Directed Technical Change, **Acemoglu**

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Directed technological change: endogenize the direction and bias of

new technologies that are developed and adopted.

Mass produced labor intensive products replace artisanal products? Machines replace labor?

What determines the relative profitability of developing different technologies? It is more profitable to develop technologies...

1) when the goods produced by these technologies command higher prices (price effect);

2) that have a larger market (market size effect).

Preferences

$$\int_0^\infty \frac{c^{1-\theta}-1}{1-\theta} e^{-\rho t} dt$$

$$c + I + R \leq Y$$

$$= \left[\gamma (A_L Y_L)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)(A_Z Y_Z)^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

R is R&D expenditures, Y_L and Y_Z are intermediate goods producing final good Y.

A_L is Y_L augmenting, A_Z is Y_Z augmenting.

($\varepsilon = 1$ is Cobb-Douglas, $\varepsilon = \infty$ is perfect substitues, $\varepsilon > 1$ is the case of gross substitutes.)

Normalizing the final good price to $1,$ the CES unit cost function, gives:

$$\left[\gamma^{\varepsilon} p_L^{1-\varepsilon} + (1-\gamma^{\varepsilon}) p_Z^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} = 1 \qquad \#$$

The relative prices of Y_Z and Y_L are

$$p = \frac{p_Z}{p_L} = \frac{MP_Z}{MP_L} = \frac{(1-\gamma)}{\gamma}\left(\frac{A_Z}{A_L}\right)^{\frac{\varepsilon-1}{\varepsilon}}\left(\frac{Y_Z}{Y_L}\right)^{-\frac{1}{\varepsilon}}$$

Note the effect of $\varepsilon \gtrless 1$ on derivatives of $\frac{MP_Z}{MP_L}$ wrt A_Z and $A_L.$ When Y_z, Y_L are substitutes, $\varepsilon > 1,$ A_Z is Z –biased and A_L is labor biased.

Intuition for why, when $\varepsilon < 1$,
Z-augmenting technical change is

L-biased:

With gross complementarity ($\varepsilon < 1$), an increase in the productivity of Z increases the demand for labor, L, by more than the demand for Z, creating "excess demand"for labor.

The marginal product of labor increases by more than the marginal

product of Z.

Take case where $\varepsilon = 0$ (Leontief, fixed proportions): starting from a situation in which $\gamma(A_L Y_L) = (1 - \gamma)(A_Z Y_Z)$, a small increase in A_Z will create an excess of the services of the Z factor, and its price will fall to 0.

Intermediate Goods

Intermediate goods, which depreciate after use, have production functions

$$Y_L = (1-\beta)^{-1}\left(\int_0^{N_L}(x_L(j))^{1-\beta}dj\right)L^{\beta}$$

$$Y_Z = (1-\beta)^{-1}\left(\int_0^{N_Z}(x_z(j))^{1-\beta}dj\right)Z^{\beta}$$

with N_L and N_Z the range of machines. Machines $x_L(j)$ and $x_Z(j)$ have prices $\chi_L(j),\ \ \chi_Z(j)$.

$$Max_{L,\{x_L(j)\}} p_L Y_L - w_L L - \int_0^{N_L} \chi_L(j)(x_L(j))dj$$

$$Max_{Z,\{x_L(j)\}} p_Z Y_Z - w_Z Z - \int_0^{N_Z} \chi_Z(j)(x_Z(j))dj$$

FOC

$$w_L = \frac{\beta}{1-\beta}p_L\left(\int_0^{N_L}(x_L(j))^{1-\beta}dj\right)L^{\beta-1}$$

$$w_Z = \frac{\beta}{1-\beta}p_Z\left(\int_0^{N_Z}(x_z(j))^{1-\beta}dj\right)Z^{\beta-1}$$

$$
\begin{aligned}
p_L L^\beta (x_z(j))^{-\beta} &= \chi_L(j) \\
\frac{p_L}{\chi_L(j)} L^\beta &= (x_L(j))^\beta \\
x_L(j) &= \left(\frac{p_L}{\chi_L(j)}\right)^{\frac{1}{\beta}} L \\
p_Z Z^\beta (x_Z(j))^{-\beta} &= \chi_Z(j) \\
\frac{p_Z}{\chi_Z(j)} Z^\beta &= (x_Z(j))^\beta \\
x_Z(j) &= \left(\frac{p_Z}{\chi_Z(j)}\right)^{\frac{1}{\beta}} Z
\end{aligned}
$$

So demand curves are

$$
\begin{aligned}
x_L(j) &= \left(\frac{p_L}{\chi_L(j)}\right)^{\frac{1}{\beta}} L \\
x_Z(j) &= \left(\frac{p_Z}{\chi_Z(j)}\right)^{\frac{1}{\beta}} Z
\end{aligned}
$$

Machines

Monopolists produce machines $x_Z(j)$ and $x_L(j)$, facing marginal cost of production $1-\beta,$ and maximize:

$$Max_{\chi_L(j)}\ \chi_L(j)\left(\left(\frac{p_L}{\chi_L(j)}\right)^{\frac{1}{\beta}}L\right)$$

$$-(1-\beta)\left(\left(\frac{p_L}{\chi_L(j)}\right)^{\frac{1}{\beta}}L\right)$$

and

$$Max_{\chi_Z(j)}\ \chi_Z(j)\left(\left(\frac{p_Z}{\chi_Z(j)}\right)^{\frac{1}{\beta}}Z\right)$$

$$-(1-\beta)\left(\left(\frac{p_Z}{\chi_Z(j)}\right)^{\frac{1}{\beta}}Z\right)$$

FOC for producing $x_L(j)$ wrt $\chi_L(j)$

$$\frac{\beta-1}{\beta}\chi_L(j)^{\frac{-1}{\beta}}(p_L)^{\frac{1}{\beta}}+\frac{(1-\beta)}{\beta}\chi_L(j)^{\frac{-1}{\beta}-1}(p_L)^{\frac{1}{\beta}}=0$$

$$-\chi_L(j)^{\frac{-1}{\beta}}+\chi_L(j)^{\frac{-1}{\beta}-1}=\chi_L(j)^{\frac{-1}{\beta}-1}(1-\chi_L(j))=0$$

So

$$1=\chi_L(j)$$

and therefore similarly

$$1=\chi_Z(j)$$

Profits

$$\pi_L = (p_L)^{\frac{1}{\beta}} L - (1-\beta)(p_L)^{\frac{1}{\beta}} L = \beta (p_L)^{\frac{1}{\beta}} L$$

$$\pi_Z = \beta (p_Z)^{\frac{1}{\beta}} Z$$

The values satisfy

$$rV_L = \pi_L + \dot{V}_L$$

$$rV_Z = \pi_Z + \dot{V}_Z$$

and steady state net discounted values of new innovations are

$$V_L = \frac{\beta (p_L)^{\frac{1}{\beta}} L}{r}, \quad V_Z = \frac{\beta (p_Z)^{\frac{1}{\beta}} Z}{r},$$

$$\frac{V_Z}{V_L} = \left(\frac{p_Z}{p_L}\right)^{\frac{1}{\beta}} \frac{Z}{L} = (p)^{\frac{1}{\beta}} \frac{Z}{L}$$

So relative incentive/profitability to innovate for Z relative to L, $\frac{V_Z}{V_L}$, has

i) relative price effect, p_Z relative to p_L, (innovate for the the expensive good)

ii) market size effect, Z relative to L, (innovate for the abundant factor)

Simplify further: Now set $A_z = A_L = 1$ because technical change will augment Y_L and Y_Z by producing new machines and adding to the range of N_L and N_Z. Using demand functions for $x_{L(j)}$, $x_Z(j)$ in the production functions

$$
\begin{aligned}
Y_L &= (1-\beta)^{-1}\left(\int_0^{N_L}\left((p_L)^{\frac{1}{\beta}}L\right)^{1-\beta}dj\right)L^{\beta} \\
&= (1-\beta)^{-1}N_L(p_L)^{\frac{1-\beta}{\beta}}L \\
Y_Z &= (1-\beta)^{-1}\left(\int_0^{N_Z}\left((p_Z)^{\frac{1}{\beta}}Z\right)^{1-\beta}dj\right)Z^{\beta} \\
&= (1-\beta)^{-1}N_Z(p_Z)^{\frac{1-\beta}{\beta}}Z
\end{aligned}
$$

and substituting these into the relative price of intermediate goods:

$$p = \frac{p_Z}{p_L} = \frac{MP_Z}{MP_L} = \frac{1-\gamma}{\gamma}\left(\frac{Y_Z}{Y_L}\right)^{-\frac{1}{\varepsilon}}$$

$$= \frac{1-\gamma}{\gamma}\left(\frac{N_Z (p_Z)^{\frac{1-\beta}{\beta}} Z}{N_L (p_L)^{\frac{1-\beta}{\beta}} L}\right)^{-\frac{1}{\varepsilon}}$$

$$p = \frac{1-\gamma}{\gamma}\left(\frac{N_Z Z}{N_L L}\right)^{-\frac{1}{\varepsilon}} (p)^{-\left(\frac{1-\beta}{\beta\varepsilon}\right)}$$

$$p = \left[\frac{1-\gamma}{\gamma}\left(\frac{N_Z Z}{N_L L}\right)^{-\frac{1}{\varepsilon}}\right]^{\frac{\beta\varepsilon}{\beta\varepsilon+1-\beta}}$$

$$= \left[\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\beta\varepsilon}{\beta\varepsilon+1-\beta}}\left(\frac{N_Z Z}{N_L L}\right)^{-\frac{1}{\varepsilon}\frac{\beta\varepsilon}{\beta\varepsilon+1-\beta}}\right]$$

$$p = \left[\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\beta\varepsilon}{\sigma}}\left(\frac{N_Z}{N_L}\right)^{-\frac{\beta}{\sigma}}\left(\frac{Z}{L}\right)^{-\frac{\beta}{\sigma}}\right] \qquad \#$$

$$\sigma = \beta\varepsilon + 1 - \beta,\ \sigma > 1 \ iff \ \varepsilon > 1$$

Now

$$\frac{V_Z}{V_L} = (p)^{\frac{1}{\beta}} \frac{Z}{L} = \left[\left(\frac{1-\gamma}{\gamma} \right)^{\frac{\varepsilon}{\sigma}} \left(\frac{N_Z Z}{N_L L} \right)^{-\frac{1}{\sigma}} \right] \frac{Z}{L}$$

$$= \left(\frac{1-\gamma}{\gamma} \right)^{\frac{\varepsilon}{\sigma}} \left(\frac{N_Z}{N_L} \right)^{-\frac{1}{\sigma}} \left(\frac{Z}{L} \right)^{\frac{\sigma-1}{\sigma}}$$

shows relative profitability:

An increase in relative supply of Z to L, increases relative profitability of Z iff $\sigma > 1$ (that is iff $\varepsilon > 1$), that is if factors are gross substitutes: market effect dominates price effect, and there is an incentive to innovate for Z, even though the price effect goes counter to it. We will model innovation of new machines in a minute.

Also consider factor rewards. Since $\chi_L(j) = \chi_Z(j) = 1$,

$$x_L(j) = \left(\frac{p_L}{\chi_L(j)}\right)^{\frac{1}{\beta}} L = (p_L)^{\frac{1}{\beta}} L$$

$$x_Z(j) = \left(\frac{p_Z}{\chi_Z(j)}\right)^{\frac{1}{\beta}} Z = (p_Z)^{\frac{1}{\beta}} Z$$

$$w_L = \frac{\beta}{1-\beta} p_L \left(\int_0^{N_L} (x_L(j))^{1-\beta} dj\right) L^{\beta-1}$$

$$= \frac{\beta}{1-\beta} p_L \left(\int_0^{N_L} \left((p_L)^{\frac{1}{\beta}} L\right)^{1-\beta} dj\right) L^{\beta-1}$$

$$= \frac{\beta}{1-\beta} p_L \left(\int_0^{N_L} (p_L)^{\frac{1-\beta}{\beta}} dj\right)$$

$$w_L = \frac{\beta}{1-\beta} (p_L)^{\frac{1}{\beta}} N_L$$

$$w_Z = \frac{\beta}{1-\beta} (p_Z)^{\frac{1}{\beta}} N_Z$$

$$\frac{w_Z}{w_L} = p^{\frac{1}{\beta}} \frac{N_Z}{N_L} = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}} \left(\frac{N_Z}{N_L}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{Z}{L}\right)^{-\frac{1}{\sigma}}$$

So $\frac{w_Z}{w_L}$ is decreasing in relative supply $\frac{Z}{L}$.

But for "relative innovation" it depends on $\frac{\sigma - 1}{\sigma}$, because the relative value of the marginal products also depends on the elasticity of substitution.

Supply of Innovations:

Production of new machines blueprints (innovation):

$$\dot{N}_L = \eta_L R_L \qquad \dot{N}_Z = \eta_Z R_Z$$

$$R_L + R_Z = R$$

$\$1$ gets η_L machine blueprints, so inventing one L machine costs $(\eta_L)^{-1}$, and profits for a machine per cost of machine, that is profits per $\$$, is $\frac{\pi_L}{(\eta_L)^{-1}}$. So the relative per dollar costs of innovations for machines, $\frac{(\eta_Z)^{-1}}{(\eta_L)^{-1}}$, is constant. On a BGP, p_L, p_Z, $\frac{V_Z}{V_L}$ are constant and N_L, N_Z grow at the same rate. On a BGP, to innovate in both sectors, investing a dollar in one sector must be as profitable as the other:

$$\frac{\pi_Z}{(\eta_Z)^{-1}} = \frac{\pi_L}{(\eta_L)^{-1}}$$

So

$$\eta = \frac{\eta_Z}{\eta_L} = \frac{\pi_L}{\pi_Z} = \frac{(p_L)^{\frac{1}{\beta}} L}{(p_Z)^{\frac{1}{\beta}} Z} \quad \#$$

$$= \left(\frac{1-\gamma}{\gamma}\right)^{\frac{-\varepsilon}{\sigma}} \left(\frac{N_Z}{N_L}\right)^{\frac{1}{\sigma}} \left(\frac{Z}{L}\right)^{\frac{1-\sigma}{\sigma}}$$

$$\frac{N_Z}{N_L} = \eta^{\sigma}\left(\frac{1-\gamma}{\gamma}\right)^{\varepsilon}\left(\frac{Z}{L}\right)^{\sigma-1} \quad \#$$

This is the innovation possibility frontier.

So with endogenous technical change along a balanced growth path, the relative bias of technology is determined by $\sigma - 1$: If factors are gross substitutes, an increase in $\frac{Z}{L}$ raises $\frac{N_Z}{N_L}$ so the relative range of Z expands.

In terms of relative wages

$$
\begin{aligned}
\frac{w_Z}{w_L} &= p^{\frac{1}{\beta}}\frac{N_Z}{N_L} \\
&= \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}}\left(\frac{N_Z}{N_L}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{Z}{L}\right)^{-\frac{1}{\sigma}} \\
&= \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}}\left(\eta^{\sigma}\left(\frac{1-\gamma}{\gamma}\right)^{\varepsilon}\left(\frac{Z}{L}\right)^{\sigma-1}\right)^{\frac{\sigma-1}{\sigma}} \\
&\quad \cdot\left(\frac{Z}{L}\right)^{-\frac{1}{\sigma}} \\
&= \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}}\left(\eta^{\sigma}\left(\frac{1-\gamma}{\gamma}\right)^{\varepsilon}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{Z}{L}\right)^{\frac{\left(\sigma^2-2\sigma+1\right)-1}{\sigma}} \\
&= \left(\frac{1-\gamma}{\gamma}\right)^{\varepsilon}\eta^{\sigma-1}\left(\frac{Z}{L}\right)^{\sigma-2}
\end{aligned}
$$

whereas with fixed supplies of machines

$$
\frac{w_Z}{w_L} = p^{\frac{1}{\beta}}\frac{N_Z}{N_L} = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}}\left(\frac{N_Z}{N_L}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{Z}{L}\right)^{-\frac{1}{\sigma}}
$$

Elasticity is $\sigma - 2 > -\frac{1}{\sigma}$: relative demand curve becomes more elastic when other factors, relative machine ranges $\frac{N_Z}{N_L}$, adjust.

If $\sigma > 2,$ positive relation between factor supplies and rewards is more surprising. With fixed $\frac{N_Z}{N_L},$ relative $\frac{w_Z}{w_L}$ wage decreases with $\frac{Z}{L},$ as in the first line of the equations above, but if the technology is biased towards abundant factors, the overall effect is ambigous.

Long run consumption growth

$$g_c = g = \theta^{-1}(r - \rho)$$

$$\theta g = r - \rho$$

If the free entry condition, that the cost of the machine is equal to discounted profits holds (value of machine equals its cost)

$$V_L = \frac{1}{\eta_L}$$

$$\eta_L \frac{\beta(p_L)^{\frac{1}{\beta}} L}{r} = 1$$

Solve for r and substitute into g to give

$$g = \theta^{-1}\left(\eta_L \beta (p_L)^{\frac{1}{\beta}} L - \rho\right)$$

Now solve using for p_L using the three equations marked #, and substitute into the equation above:

$$g = \theta^{-1}\left(\beta\left[(1-\gamma)^{\varepsilon}(\eta_Z Z)^{\sigma-1} + \gamma^{\varepsilon}(\eta_L L)^{\sigma-1}\right]^{\frac{1}{\sigma-1}} - \rho\right)$$

Stability

When $\eta \frac{\pi_Z}{\pi_L} = \eta \frac{V_Z}{V_L} > 1$ only $Z-$ complimentary machines are built and vice versa. Since $\frac{V_Z}{V_L}$ is decreasing in $\frac{N_Z}{N_L}$ (given factor supplies $\frac{Z}{L}$) from

$$\frac{V_Z}{V_L} = \left(\frac{p_Z}{p_L}\right)^{\frac{1}{\beta}} \frac{Z}{L}$$

$$= \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}} \left(\frac{N_Z}{N_L}\right)^{-\frac{1}{\sigma}} \left(\frac{Z}{L}\right)^{\frac{\sigma-1}{\sigma}}$$

the system should be stable.

Why we need Harrod-neutral techical change for a BGP?

Assume CRS.

$$Y = F(B(t)K, A(t)L)$$

$$A(t) = A(0)e^{xt}, \;\; B(t) = B(0)e^{zt} \;\;\; L(t) = L(0)e^{nt}$$

$$A(0) = B(0) = L(0) = 1$$

$$Y = Ke^{zt}F\left(1, \frac{L}{K}\frac{A(t)}{B(t)}\right) = Ke^{zt}\phi\left(1, \frac{L}{K}e^{(x-z)t}\right)$$

Let capital grow at γ_K on the BGP, normalizing $K(0) = 1$ on the BGP. Then

$$\frac{Y}{K} = e^{zt}\phi(1, e^{(x-z+n-\gamma_K)t})$$

On the BGP

$$\gamma_K = \frac{\dot{K}}{K} = s\left(\frac{Y}{K}\right) - \delta$$

where δ is depreciation. Then $\frac{Y}{K}$ is constant, as is $e^{zt}\phi(1, \frac{L}{K}e^{(x-z+n-\gamma_K)t})$.

Either

I. $z = 0,$ and the right side constant if

$$x + n = \gamma_K$$

This is Harrod neutral technologial change. Show factor shares constant.

or

II. if $z > 0$, we still need $\frac{Y}{K}$ constant, so we need

$$\frac{d(e^{zt}\phi(1, e^{(x-z+n-\gamma_K)t}))}{dt} = 0$$

True if

$$0 = ze^{zt}\phi(1, e^{(x-z+n-\gamma_K)t})$$
$$+ e^{zt}\phi'(1, e^{(x-z+n-\gamma_K)t})(x - z + n - \gamma_K)e^{(x-z+n-\gamma_K)t}$$

$$\frac{\phi'(1, e^{(x-z+n-\gamma_K)t})}{\phi(1, e^{(x-z+n-\gamma_K)t})}e^{(x-z+n-\gamma_K)t} = \frac{-z}{(x - z + n - \gamma_K)}$$

$$\frac{\phi'(1, \chi)}{\phi(1, \chi)}\chi = 1 - \alpha$$

$$\phi(1, \chi) = C\chi^{1-\alpha}$$

Then

$$\frac{Y}{K} = e^{zt}\phi\left(1, \frac{L}{K}e^{(x-z)t}\right) = e^{zt}(Ce^{(x-z+n-\gamma_K)t})^{1-\alpha}$$

$$= e^{zt}C^{1-\alpha}((K(0))e^{\gamma_K t})^{\alpha-1}B(0)(e^{-zt})^{1-\alpha}$$

$$\cdot\, (L(0)e^{nt})^{1-\alpha}(A(0)e^{xt})^{1-\alpha}$$

and since $A(0) = B(0) = 1$,

$$Y = e^{zt}C^{1-a}(K(t))^{\alpha}(e^{zt})^{\alpha-1}(A(t)L(t))^{1-\alpha}$$

$$Y = e^{zt}C^{1-a}(K(t)e^{zt})^{\alpha}(e^{zt})^{-1}(A(t)L(t))^{1-\alpha}$$

$$Y = C^{1-a}(K(t)B(t))^{\alpha}(A(t)L(t))^{1-\alpha}$$

that is, the production function must be Cobb-Douglas.

New Supply of Innovations

$$\dot{N}_L = \eta_L N_L R_L \quad \dot{N}_Z = \eta_Z N_R R_Z$$

$1 gets $N_L \eta_L$ machine blueprints, so inventing one L machine costs $(N_L \eta_L)^{-1}$, and profits for a machine per cost of machine, that is profits per $\$$, is $\frac{\pi_L}{(N_L \eta_L)^{-1}}$.

So the relative per dollar costs of innovations for machines is $\frac{(N_Z \eta_Z)^{-1}}{(N_L \eta_L)^{-1}}$ On a BGP, p_L, p_Z, $\frac{V_Z}{V_L}$ are constant and N_L, N_Z grow at the same rate. On a BGP, to innovate in both sectors, investing in one sector must be as profitable as the other:

$$\frac{\pi_z}{(N_Z \eta_Z)^{-1}} = \pi_z N_Z \eta_Z = \frac{\pi_L}{(N_L \eta_L)^{-1}} = \pi_L N_L \eta_L$$

$$\eta = \frac{\eta_Z}{\eta_L} = \frac{N_L \pi_L}{N_Z \pi_Z} = \frac{N_L (p_L)^{\frac{1}{\beta}} L}{N_Z (p_Z)^{\frac{1}{\beta}} Z}$$

$$= \left(\frac{1-\gamma}{\gamma}\right)^{\frac{-\varepsilon}{\sigma}} \left(\frac{N_Z}{N_L}\right)^{\frac{1-\sigma}{\sigma}} \left(\frac{Z}{L}\right)^{\frac{1-\sigma}{\sigma}}$$

BGP Stability if $\sigma < 1$?

$$\frac{N_Z}{N_L} = \eta^{\frac{\sigma}{1-\sigma}} \left(\frac{1-\gamma}{\gamma} \right)^{\frac{\varepsilon}{1-\sigma}} \left(\frac{Z}{L} \right)^{-1}$$

Substituting into $\frac{w_Z}{w_L}$

$$\frac{w_Z}{w_L} = p^{\frac{1}{\beta}} \frac{N_Z}{N_L}$$

$$= \left(\frac{1-\gamma}{\gamma} \right)^{\frac{\varepsilon}{\sigma}} \left(\frac{N_Z}{N_L} \right)^{\frac{\sigma-1}{\sigma}} \left(\frac{Z}{L} \right)^{-\frac{1}{\sigma}}$$

$$= \left(\frac{1-\gamma}{\gamma} \right)^{\frac{\varepsilon}{\sigma}} \left(\eta^{\frac{\sigma}{1-\sigma}} \left(\frac{1-\gamma}{\gamma} \right)^{\frac{\varepsilon}{1-\sigma}} \left(\frac{Z}{L} \right)^{-1} \right)^{\frac{\sigma-1}{\sigma}}$$

$$\cdot \left(\frac{Z}{L} \right)^{-\frac{1}{\sigma}}$$

$$= \left(\frac{1-\gamma}{\gamma} \right)^{\frac{\varepsilon}{\sigma}} \left(\eta^{\frac{\sigma}{1-\sigma}} \left(\frac{1-\gamma}{\gamma} \right)^{\frac{\varepsilon}{1-\sigma}} \right)^{\frac{\sigma-1}{\sigma}} \left(\frac{Z}{L} \right)^{-1}$$

$$= \eta \left(\frac{Z}{L} \right)^{-1}$$

But now factor shares are constant:

$$\frac{s_Z}{s_L} = \frac{w_Z Z}{w_L L} = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}} \left(\eta^{\frac{\sigma}{1-\sigma}} \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{1-\sigma}}\right)^{\frac{\sigma-1}{\sigma}}$$

Also note from above

$$\frac{s_Z}{s_L} = \frac{w_Z Z}{w_L L} = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}} \left(\frac{N_Z}{N_L}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{Z}{L}\right)^{\frac{\sigma-1}{\sigma}}$$

$$= \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}} \left(\frac{N_Z}{N_L}\frac{Z}{L}\right)^{\frac{\sigma-1}{\sigma}}$$

so if factor shares are constant, $\frac{Z}{L}$ must grow as $\frac{N_L}{N_Z}$.