − cospectrum

$q_{12}(\omega)$ − quadratum spectrum

Alternatively:

$$f_{12}(\omega) = \alpha_{12}(\omega)\,\mathrm{Exp}(i\phi_{12}(\omega))$$

$$\alpha_{12}(\omega) = \left[c_{12}(\omega)^2 + q_{12}(\omega)^2\right]^{\frac{1}{2}}$$

$$\phi_{12}(\omega) = \arg\{c_{12}(\omega) - iq_{12}(\omega)\}$$

$$= \arctan\left(\frac{q_{12}}{c_{12}}\right)$$

α_{12} − amplitude spectrum
ϕ_{12} − phase spectrum

$$K^2_{12} = \frac{\alpha_{12}(\omega)^2}{f_{11}(\omega)\,f_{22}(\omega)}$$

1.1 Some properties of Squared Coherency, $K^2_{12}(\omega)$

$K^2_{12}(\omega) = 1$ if X_{it}, X_{jt} are related by a time invariant linear filter; i.e. if

$$X_{jt} = \sum_{k=-\infty}^{\infty} \psi_k X_{it-k},$$

for convergent sequences ψ_k.

Proof:

$$X_{jt} = \int_{-\pi}^{\pi}\left[\sum_{-\infty}^{\infty} \psi_k e^{-ik\omega}\right] e^{it\omega}\,dZ_i$$

or

$$dZ_j(\omega) = \sum_k \psi_k e^{-ij\omega} dZ_i(\omega)$$

and as dZ_j, dZ_i are linearly related, the squared correlation must be 1.

$K^2_{12}(\omega)$ is invariant to use of linear filters on X_{1t}, X_{2t}.

$$K^2_{12}(\omega) = 0 \qquad -\pi < \omega < \pi$$

if X_{1t}, X_{2t} are uncorrelated.

(Saved on v:crossSpectral, a:crossSpec)