In Aghion-Howitt book, page 57:

Poisson Process modeled as:

$$\begin{aligned} & \quad \textit{Cumulative distr. for event before } T \\ F(T) &= 1 - e^{-\mu T} \\ f(T) &= \mu e^{-\mu T} \quad (\textit{Density}) \\ g(x) &= \textit{prob. that } x \textit{ events occurs in the } \Delta \textit{ inteval} \\ &= \frac{(\mu\Delta)^x e^{-\mu\Delta}}{x!} \end{aligned}$$

Final product:

$$y = Ax^{\alpha}, \quad \alpha < 1$$

$$p_t = A_t \alpha (x_t)^{\alpha - 1}$$

because intermediate good price is its marginal product in final good production.

Intermediate product:

Uses one unit of labor to produce one unit of x with a linear technology $x = L_x$, so:

$$\pi = p_t x - w_t x = A_t \alpha (x_t)^{\alpha} - w_t x_t$$

$$x_t = Arg \max\{A_t \alpha (x_t)^{\alpha} - w_t x_t\}$$

$$\alpha^2 A x^{a-1} = w, \quad \frac{w_t x_t}{\alpha} = \frac{\alpha A_t}{x_t^{-\alpha}} = \alpha A_t x_t^{\alpha};$$

$$x_t = \left(\frac{\alpha^2}{\left(\frac{w_t}{A_t}\right)} \right)^{\frac{1}{1-\alpha}} \equiv x(\omega_t); \ where \ \omega = \frac{w}{A}$$

$$\pi = A_t \alpha x^{\alpha} - w_t x = (\alpha^{-1} - 1) w_t x_t$$

If $\alpha = 1$, $\frac{dp}{dx} = 0$, no profits.

$$w_t x_t = \alpha^2 A_t x_t^\alpha$$

$$\pi = (\alpha^{-1} - 1) w_t x_t = A_t (\alpha^{-1} - 1) \alpha^2 x_t^\alpha$$

$$\pi = A_t (\alpha^{-1} - 1) \left(\frac{w_t}{A_t} \right) (x_t(\omega_t)) \equiv A_t \tilde{\pi}_t(\omega_t)$$

NOTE $\tilde{x}(\omega_t), \tilde{\pi}_t(\omega_t)$ are decreasing in ω_t :

$$\pi_t = A_t (\alpha^{-1} - 1) \frac{w_t}{A_t} x_t(\omega_t)$$

$$= A_t (\alpha^{-1} - 1)(\alpha)^{\frac{2}{1-\alpha}} (\omega_t)^{\frac{\alpha}{\alpha-1}}$$

1. Distributional Assumption: If n persons do research, probabilty density function of an invention at time τ is given by the density

$$f(\tau) = (\lambda n)e^{-(\lambda n)\tau}$$

with cdf, for probability of an invention by time τ :

$$F(\tau) = 1 - e^{-(\lambda n)\tau}$$

$$F(0) = 0, \;\; F(\infty) = 1$$

2. Profits if firm survives until T, when new invention arrives:

$$\int_0^T \pi e^{-rs} ds = \frac{\pi}{r}(1 - e^{-rT})$$

3.Expected value of firm is discounted profits summed over survival probabilties to each T (that is by probabilities that there is an invention exactly at T, given by the density function above):

$$V = \int_0^\infty (\lambda n) e^{-(\lambda n)T} \int_0^T \pi e^{-rs} ds dT$$

$$= \int_0^\infty (\lambda n) e^{-(\lambda n)T} \left(\frac{\pi}{r} (1 - e^{-rT}) \right) dT$$

$$V = \frac{\pi}{r} \left(\int_0^\infty (\lambda n) e^{-(\lambda n)T} (1 - e^{-rT}) dT \right)$$

$$= \frac{\pi}{r} \left(\int_0^\infty (\lambda n) e^{-(\lambda n)T} dT - \int_0^\infty (\lambda n) e^{-(r+\lambda n)T} dT \right)$$

$$= \frac{\pi}{r} \left((0 + 1) - \left(0 + \frac{\lambda n}{r + \lambda n} \right) \right)$$

$$= \frac{\pi}{r} \left(1 - \frac{\lambda n}{r + \lambda n} \right) = \frac{\pi}{r} \left(\frac{r}{r + \lambda n} \right) = \frac{\pi}{r + \lambda n}$$

4. Therefore, the expected value of a firm:

$$V = \frac{\pi}{r + \lambda n}$$

$$(r + \lambda n)V = \pi$$

$$rV = \pi - (\lambda n)V$$

Labor is divided between research and production

$$L = n_t + x_t$$

Arbitrage across labor markets for research and manufacturing (assuming $A_{t+1} = \gamma A_t$ because improvements in technology are by a factor $\gamma)$:Now if t is the time between innovations,
$Max_{n_t}\ n_t \lambda V_{t+1} - w_t n_t$ yields

$$w_t = \lambda V_{t+1}; \quad \omega_t = \frac{w_t}{A_t} = \frac{\lambda}{A_t} V_{t+1}$$

$$rV_{t+1} = \pi_{t+1} - n_{t+1}\lambda V_{t+1}$$

$$V_{t+1} = \frac{\pi_{t+1}}{r + n_{t+1}\lambda} = \frac{(\alpha^{-1} - 1) w_{t+1} x_{t+1}}{r + n_{t+1}\lambda}$$

$$\omega_t = \frac{\lambda}{A_t} V_{t+1} = \frac{\lambda}{A_t} \left(\frac{(\alpha^{-1} - 1)\alpha^2 A_{t+1} x_{t+1}^{\alpha}}{r + n_{t+1}\lambda} \right)$$

$$\omega_t = \lambda \frac{\gamma(\alpha^{-1} - 1)\alpha^2 x_{t+1}^{\alpha}}{r + n_{t+1}\lambda}$$

$$= \lambda \frac{\gamma(\alpha^{-1} - 1)\alpha^2 \left(\frac{\alpha^2}{\left(\frac{w_{t+1}}{A_{t+1}} \right)} \right)^{\frac{\alpha}{1-\alpha}}}{r + n_{t+1}\lambda}$$

So dynamics in ω and n :

$$\omega_t = \lambda \frac{\gamma \tilde{\pi}(\omega_{t+1})}{r + n_{t+1}\lambda}$$

$$L = n_t + \tilde{x}_t(\omega_t)$$

Steady State (n, ω) is unique (draw):

$$r + n_{t+1}\lambda = \lambda \frac{\gamma \tilde{\pi}(\omega_{t+1})}{\omega_t}; \qquad \frac{dn}{d\omega} < 0$$

$$L = n_t + \tilde{x}_t(\omega_t); \qquad \frac{dn}{d\omega} > 0$$

$$\tilde{\pi}(\omega_{t+1}) = \frac{(r + (L - \tilde{x}_t(\omega_{t+1}))\lambda)\omega_t}{\lambda\gamma}$$

$$\left(r + \left(L - (\alpha^2)^{\frac{1}{1-\alpha}}(\omega_{t+1})^{\frac{1}{\alpha-1}}\right)\lambda\right)\omega_t$$

$$= \lambda\gamma(\alpha^{-1} - 1)(\alpha)^{\frac{2}{1-\alpha}}(\omega_{t+1})^{\frac{\alpha}{\alpha-1}}$$

$$\left(r + \left(L - (\alpha)^{\frac{2}{1-\alpha}}\omega^{\frac{1}{\alpha-1}}\right)\lambda\right)\omega = \lambda\gamma(\alpha^{-1} - 1)(\alpha)^{\frac{2}{1-\alpha}}(\omega_t$$

Local dynamics:

$$\tilde{\pi}'(\omega)d\omega_{t+1} = \frac{(r + (L - \tilde{x}(\omega))\lambda)d\omega_t - \tilde{x}'(\omega)\lambda\omega d\omega_{t+1}}{\lambda\gamma}$$

$$\left(\tilde{\pi}'(\omega) + \frac{\tilde{x}'(\omega)\lambda\omega}{\lambda\gamma}\right)d\omega_{t+1} = \frac{(r + (L - \tilde{x}(\omega))\lambda)d\omega_t}{\lambda\gamma}$$

$$d\omega_{t+1} = \left[\frac{(r + (L - \tilde{x}(\omega))\lambda)}{\lambda\gamma\tilde{\pi}'(\omega) + \tilde{x}'(\omega)\lambda\omega}\right]d\omega_t$$

Oscillatory since $\frac{(r+(L-\tilde{x}(\omega))\lambda)}{\lambda\gamma\tilde{\pi}'(\omega)+\tilde{x}'(\omega)} < 0.$

Evaluated at SS:

$$\tilde{\pi} = (\alpha^{-1} - 1)(\alpha)^{\frac{2}{1-\alpha}}(\omega)^{\frac{\alpha}{\alpha-1}}$$

$$\tilde{\pi}'(\omega) = \frac{\alpha}{\alpha - 1}\left(\frac{1-\alpha}{\alpha}\right)(\alpha)^{\frac{2}{1-\alpha}}(\omega)^{\frac{1}{\alpha-1}}$$

$$= -(\alpha)^{\frac{2}{1-\alpha}}(\omega)^{\frac{1}{\alpha-1}} < 0$$

and

$$x = \left(\frac{\alpha^2}{\omega}\right)^{\frac{1}{1-\alpha}}$$

$$x(\omega) = (\alpha^2)^{\frac{1}{1-\alpha}}\omega^{\frac{1}{\alpha-1}}$$

$$x'(\omega) = \frac{1}{\alpha - 1}(\alpha^2)^{\frac{1}{1-\alpha}}\omega^{\frac{2-\alpha}{\alpha-1}} < 0$$

$$(r + (L - \tilde{x}_t(\omega_{t+1}))\lambda) = \frac{\lambda\gamma\tilde{\pi}(\omega_{t+1})}{\omega_t}$$

$$\frac{d\omega_{t+1}}{d\omega_t} = \left[\frac{(r + (L - \tilde{x}(\omega))\lambda)}{\lambda\gamma\tilde{\pi}'(\omega) + \tilde{x}'(\omega)\lambda\omega} \right]$$

$$= \left[\frac{\frac{\lambda\gamma\tilde{\pi}(\omega_{t+1})}{\omega_t}}{-(\alpha)^{\frac{2}{1-\alpha}}(\omega)^{\frac{1}{\alpha-1}}\lambda\gamma + \frac{1}{\alpha-1}(\alpha^2)^{\frac{1}{1-\alpha}}\omega_t^{\frac{2-\alpha}{\alpha-1}}\lambda\omega} \right]$$

$$= \left[\frac{\lambda\gamma(\alpha^{-1} - 1)(\alpha)^{\frac{2}{1-\alpha}}(\omega_t)^{\frac{1}{\alpha-1}}}{(\alpha)^{\frac{2}{1-\alpha}}\omega_t^{\frac{1}{\alpha-1}}(-\lambda\gamma + \lambda\frac{1}{\alpha-1})} \right]$$

$$= \left[\frac{\lambda\gamma(\alpha^{-1} - 1)}{-\lambda\gamma + \lambda\frac{1}{\alpha-1}} \right]$$

$$= \left[\frac{(\alpha^{-1} - 1)}{\left(\frac{1}{(\alpha-1)\gamma} - 1\right)} \right] = \left[-\frac{(\alpha^{-1} - 1)}{\left(\frac{1}{(1-\alpha)\gamma} + 1\right)} \right]$$

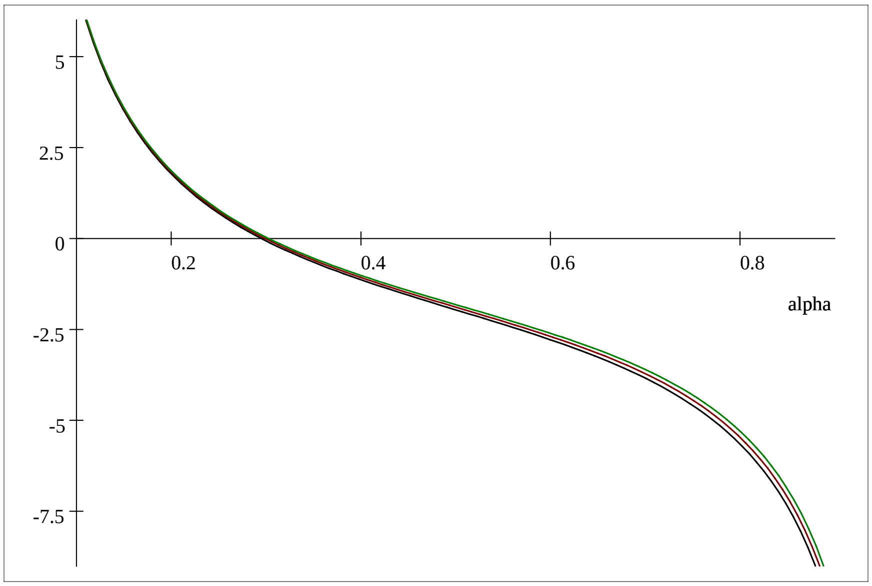
Determinacy:

Since ω is not a predetermined variable, for determinacy we need $\frac{d\omega_{t+1}}{d\omega_t} < -1$, or

$$(\alpha^{-1} - 1) > \left(\frac{1}{(1-\alpha)\gamma} + 1 \right)$$

Determinacy

$$(\alpha^{-1} - 1) > \left(\frac{1}{(1-\alpha)\gamma} + 1 \right)$$



$$\gamma = 1.02,\ 1.06,\ 1.1$$

Determinacy region for α to the left of the intersection.

Do comparative statics. At steady state

$$\tilde{\pi} = (\alpha^{-1} - 1)\omega x = \frac{1-\alpha}{\alpha}\omega x = \frac{1-\alpha}{\alpha}\omega(L - \hat{n})$$

$$\omega = \lambda\frac{\gamma\tilde{\pi}(\omega)}{r + \hat{n}\lambda} = \lambda\frac{\gamma\frac{1-\alpha}{\alpha}\omega(L - \hat{n})}{r + \hat{n}\lambda}$$

$$1 = \lambda\frac{\gamma\frac{1-\alpha}{\alpha}(L - \hat{n})}{r + \hat{n}\lambda}; \quad \frac{dn}{d\alpha} < 0 \qquad \#$$

1. n decreases with the elasticty of demand (α) faced by monopolist: lower profits, less innovation.

2. Lower $r,$ and higher L, γ or λ increases discounted stream of profits, and cause an increase in n.

Growth:

$$y_t = A_t x^\alpha = A_t(L-n)^\alpha$$

$$y_{t+1} = A_{t+1} x^\alpha = A_{t+1}(L-n)^\alpha$$

$$y_{t+1} = \gamma y_t$$

since $A_{t+1} = \gamma A_t$ where t to $t+1$ reflects the random time between innovations and gives rise to a discrete jump in productivity and output- a quality ladder. Taking logs:

$$\ell n\ y_{t+1} = \ell n\ y_t + \ell n\ \gamma$$

Log output increases by steps of γ. Converting to real time, between τ and $\tau+1$, we get $\varepsilon(\tau)$ innovations, with $\varepsilon(\tau)$ is distributed with Poisson parameter $\lambda n.$ So,

$$E\ \ell n\ y(\tau+1) - \ell n\ y(\tau) = \lambda n\ \ell n\ \gamma$$

where the LHS is the average growth rate $g = \lambda n\ \ell n\ \gamma.$ This is a random walk with drift, and leads to divergence.

Convergence:

Assume world grows at average rate g.

$$A_t^* = e^{gt}$$

Between τ and $\tau + d\tau,$ assume

$$A_{t+1} = F(A_t, e^{gt}), \quad F_1 > 0, \; F_2 > 0$$

so $F_1 > 0$ implies domestic spillovers, $F_2 > 0$ implies international spillovers.

$$F(A, \; A) = \gamma A, \quad \gamma > 1$$

which means there are only domestic spillovers. This implies,since

$$F(A_\tau, \; A_\tau^*) > F(A_\tau, \; A_\tau) = \gamma A_t$$

$$\frac{F(A_\tau, \; A_\tau^*)}{A_\tau} > \gamma$$

if $A_\tau < A_\tau^*$: country whose average productivity is low will grow faster. Without it, you have random walk with drift: $\ell n \; y_{t+1} = \ell n \; y_{t+1} + \ell n \; \gamma,$ so divergence. Not too deep in terms of technology diffusion!

Welfare

τ is time. t is the number of innovations, $\pi(t,\tau)$ is the probability that there are exactly t innovations time 0 up to τ. Expected Welfare U is

$$U = \int_0^\infty e^{-r\tau} y(\tau) d\tau = \int_0^\infty e^{-r\tau} \left(\sum_{t=0}^{\infty} \pi(t,\tau) A_t (x_t)^\alpha \right) d\tau$$

The distribution for the Poisson process is

$$\pi(t,\tau) = \frac{(\lambda n \tau)^t}{t!} e^{-\lambda n \tau}, \quad E\pi(t,\tau) = \lambda n \tau$$

Remembering that
$A_t = \gamma^t A_0,\ x_t = L - n_t,$ and

$$e^{-\lambda n \tau} \sum_{t=0}^{\infty} \frac{(\gamma \lambda n \tau)^t}{t!} = e^{-\lambda n \tau} e^{\gamma \lambda n \tau} = e^{(1-\gamma)\lambda n \tau}$$

we have

$$U(n) = \frac{A_0 (L - n)^\alpha}{r - \lambda n (\gamma - 1)}$$

and $U'(n^*) = 0$ implies:

$$\frac{-\alpha A_0(L-n)(r-\lambda n(\gamma-1)) + A_0(L-n)^{\alpha}\lambda(\gamma-1)}{(r-\lambda n(\gamma-1))} = 0$$

$$1 = \frac{\lambda(\gamma-1)\alpha^{-1}(L-n^*)}{r-\lambda n^*(\gamma-1)}$$

$$so \;\; g^* = \lambda n^* \ell n \; \gamma$$

Note under laissez-faire ($(*)$ above)

$$1 = \frac{\lambda\gamma\alpha^{-1}(1-\alpha)(L-\hat{n})}{r+\lambda\hat{n}}$$

instead of

$$1 = \frac{\lambda(\gamma-1)\alpha^{-1}(L-n^*)}{r-\lambda n^*(\gamma-1)}$$

1. Social discount rates are $r-\lambda n^*(\gamma-1) < r,$ not $r+\lambda\hat{n} > r.$ This reflects positive effect of innovations on future productivity, or spillover effect: under-laissez faire too little research.

2. $(1-\alpha)$ that appears in the private case reflects the distortion from the monopoly market, where output is curtailed. Appropriability effect: too little research under-laissez faire .

3. $(\gamma-1)$ instead of γ in the numerator reflects the internalization of the business stealing effect (obsolescence) by the social planner. Too much research under-laissez

faire: business stealing effect.

Charles I Jones, "Time Series Tests of

Endogenous Growth Models;" QJE, v. 110, 1995, 495-525

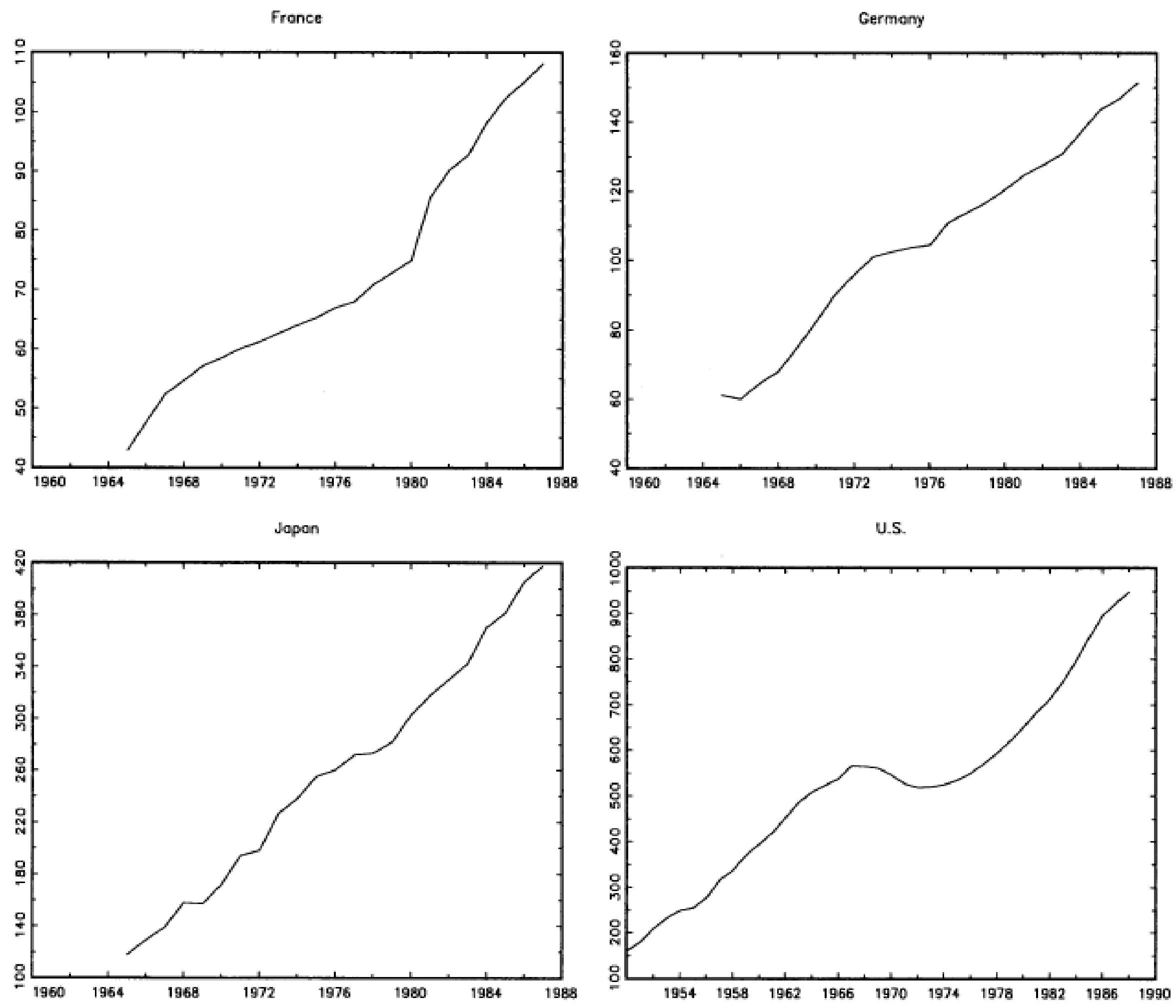


FIGURE IV
Scientists and Engineers Engaged in R&D (1000s)

Source. NSF Science and Engineering Indicators 1989 and Bureau of the Census (various).

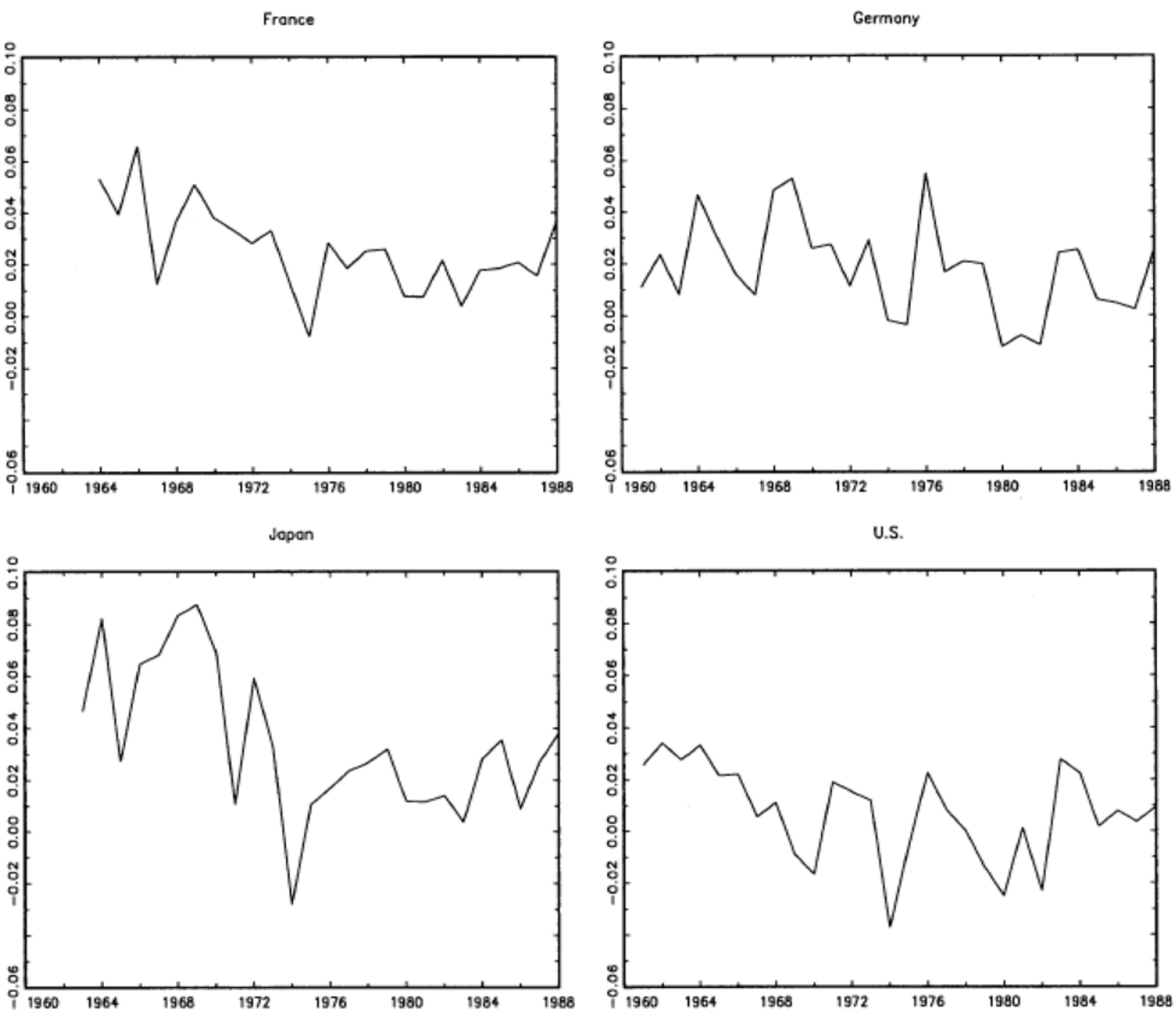


FIGURE V
Aggregate Total Factor Productivity Growth

Source. OECD Department of Economics and Statistics Analytic Database. Data provided by Steven Englander.