

1 Some Notes On Continuity And Differentiability

1.1 Lebesgue Decomposition Theorem

Any probability distribution function, $F(x)$, may be written in the form:

$$F(x) = \alpha_1 F_1(x) + \alpha_2 F_2(x) + \alpha_3 F_3(x)$$
$$\alpha_1, \alpha_2, \alpha_3 \geq 0, \alpha_i \leq 1, \alpha_1 + \alpha_2 + \alpha_3 = 1$$

$F_1(x)$ is absolutely continuous; i.e. is continuous everywhere, differentiable for almost all x .

$F_2(x)$ is a step function with at most a countable infinity of jumps- type I discontinuities.

$F_3(x)$ is a singular function, that is, is continuous and has a zero derivative a.e.:

1.2 Stochastic Limit Operations

The idea of this addendum is to indicate the extension of the ideas of continuity and differentiation to stochastic processes.

In the following, the **preferred definition** of convergence is “mean square” as opposed to almost certain convergence or convergence in probability.

Def: U_1, U_2, \dots a sequence of r.v.s:

$\{U_i\}_{i=1, \dots}$ converges in mean square to a random variable U if and only if

$$\lim_{i \rightarrow \infty} E (U_i - U)^2 = 0$$

which is written:

$$\lim_{i \rightarrow \infty} \mu_i = \mu$$

This is a convergence in 2nd. moments.

1.2.1 Stochastic Continuity

$fX(t)$ is stochastically continuous at $t = t_0$ if and only if:

$$\lim_{t \rightarrow t_0} X(t) = X(t_0)$$

that is, if and only if:

$$\lim_{t \rightarrow t_0} E (X(t) - X(t_0))^2 = 0$$

1.2.2 An Implication

If $fX(t)$ is stationary with variance $\frac{3}{4} \sigma_x^2$ and a:cf: $\frac{1}{2}(\zeta)$, then $X(t)$ is stochastically continuous if and only if

$$\lim_{t \rightarrow t_0} \frac{1}{2}(\tau_i - \tau_0) = 1$$

Note: this is a functional limit in the usual sense.

Proof:

$$\begin{aligned} E (X(t) - X(t_0))^2 &= \text{var}(X(t)) + \text{var}(X(t_0)) \\ &\quad - 2 \text{cov}(X(t), X(t_0)) \\ &= 2 \frac{3}{4} \sigma_x^2 [1 - \frac{1}{2}(\tau_i - \tau_0)] \end{aligned}$$

i.e. we require $\frac{1}{2}(\zeta)$ to be continuous at $\zeta = 0$ in the ordinary sense.

Example- to illustrate that this definition depends heavily on taking “averages.”

Let $f_X(t)$ be a signal with only two values $\{0, 1\}$ that occur at instants of time t according to a Poisson process with parameter λ .

Diagram

$\gamma(\zeta) = \exp(-\lambda|\zeta|)$
 so $\gamma(\zeta)$ is continuous at $\zeta = 0$. The process is still stochastically continuous in that the “average process” involves “nearby points” that are perfectly correlated.

1.2.3 Stochastic Differentiability

$f_X(t)$ is stochastically differentiable at $t = t_0$, with derivative $X'(t_0)$, if and only if:

$$\lim_{h \rightarrow 0} \frac{X(t_0 + h) - X(t_0)}{h} = X'(t_0)$$

But for the limit to exist, we need:

$$\lim_{h \rightarrow 0} E \left[\frac{X(t_0 + h) - X(t_0)}{h} - X'(t_0) \right] = 0$$

is true if and only if:

$$\lim_{h \rightarrow 0} E \left[\frac{X(t_0 + h) - X(t_0) - X'(t_0)h}{h^2} \right]$$

exists.

+

$$\frac{2 \frac{3}{4} h^2}{h^2} (1 - \frac{1}{2} h)$$

i.e. $p(h) \rightarrow 1$ at least as fast as $h^2 \rightarrow 0$

i.e. $\frac{1}{2} h \gg 0$ h^2

We can gain more insight:

Let

$$\frac{3}{4} h^2 = 1:$$

$$\frac{2(1 - \frac{1}{2} h)}{h^2} = \frac{f(\frac{1}{2} h) - \frac{1}{2} f(0) - \frac{1}{2} (1-h) f'(0)}{h^2}$$

which is a **2nd**. order difference.

The limit of the expression is:

$$f''(0)$$

If $f''(0)$ exists, $f''(\zeta)$ exists for all ζ .

In short, $X'(t)$ exists if and only if $f''(\zeta)$ exists.

If $X_{-}(t)$ exists:

$$\begin{aligned} \text{cov}^i X_{-}(t), X_{-}(t) &= 0 \quad \delta t \\ \text{cov}^i X_{-}(t), X_{-}(t+\zeta) &= i \frac{\partial^2}{\partial x^2} \underline{\lambda}(\zeta) \end{aligned}$$

or

$$\frac{\underline{\lambda}(\zeta)}{\underline{\lambda}(0)} = \circ(\zeta)$$

is the a:c:f: for $X_{-}(t)$.

Example: Random signal with Poisson arrivals

$$\underline{\lambda}(\zeta) = \exp(i 2^\circ j \zeta j):$$

$\underline{\lambda}(\zeta)$ is continuous, but **not** differentiable at $\zeta = 0$.

So this process is **not** stochastically differentiable.

Let us consider the derivatives of the a:c:f:

Let $E fX_{-}(t)g = 0$ for simplicity

$$\frac{\partial^2}{\partial x^2} \underline{\lambda}(\zeta) = E fX_{-}(t)X_{-}(t+\zeta)g.$$

If $X_{-}(t)$ exists, we can differentiate with respect to ζ .

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \underline{\lambda}(\zeta) &= E \left[\begin{matrix} \circledast \\ \circledast \end{matrix} X_{-}(t)X_{-}(t+\zeta) \right]_a \\ &= E X_{-}(t_i \zeta)X_{-}(t) \end{aligned}$$

by stationarity.

Differentiate again:

$$\frac{\partial^2}{\partial x^2} \underline{\lambda}(\zeta) = i E \left[\begin{matrix} \circledast \\ \circledast \end{matrix} X_{-}(t_i \zeta)X_{-}(t) \right]_a$$

or $i \frac{\partial^2}{\partial x^2} \underline{\lambda}(\zeta)$ is the a:cov:f: for $X_{-}(t)$

$$\underline{\lambda}(0) = 0$$

by the symmetry of $\frac{1}{2} (j)$.

(Saved under a:,v: CtnDyDiff)