

In Aghion-Howitt book, page 57:

Poisson Process modeled as:

*Cumulative distr. for event before T*

$$F(T) = 1 - e^{-\mu T}$$

$$f(T) = \mu e^{-\mu T} \quad (\text{Density})$$

$$g(x) = \text{prob. that } x \text{ events occurs in the } \Delta \text{ interval}$$
$$= \frac{(\mu\Delta)^x e^{-\mu\Delta}}{x!}$$

Final product:

$$y = Ax^\alpha, \quad \alpha < 1$$

$$p_t = A_t \alpha (x_t)^{\alpha-1}$$

because intermediate good price is its marginal product in final good production.

Intermediate product:

Uses one unit of labor to produce one unit of  $x$  with a linear technology  $x = L_x$ , so:

$$\pi = p_t x - w_t x = A_t \alpha (x_t)^\alpha - w_t x_t$$

$$x_t = \text{Arg max} \{A_t \alpha (x_t)^\alpha - w_t x_t\}$$

$$\alpha^2 A x^{a-1} = w, \quad \frac{w_t x_t}{\alpha} = \frac{\alpha A_t}{x_t^{-\alpha}} = \alpha A_t x_t^\alpha;$$

$$x_t = \left( \frac{\alpha^2}{\left(\frac{w_t}{A_t}\right)} \right)^{\frac{1}{1-\alpha}} \equiv x(\omega_t); \text{ where } \omega = \frac{w}{A}$$

$$\pi = A_t \alpha x^\alpha - w_t x = (\alpha^{-1} - 1) w_t x_t$$

If  $\alpha = 1$ ,  $\frac{dp}{dx} = 0$ , no profits.

Expressing profits as a function of  $\omega$  :

$$w_t x_t = \alpha^2 A_t x_t^\alpha$$

$$\pi = (\alpha^{-1} - 1) w_t x_t = A_t (\alpha^{-1} - 1) \alpha^2 x_t^\alpha$$

*or*

$$\pi = A_t (\alpha^{-1} - 1) \left( \frac{w_t}{A_t} \right) (x_t(\omega_t)) \equiv A_t \tilde{\pi}_t(\omega_t)$$

**NOTE**  $\tilde{x}(\omega_t), \tilde{\pi}_t(\omega_t)$  are decreasing in  $\omega_t$  :

$$\pi_t = A_t (\alpha^{-1} - 1) \frac{w_t}{A_t} x_t(\omega_t) = A_t (\alpha^{-1} - 1) \alpha^2 x_t^\alpha$$

$$= A_t (\alpha^{-1} - 1) \alpha^2 \left( \frac{\alpha^2}{\left( \frac{w_t}{A_t} \right)} \right)^{\frac{\alpha}{1-\alpha}}$$

$$= A_t (\alpha^{-1} - 1) (\alpha)^{\frac{2}{1-\alpha}} (\omega_t)^{\frac{\alpha}{\alpha-1}}$$

1. Distributional Assumption: If  $n$  persons do research, probability density function of an invention at time  $\tau$  is given by the density

$$f(\tau) = (\lambda n)e^{-(\lambda n)\tau}$$

with cdf, for probability of an invention by time  $\tau$  :

$$F(\tau) = 1 - e^{-(\lambda n)\tau}$$

$$F(0) = 0, \quad F(\infty) = 1$$

2. Profits if firm survives until  $T$ , when new invention arrives:

$$\int_0^T \pi e^{-rs} ds = \frac{\pi}{r} (1 - e^{-rT})$$

3. Expected value of firm is discounted profits summed over survival probabilities to each  $T$  (that is by probabilities that there is an invention exactly at  $T$ , given by the density function above):

$$\begin{aligned}
 V &= \int_0^{\infty} (\lambda n) e^{-(\lambda n)T} \int_0^T \pi e^{-rs} ds dT \\
 &= \int_0^{\infty} (\lambda n) e^{-(\lambda n)T} \left( \frac{\pi}{r} (1 - e^{-rT}) \right) dT \\
 V &= \frac{\pi}{r} \left( \int_0^{\infty} (\lambda n) e^{-(\lambda n)T} (1 - e^{-rT}) dT \right) \\
 &= \frac{\pi}{r} \left( \int_0^{\infty} (\lambda n) e^{-(\lambda n)T} dT - \int_0^{\infty} (\lambda n) e^{-(r+\lambda n)T} dT \right) \\
 &= \frac{\pi}{r} \left( (0 + 1) - \left( 0 + \frac{\lambda n}{r + \lambda n} \right) \right) \\
 &= \frac{\pi}{r} \left( 1 - \frac{\lambda n}{r + \lambda n} \right) = \frac{\pi}{r} \left( \frac{r}{r + \lambda n} \right) = \frac{\pi}{r + \lambda n}
 \end{aligned}$$

4. Therefore, the expected value of a firm:

$$V = \frac{\pi}{r + \lambda n}$$

$$(r + \lambda n)V = \pi$$

$$rV = \pi - (\lambda n)V$$

Labor is divided between research and production

$$L = n_t + x_t$$

Arbitrage across labor markets for research and manufacturing (assuming  $A_{t+1} = \gamma A_t$  because improvements in technology are by a factor  $\gamma$ ) :

Now if  $t$  is the time between innovations,

$$\text{Max}_{n_t} n_t \lambda V_{t+1} - w_t n_t$$

$$\text{FOC} : w_t = \lambda V_{t+1}; \quad \text{or} \quad \omega_t = \frac{w_t}{A_t} = \frac{\lambda}{A_t} V_{t+1}$$

Also

$$V_{t+1} = \frac{\pi_{t+1}}{r + n_{t+1} \lambda} = \frac{(\alpha^{-1} - 1) w_{t+1} x_{t+1}}{r + n_{t+1} \lambda}$$

Therefore

$$\omega_t = \frac{\lambda}{A_t} V_{t+1} = \frac{\lambda}{A_t} \left( \frac{(\alpha^{-1} - 1) \alpha^2 A_{t+1} x_{t+1}^\alpha}{r + n_{t+1} \lambda} \right)$$

$$\omega_t = \lambda \frac{\gamma (\alpha^{-1} - 1) \alpha^2 x_{t+1}^\alpha}{r + n_{t+1} \lambda}$$

$$= \lambda \frac{\gamma (\alpha^{-1} - 1) \alpha^2 \left( \frac{\alpha^2}{\left( \frac{w_{t+1}}{A_{t+1}} \right)} \right)^{\frac{\alpha}{1-\alpha}}}{r + n_{t+1} \lambda} = \lambda \frac{\gamma \tilde{\pi}(\omega_{t+1})}{r + n_{t+1} \lambda}$$



So dynamics in  $\omega$  and  $n$  :

$$\omega_t = \lambda \frac{\gamma \tilde{\pi}(\omega_{t+1})}{r + n_{t+1} \lambda}$$

$$L = n_t + \tilde{x}_t(\omega_t)$$

Steady State  $(n, \omega)$  is unique (draw):

$$r + n_{t+1} \lambda = \lambda \frac{\gamma \tilde{\pi}(\omega_{t+1})}{\omega_t}; \quad \frac{dn}{d\omega} < 0$$

$$L = n_t + \tilde{x}_t(\omega_t); \quad \frac{dn}{d\omega} > 0$$

$$\frac{(r + (L - \tilde{x}_t(\omega_{t+1})) \lambda) \omega_t}{\lambda \gamma} = \tilde{\pi}(\omega_{t+1})$$

$$\begin{aligned} & \left( r + \left( L - (\alpha^2)^{\frac{1}{1-\alpha}} \omega^{\frac{1}{\alpha-1}} \right) \lambda \right) \omega \\ &= -\lambda \gamma (\alpha^{-1} - 1) (\alpha)^{\frac{2}{1-\alpha}} (\omega)^{\frac{\alpha}{\alpha-1}} \end{aligned}$$

Local dynamics:

$$\tilde{\pi}'(\omega)d\omega_{t+1} = \frac{(r + (L - \tilde{x}(\omega))\lambda)d\omega_t - \tilde{x}'(\omega)\lambda\omega d\omega_{t+1}}{\lambda\gamma}$$

$$\left( \tilde{\pi}'(\omega) + \frac{\tilde{x}'(\omega)\lambda\omega}{\lambda\gamma} \right) d\omega_{t+1} = \frac{(r + (L - \tilde{x}(\omega))\lambda)d\omega_t}{\lambda\gamma}$$

$$d\omega_{t+1} = \left[ \frac{(r + (L - \tilde{x}(\omega))\lambda)}{\lambda\gamma\tilde{\pi}'(\omega) + \tilde{x}'(\omega)\lambda\omega} \right] d\omega_t = Jd\omega_t$$

Oscillatory since  $J = \frac{(r+(L-\tilde{x}(\omega))\lambda)}{\lambda\gamma\tilde{\pi}'(\omega)+\tilde{x}'(\omega)} < 0$ .

Higher research,  $n_{t+1}$  tomorrow increases creative destruction by raising probability of innovation and also less profit because of higher wages. This leads to low research today,  $n_t$ , because the value of today's innovation is reduced.

$J$  evaluated at SS:

$$\tilde{\pi} = (\alpha^{-1} - 1)(\alpha)^{\frac{2}{1-\alpha}} (\omega)^{\frac{\alpha}{\alpha-1}}$$

$$\begin{aligned}\tilde{\pi}'(\omega) &= \frac{\alpha}{\alpha - 1} \left( \frac{1 - \alpha}{\alpha} \right) (\alpha)^{\frac{2}{1-\alpha}} (\omega)^{\frac{1}{\alpha-1}} \\ &= -(\alpha)^{\frac{2}{1-\alpha}} (\omega)^{\frac{1}{\alpha-1}} < 0\end{aligned}$$

and

$$x = \left( \frac{\alpha^2}{\omega} \right)^{\frac{1}{1-\alpha}}$$

$$x(\omega) = (\alpha^2)^{\frac{1}{1-\alpha}} \omega^{\frac{1}{\alpha-1}}$$

$$x'(\omega) = \frac{1}{\alpha - 1} (\alpha^2)^{\frac{1}{1-\alpha}} \omega^{\frac{2-\alpha}{\alpha-1}} < 0$$

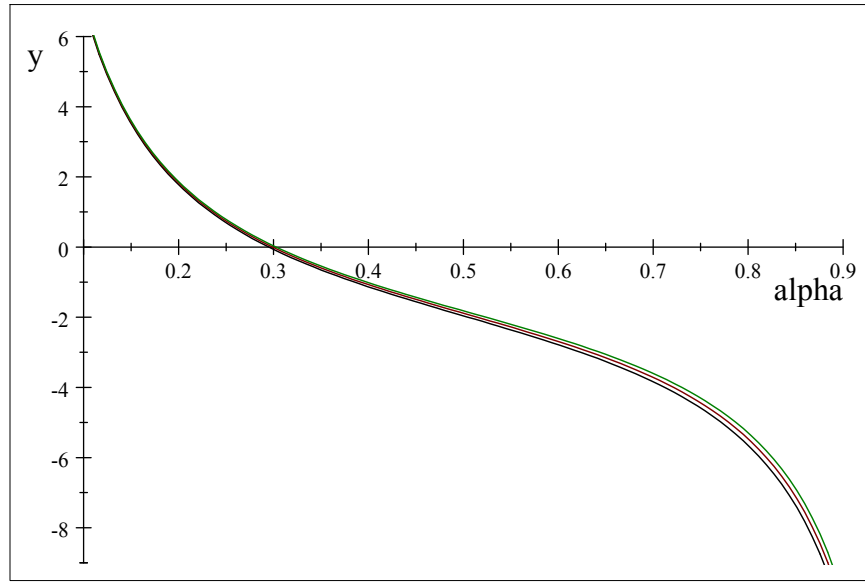
$$\begin{aligned}
\frac{d\omega_{t+1}}{d\omega_t} &= \left[ \frac{(r + (L - \tilde{x}(\omega))\lambda)}{\lambda\gamma\tilde{\pi}'(\omega) + \tilde{x}'(\omega)\lambda\omega} \right] \\
&= \left[ \frac{\frac{\lambda\gamma\tilde{\pi}(\omega_{t+1})}{\omega_t}}{-\left(\alpha\right)^{\frac{2}{1-\alpha}} \left(\omega\right)^{\frac{1}{\alpha-1}} \lambda\gamma + \frac{1}{\alpha-1} \left(\alpha^2\right)^{\frac{1}{1-\alpha}} \omega_t^{\frac{2-\alpha}{\alpha-1}} \lambda\omega} \right] \\
&= \left[ \frac{\lambda\gamma\left(\alpha^{-1} - 1\right)\left(\alpha\right)^{\frac{2}{1-\alpha}} \left(\omega_t\right)^{\frac{1}{\alpha-1}}}{\left(\alpha\right)^{\frac{2}{1-\alpha}} \omega_t^{\frac{1}{\alpha-1}} \left(-\lambda\gamma + \lambda\frac{1}{\alpha-1}\right)} \right] \\
&= \left[ \frac{\lambda\gamma\left(\alpha^{-1} - 1\right)}{-\lambda\gamma + \lambda\frac{1}{\alpha-1}} \right] \\
&= \left[ \frac{\left(\alpha^{-1} - 1\right)}{\left(\frac{1}{(\alpha-1)\gamma} - 1\right)} \right] = \left[ -\frac{\left(\alpha^{-1} - 1\right)}{\left(\frac{1}{(1-\alpha)\gamma} + 1\right)} \right]
\end{aligned}$$

Determinacy:  $\omega$  is not a predetermined, so determinacy requires  $\frac{d\omega_{t+1}}{d\omega_t} < -1$ , or

$$\left(\alpha^{-1} - 1\right) > \left(\frac{1}{(1-\alpha)\gamma} + 1\right)$$

# Determinacy

$$(\alpha^{-1} - 1) > \left( \frac{1}{(1 - \alpha)\gamma} + 1 \right)$$



$$\gamma = 1.02, 1.06, 1.1$$

Do comparative statics. At steady state

$$\tilde{\pi} = (\alpha^{-1} - 1)\omega x = \frac{1 - \alpha}{\alpha} \omega x = \frac{1 - \alpha}{\alpha} \omega(L - \hat{n})$$

$$\omega = \lambda \frac{\gamma \tilde{\pi}(\omega)}{r + \hat{n}\lambda} = \lambda \frac{\gamma \frac{1 - \alpha}{\alpha} \omega(L - \hat{n})}{r + \hat{n}\lambda}$$

$$1 = \lambda \frac{\gamma \frac{1 - \alpha}{\alpha} (L - \hat{n})}{r + \hat{n}\lambda}; \quad \frac{dn}{d\alpha}, \frac{dn}{dr} < 0, \frac{dn}{dL} > 0 \quad \#$$

1.  $n$  decreases with the elasticity of demand ( $\alpha$ ) faced by monopolist: lower profits, less innovation.

2. Lower  $r$ , and higher  $L, \gamma$  or  $\lambda$  increases discounted stream of profits, and cause an increase in  $n$ .

Growth:

$$y_t = A_t x^\alpha = A_t (L - n)^\alpha$$

$$y_{t+1} = A_{t+1} x^\alpha = A_{t+1} (L - n)^\alpha$$

$$y_{t+1} = \gamma y_t$$

since  $A_{t+1} = \gamma A_t$  where  $t$  to  $t + 1$  reflects the random time between innovations and gives rise to a discrete jump in productivity and output- a quality ladder. Taking logs:

$$\ln y_{t+1} = \ln y_t + \ln \gamma$$

Log output increases by steps of  $\gamma$ .

Converting to real time, between  $\tau$  and  $\tau + 1$ , we get  $\varepsilon(\tau)$  innovations, with  $\varepsilon(\tau)$  is distributed with Poisson parameter  $\lambda n$ . So,

$$E \ln y(\tau + 1) - \ln y(\tau) = \lambda n \ln \gamma$$

where the LHS is the average growth rate  $g = \lambda n \ln \gamma$ . This is a random walk with drift, and leads to divergence.

Convergence:

Assume world grows at average rate  $g$ .

$$A_t^* = e^{gt}$$

Between  $\tau$  and  $\tau + d\tau$ , assume

$$A_{t+1} = F(A_t, e^{gt}), \quad F_1 > 0, \quad F_2 > 0$$

so  $F_1 > 0$  implies domestic spillovers,  
 $F_2 > 0$  implies international spillovers.

$$F(A, A) = \gamma A, \quad \gamma > 1$$

which means there are only domestic spillovers. This implies, since

$$F(A_\tau, A_\tau^*) > F(A_\tau, A_\tau) = \gamma A_\tau$$

$$\frac{F(A_\tau, A_\tau^*)}{A_\tau} > \gamma$$

if  $A_\tau < A_\tau^*$  : country whose average productivity is low will grow faster. Without it, you have random walk with drift:

$\ln y_{t+1} = \ln y_t + \ln \gamma$ , so divergence. Not too deep in terms of technology diffusion!



## Welfare

$\tau$  is time.  $t$  is the number of innovations,  $\pi(t, \tau)$  is the probability that there are exactly  $t$  innovations time 0 up to  $\tau$ .

Expected Welfare  $U$  is

$$U = \int_0^{\infty} e^{-r\tau} y(\tau) d\tau = \int_0^{\infty} e^{-r\tau} \left( \sum_{t=0}^{\infty} \pi(t, \tau) A_t(x_t)^\alpha \right) d\tau$$

The distribution for the Poisson process is

$$\pi(t, \tau) = \frac{(\lambda n \tau)^t}{t!} e^{-\lambda n \tau}, \quad E\pi(t, \tau) = \lambda n \tau$$

Remembering that  $A_t = \gamma^t A_0$ ,  $x_t = L - n_t$ ,

$$U(n) = \frac{A_0(L - n)^\alpha}{r - \lambda n(\gamma - 1)}$$

and  $U'(n^*) = 0$  implies:

$$\frac{-\alpha A_0(L - n)(r - \lambda n(\gamma - 1)) + A_0(L - n)^\alpha \lambda(\gamma - 1)}{(r - \lambda n(\gamma - 1))} = 0$$

$$1 = \frac{\lambda(\gamma - 1)\alpha^{-1}(L - n^*)}{r - \lambda n^*(\gamma - 1)}$$

$$\text{so } g^* = \lambda n^* \ln \gamma$$

Note under laissez-faire (equation (\* \*))

$$1 = \frac{\lambda\gamma\alpha^{-1}(1-\alpha)(L-\hat{n})}{r+\lambda\hat{n}}$$

instead of

$$1 = \frac{\lambda(\gamma-1)\alpha^{-1}(L-n^*)}{r-\lambda n^*(\gamma-1)}$$

1. Social discount rates are  $r - \lambda n^*(\gamma - 1) < r$ , not  $r + \lambda \hat{n} > r$ . This reflects positive effect of innovations on future productivity, or spillover effect: under-laissez faire too little research.
2.  $(1 - \alpha)$  appears in the private case: reflects the monopoly distortion, with output curtailed. Appropriability effect: too little research under-laissez faire .
3.  $(\gamma - 1)$  instead of  $\gamma$  in the numerator reflects the internalization of the business stealing effect (obsolescence) by the social planner. Too much research under-laissez faire: business stealing effect.

Charles I Jones, "Time Series Tests of Endogenous Growth Models;" *QJE*, v. 110, 1995, 495-525

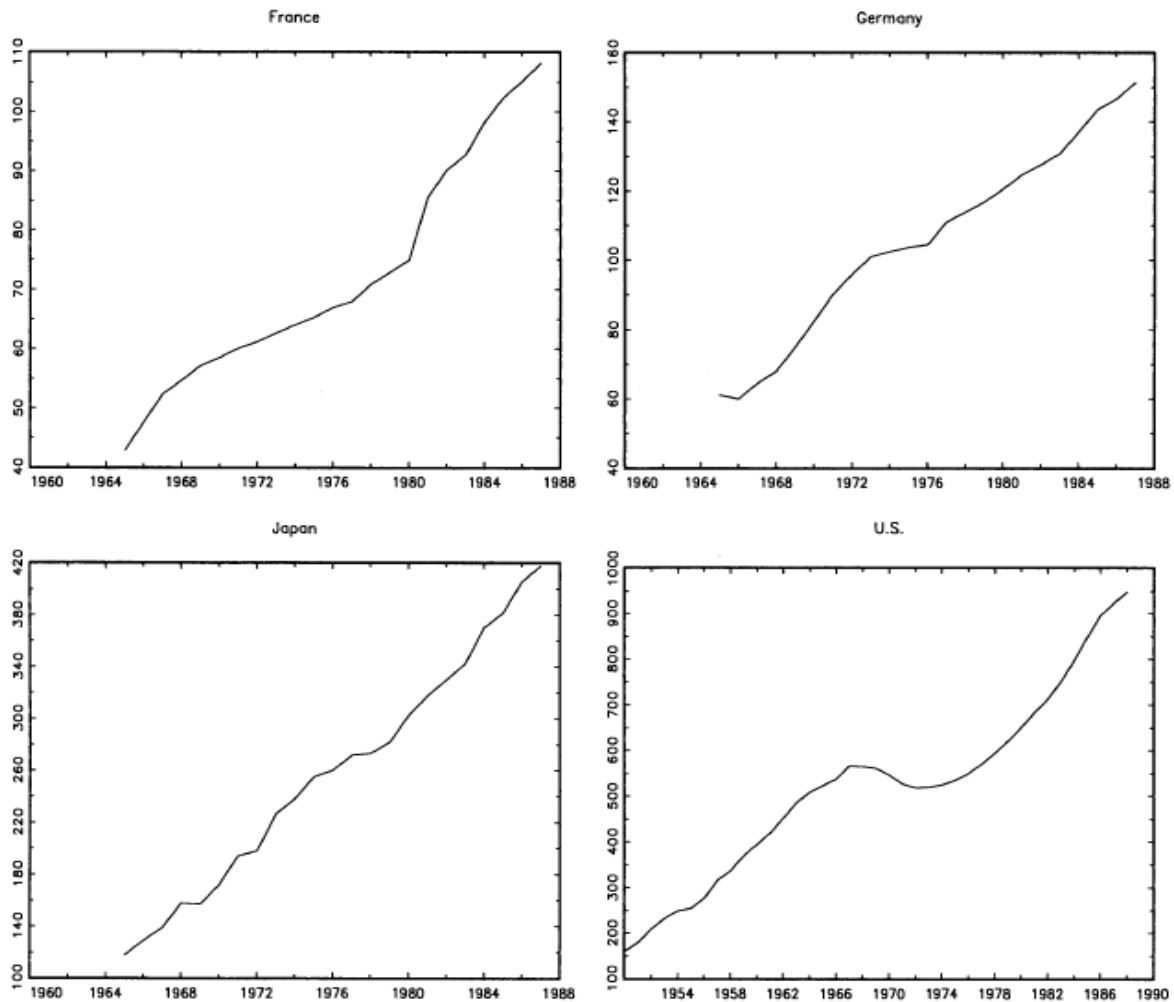
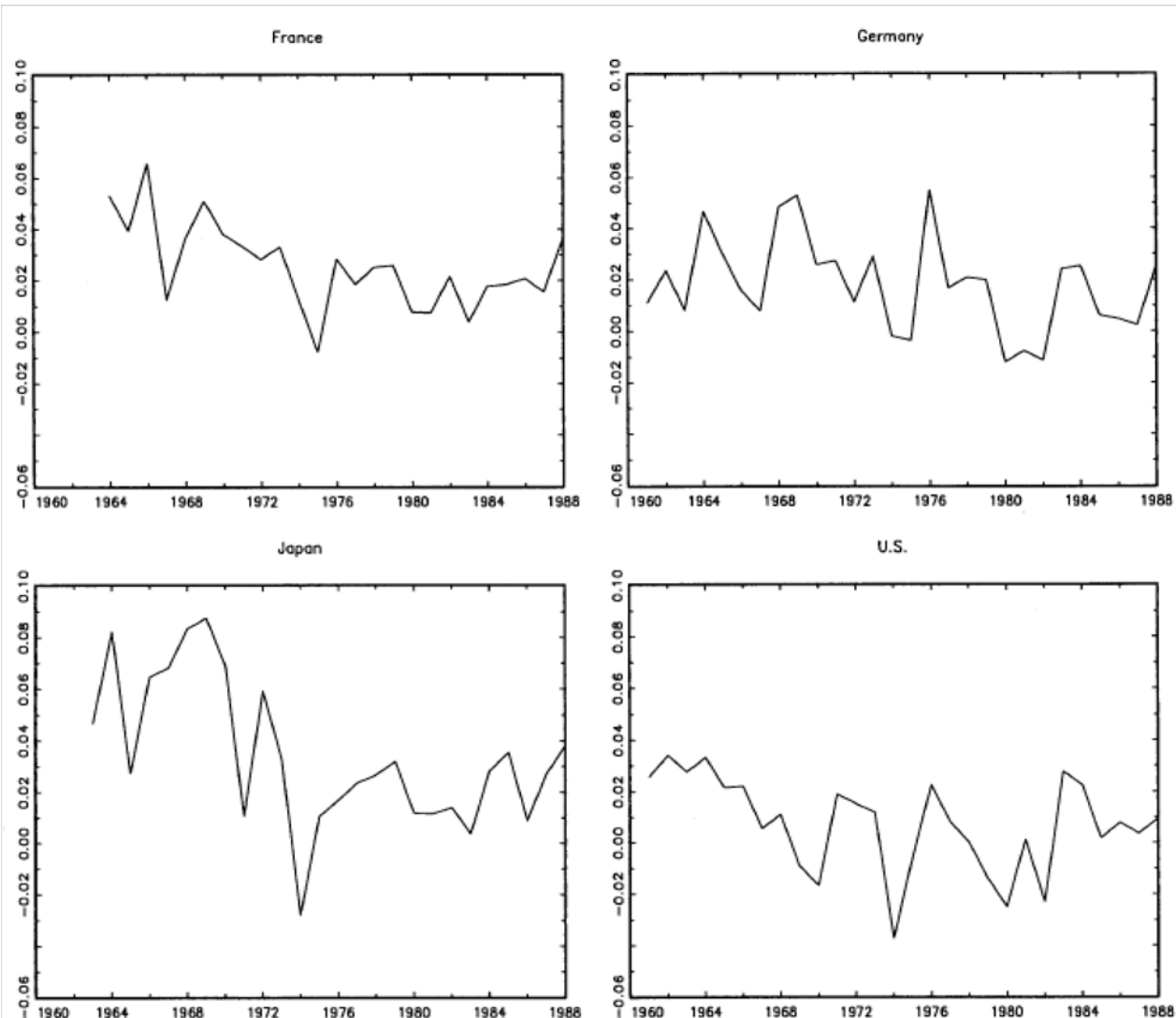


FIGURE IV

Scientists and Engineers Engaged in R&D (1000s)

Source. *NSF Science and Engineering Indicators 1989* and *Bureau of the Census* (various).



**FIGURE V**

**Aggregate Total Factor Productivity Growth**

*Source.* OECD Department of Economics and Statistics Analytic Database.  
Data provided by Steven Englander.

## Scale Effects

Final good:

$$Y_t = \left( \frac{L}{M} \right)^{1-\alpha} \int_0^M (A_{it})^{1-\alpha} (x_{it})^\alpha di$$

with each product having its own productivity parameter  $A_{it}$ . Output depends not on total labor but on labor per intermediate good,  $\frac{L}{M}$ . Note that before integrating wrt  $i$ , we can say each intermediate good contributes

$$Y_{it} = \left( \frac{L}{M} \right)^{1-\alpha} (A_{it})^{1-\alpha} (x_{it})^\alpha$$

Now for simplicity assume each person can with probability  $\psi$  can invent a new intermediate product, and a fraction  $\varepsilon$  of intermediate products become obsolete, so change is

$$\psi L - \varepsilon M$$

$$M_{t+1} = M_t + \psi L - \varepsilon M_t$$

$$M_t = \frac{\psi}{\varepsilon}L + (1 - \varepsilon)^t \left[ M_0 - \frac{\psi}{\varepsilon}L \right]$$
$$\rightarrow \frac{\psi}{\varepsilon}L \text{ as } t \rightarrow \infty$$

so product variety  $M$  is proportional to  $L$ .

Thus

$$Y_t = \left(\frac{\psi}{\varepsilon}\right)^{1-\alpha} \int_0^M (A_{it})^{1-\alpha} (x_{it})^\alpha di$$

$$Y_{it} = \left(\frac{\psi}{\varepsilon}\right)^{1-\alpha} (A_{it})^{1-\alpha} (x_{it})^\alpha$$

$$p_{it} = \frac{\partial Y_{it}}{\partial x_{it}} = \alpha \left(\frac{\psi}{\varepsilon}\right)^{1-\alpha} (A_{it})^{1-\alpha} (x_{it})^{\alpha-1}$$

Normalizing prices by wages so  $w_{it} = 1$

$$\Pi_{it} = p_{it}x_{it} - x_{it} = \alpha \left(\frac{\psi}{\varepsilon}\right)^{1-\alpha} (A_{it})^{1-\alpha} (x_{it})^\alpha - x_{it}$$

so optimizing wrt  $x_{it}$ ,

$$x_{it} = \alpha^{\frac{2}{1-\alpha}} \frac{\varepsilon}{\psi} A_{it}$$

$$\Pi_{it} = \left( (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} \right) \frac{\varepsilon}{\psi} A_{it} \equiv \pi \frac{\varepsilon}{\psi} A_{it}$$

So all independent of  $L$ .



Now if  $\phi(n_{it})$  is the probability of a successful invention of an intermediate good while spending  $n_{it}$  on research, the net profit will be

$$\phi(n_{it})\Pi_{it} - R_{it} = \left[ \phi(n_{it})\pi \frac{\varepsilon}{\psi} - n_{it} \right] A_{it}$$

where  $n_{it} = \frac{R_{it}}{A_{it}^*}$ , expenditure for research needed to attain productivity  $A_{it}^*$ .

Optimizing this,

$$\phi'(n_{it})\pi \frac{\varepsilon}{\psi} = 1$$

so  $n_{it}$  is independent of  $L$  and so is growth as the rate of innovation depends on  $n_{it}$ .