The Growth Dynamics of Innovation, Diffusion, and the Technology Frontier

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Broad Question

How do technology adoption and innovation **jointly** determine: (1) the **shape of the productivity distribution**; (2) the **aggregate growth rate**; and (3) the **technology frontier**?

- Innovation = the invention of new ideas/technologies
- Adoption = using a technology that has already been invented
- Technology frontier = highest productivity technology in use

Goal: Build a simple but rich framework that allows study of the interaction between innovation and diffusion and the effect of related policies.

Specific Questions

- What determines the shape of the productivity distribution?
  - How and why would subsidizing adoption or innovation change the tail index (variance) of the productivity distribution?
  - Is there hysteresis in determining the shape of the productivity distribution?
- What determines the technology frontier?
  - How productive can the best firms be relative to the worst firms?
  - Is the ratio of the frontier to the min productivity bounded?
- What determines the aggregate growth rate?
  - Innovation expands the frontier and drives long-run growth
  - What determines the rate of innovation?
  - Would a subsidy to adoption increase, decrease, or have no impact on aggregate growth rates? Why?
  - Is this sensitive to ideas being *partially* excludable (patents, licensing)?

Outline

1. Develop stripped down model of adoption and exogenous innovation
   - See how parameters affect shape of productivity distribution
   - Establish some properties of the equilibrium
   - See what determines value of firm (option value of adoption)
2. Firms choose innovation rate $\rightarrow$ endogenous growth rates
3. Extensions
   - Leap-frogging to frontier (like quality ladders)
   - "Directed" adoption (today—half-slide)
   - Partial excludability (today—half slide)
   - Entry/Exit (not today—in paper)
   - Monopolistic Competition (not today—in paper)

Model Summary with Exogenous, Stochastic Innovation

Environment:

- Continuous time
- Unit mass of infinitely lived firms
- Firms heterogeneous over productivity Z and innovation type $i \in \{\ell, h\}$
- For simplicity, profits=output=productivity=Z

Notation:

- Partial Derivative Operator: $\boldsymbol{\partial}_t \equiv \frac{\partial}{\partial t}$, or Univariate $F'(z) \equiv \frac{\mathrm{d}F(z)}{\mathrm{d}z}$
- Idiosyncratic productivity Z with cdf $\Phi_i(t, Z)$, pdf $\boldsymbol{\partial}_Z \Phi_i(t, Z)$
  - Unconditional distribution: $\Phi(t, Z) \equiv \sum_i \Phi_i(t, Z)$ with $\Phi(t, \infty) = 1$
  - $\mathrm{support}\,\{\Phi_i(t, \cdot)\} = [M(t), B(t))$. Technology frontier $= B(t) \leq \infty$

Stochastic Innovation with a Finite Frontier

Innovation and adoption are two ways to improve Z:
Adoption depends on technology distribution, make innovation return completely idiosyncratic to highlight endogenous link.

- Innovation: stochastic process for the proportional growth in Z
  - Assume a Markov chain of innovation states, i
    - Issues with geometric Brownian motion
  - ℓ: firm is **stagnant**, waiting to be innovative or adopt technology
  - h: firm **innovative**. Z grows at rate $\gamma > 0$ (exogenous for now)
  - $\ell \to h$ jump intensity $= \lambda_\ell$. $h \to \ell$ jump intensity $= \lambda_h$

- Technology diffusion: an *immediate* draw of a new productivity
  - Incumbent can draw a productivity from the **existing** distribution in the economy, which will evolve endogenously
  - Draw from the distorted unconditional CDF: $[\Phi(t, Z)]^\kappa$ for $\kappa > 0$
  - Adoption cost: $\zeta > 0$, proportional to size of economy
  - Assume start in ℓ state (for simplicity)
  - (Random search is easy: others have teaching market, congestion)

Optimal Stopping and Endogenous Minimum of Support

- For now, all innovation is exogenous
- Firm only needs to decide when to adopt
- Optimal policy is threshold rule: $Z < M(t)$ means adopt
- Optimal stopping problem for **endogenous threshold** $M(t)$
  - Continuation value of firm $V_i(t, Z)$
  - Value matching condition at $M(t)$, the value of technology diffusion
  - Smooth pasting condition
  - Can prove ℓ and h firms choose same $M(t)$
- Reservation productivity $M(t)$ is min of support of distribution

Firm's Problem

Bellman Equations (Continuation Value):

$$rV_\ell(t,Z) = \underbrace{Z}_{\text{Flow Profits}} + \lambda_\ell \underbrace{\left(V_h(t,Z) - V_\ell(t,Z)\right)}_{\text{Jump to } h} + \underbrace{\partial_t V_\ell(t,Z)}_{\text{Capital Gains}}$$

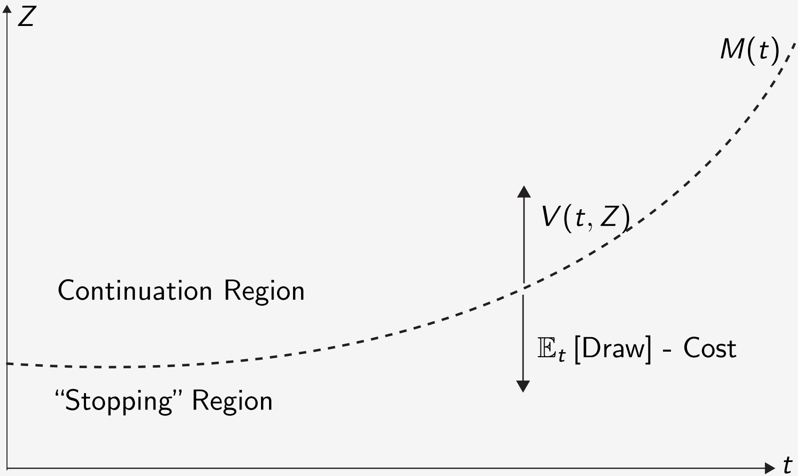
$$rV_h(t,Z) = Z + \underbrace{\gamma Z \partial_Z V_h(t,Z)}_{\text{Exogenous Innovation}} + \lambda_h \left(V_\ell(t,Z) - V_h(t,Z)\right) + \partial_t V_h(t,Z)$$

Value Matching + Smooth Pasting:

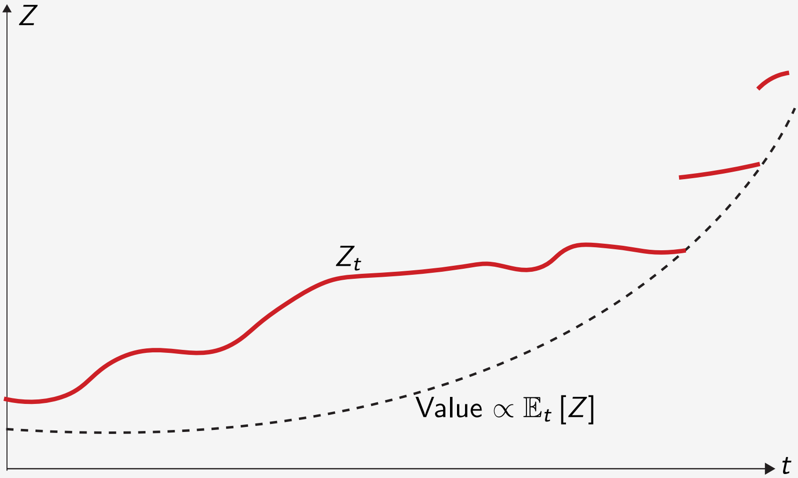
$$\underbrace{V_i(t,M(t))}_{\text{Value at Threshold}} = \underbrace{\int_{M(t)}^{B(t)} V_\ell(t,Z')\mathrm{d}\Phi(t,Z')^\kappa}_{\text{Gross Adoption Value}} - \underbrace{\zeta M(t)}_{\text{Adoption Cost}}$$

$$\partial_Z V_i(t,M(t)) = \underbrace{0}_{\text{Cost} \perp Z}$$

Value Functions and Stopping Problems



Example Path: Stochastic Innovation + Adoption Choices



- Instantaneous choice: adopt vs. operate existing technology
- Take aggregates as given (i.e., don't internalize effect of decision)

Evolution of the Distribution

Define $S_i(t)$ as the endogenous flow of adopters crossing $M(t)$.

Kolmogorov Forward Equations (for cdfs):

$$\partial_t \Phi_\ell(t,Z) = \underbrace{-\lambda_\ell \Phi_\ell(t,Z) + \lambda_h \Phi_h(t,Z)}_{\text{Net Flow from Jumps}} + \underbrace{(S_\ell(t) + S_h(t))}_{\text{Flow Adopters}} \underbrace{\Phi(t,Z)^\kappa}_{\text{Draw} \leq Z} - \underbrace{S_\ell(t)}_{\text{Adopt}}$$

$$\partial_t \Phi_h(t,Z) = \underbrace{-\gamma Z \partial_Z \Phi_h(t,Z)}_{\text{Innovation}} - \lambda_h \Phi_h(t,Z) + \lambda_\ell \Phi_\ell(t,Z) - S_h(t)$$

$$0 = \Phi_i(t, M(t))$$

$$1 = \Phi_\ell(t,\infty) + \Phi_h(t,\infty)$$

Evolution of the Technology Frontier and Adoption Threshold:

$$\frac{\partial_t B(t)}{B(t)} = \gamma \quad ; \quad \frac{\partial_t M(t)}{M(t)} \equiv g(t)$$

Normalization and Stationarity

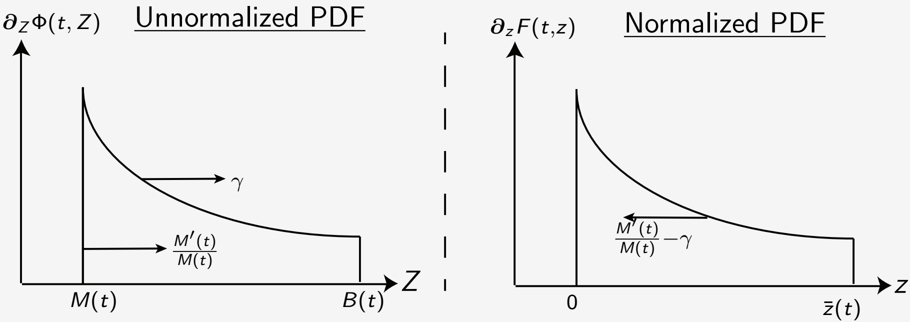
$$\boxed{z \equiv \log(Z/M(t))}$$

$$\bar{z}(t) = \log(B(t)/M(t)) \qquad \text{(Relative Frontier)}$$

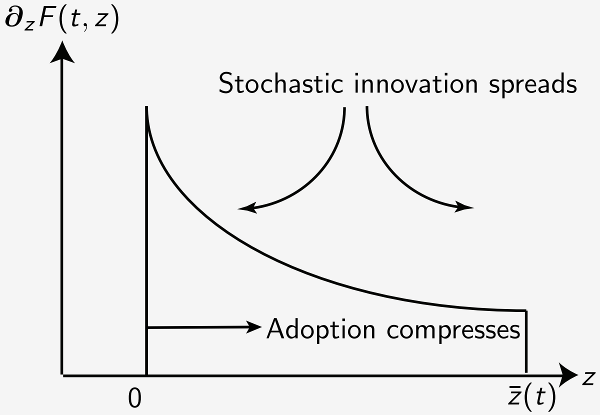
$$\log(M(t)/M(t)) = 0 \qquad \text{(Adoption Threshold)}$$

$$F_i(t,z) \equiv \Phi_i(t,Z)$$

$$v_i(t,z) \equiv V_i(t,Z)/M(t)$$

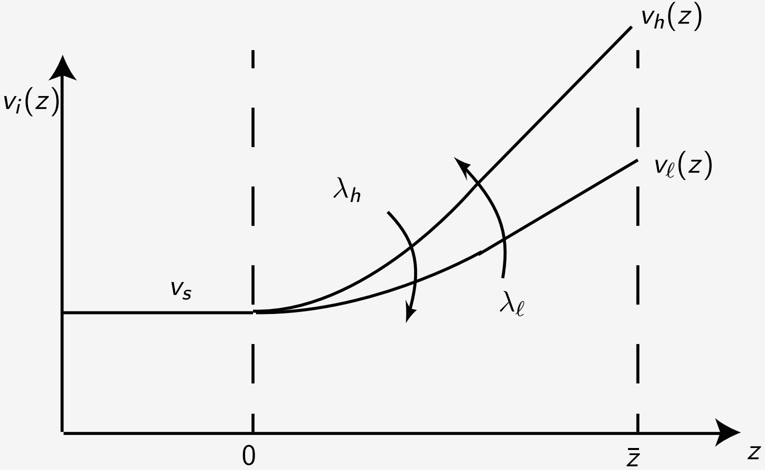


Tension between Stochastic Innovation and Adoption



- Need possibility of relative (not necessarily absolute) “bad luck”
- Adoption value > 0 + increasing dispersion $\implies$ technology diffusion
- $\partial_t \bar{z}(t) = \gamma - g(t)$, from h firms infinitesimally close to the frontier
- Finite frontier stationary distribution has $\gamma = g$ or is degenerate

Normalized, Stationary Value Functions



Stationary, Normalized Firm's Problem with Finite Support

$$(r-g)v_\ell(z) = e^z - \underbrace{g v_\ell'(z)}_{\text{Fall Backwards}} + \underbrace{\lambda_\ell(v_h(z) - v_\ell(z))}_{\text{Jump to } h}$$

$$(r-g)v_h(z) = e^z + \underbrace{\lambda_h(v_\ell(z) - v_h(z))}_{\text{Jump to } \ell}$$

$$v_\ell(0) = v_h(0) = \underbrace{\int_0^{\bar{z}} v_\ell(z)\mathrm{d}F(z)^\kappa}_{\text{Start } \ell \text{ type}} - \zeta$$

$$v_\ell'(0) = 0$$

- $M(t)$ same: use smooth pasting and value matching in bellman
- Note absence of drift and smooth pasting condition for h types
- Robust to adoption technology variations (e.g. could start h type)

KFE for the Stationary Distribution with Finite Support

$$0 = \underbrace{gF_\ell'(z)}_{\text{Drift for } \ell} + \underbrace{\lambda_h F_h(z) - \lambda_\ell F_\ell(z)}_{\text{Net Flow}} + \underbrace{SF(z)^\kappa}_{\text{Draw } < z} - S$$

$$0 = \lambda_\ell F_\ell(z) - \lambda_h F_h(z)$$

$$0 = F_\ell(0) = F_h(0)$$

$$1 = F_h(\bar{z}) + F_\ell(\bar{z})$$

$$S = \underbrace{gF_\ell'(0)}_{\text{Only } \ell \text{ cross}}$$

- Define $\hat{\lambda} \equiv \frac{\lambda_\ell}{\lambda_h}, \bar{\lambda} \equiv \frac{\lambda_\ell}{r-\gamma+\lambda_h} + 1$. Changes with model variations

Stationary Equilibrium for $\kappa = 1$

1. $\bar{z}(t) < \infty \; \forall \; t$ but $\bar{z} \to \infty$ for all κ. No **bounded** equilibrium exist.

$$F_\ell(z) = \frac{1}{1+\hat{\lambda}} e^{-\alpha z} \propto F_h(z)$$

$$v_l(z) = \underbrace{\frac{\bar{\lambda}}{\gamma + (r-\gamma)\bar{\lambda}} e^{z}}_{\text{Production in Perpetuity}} + \underbrace{\frac{1}{(r-\gamma)(\nu+1)} e^{-\nu z}}_{\text{Option Value of Diffusion}}$$

2. Where α is the tail index of the asymptotic "power law"

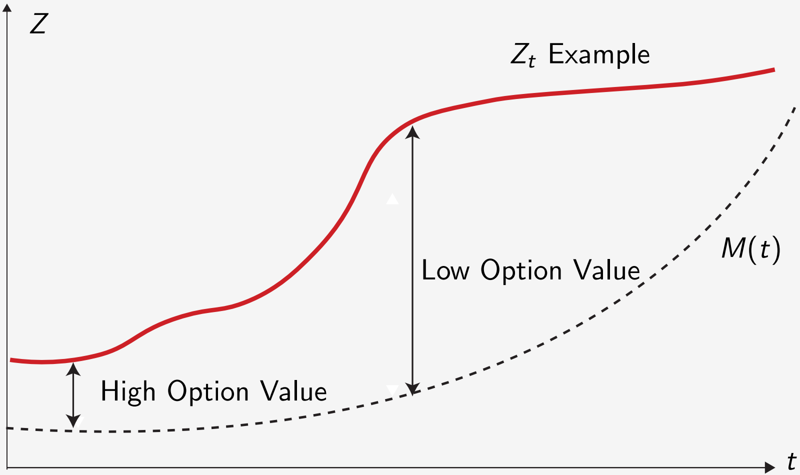
$$\alpha \equiv (1+\hat{\lambda}) F_\ell'(0)$$

3. And ν is the rate the diffusion option value decreases

$$\nu = \frac{(r-\gamma)\bar{\lambda}}{\gamma}$$

4. And an explicit equation for $F_\ell'(0)$. **Unique Stationary Equilibrium**. Endogenous productivity distribution shape.

Option Value of Diffusion



Infinite vs. Unbounded vs. Bounded Frontier

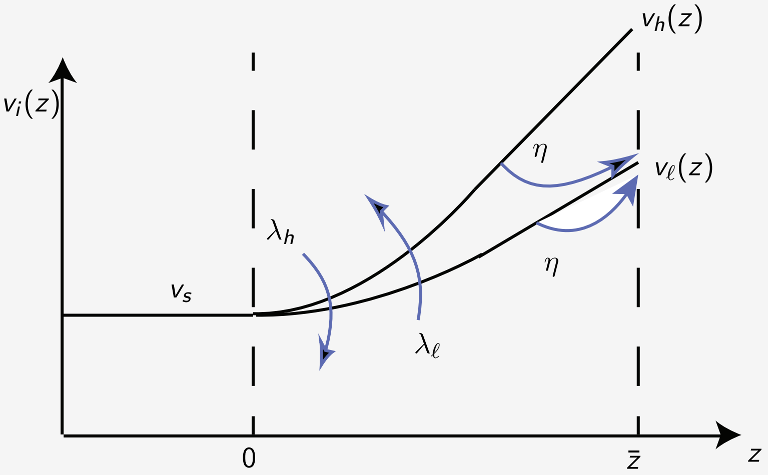
1. **Initial Fat Tail:** $B(0) = \infty$
   - $\bar{z}(t) = \infty$ for all t
   - $g > \gamma$, both ℓ and h agents fall back.
   - Hysteresis, continuum of stationary equilibrium $\{\alpha, g\}$. See paper.

2. **Finite Support:** $B(0) < \infty$
   1. **Finite, Unbounded Support:**
      - Define unbounded support as $\bar{z}(t) < \infty \; \forall \; t$ and $\lim_{t\to\infty} \bar{z}(t) = \infty$
      - $g = \gamma$, **unique solution $\neq$ infinite support!**
      - With innovation, infinite support approximation is not innocuous
      - With finite support, shape is endogenously determined by innovation/adoption costs and benefits
      - At $\bar{z}$, option value of adoption $= 0$
   2. **Finite, Bounded Support:**
      - Can compression from "adoption" balance the "spread" from stochastic innovation?
      - What is needed for $\lim_{t\to\infty} \bar{z}(t) < \infty$?
      - For what questions does bounded vs. unbounded matter?

Stochastic Innovation with a Bounded Frontier

- Fundamental reason no bounded support equilibrium exists:
  - Positive measure of firms have been lucky forever
  - Probability of adopting their technologies drops to 0
- Seems counterintuitive and against (limited) evidence.
  To "fix": Assume there is some chance for leap-frogging to frontier (analogy is non-multiplicative version of "quality ladders")
- Either innovation or adoption could have a probability of jump
  1. Innovation attempts gain a major insight and jump to the frontier
  2. Adopters firm gaining a huge spillover and jump to the frontier
- Assume arrival rate $\eta > 0$ for jumps to $\bar{z}$ for operating firms
  - For simplicity, assume disruptive jump sets firm to ℓ
  - $F_\ell(z)$ and $F_h(z)$ could be discontinuous (but won't be in this setup)

Bounded Frontier Due to Leap-Frogging



Bellman Equation and KFE

Add jumps to $\bar{z} < \infty$ to Bellman equations,

$$(r-g)v_\ell(z) = e^z - gv_\ell'(z) + \lambda_\ell(v_h(z) - v_\ell(z))) + \underbrace{\eta(v_\ell(\bar{z}) - v_l(z))}_{\text{To Frontier}}$$

$$(r-g)v_h(z) = e^z + \lambda_h(v_\ell(z) - v_h(z)) + \underbrace{\eta(v_\ell(\bar{z}) - v_h(z))}_{\text{To Frontier}}$$

KFE has insertion at $\bar{z}$ with a Heaviside: $\mathbb{H}(z-\bar{z})$

$$0 = gF_\ell'(z) + \lambda_h F_h(z) - \lambda_\ell F_\ell(z) - \underbrace{\eta F_\ell(z)}_{\text{Jump Out}} + \underbrace{\eta\mathbb{H}(z-\bar{z})}_{\text{Jump to } \ell} + \underbrace{SF(z)^\kappa - S}_{\text{Same Adoption}}$$

$$0 = \lambda_\ell F_\ell(z) - \lambda_h F_h(z) - \underbrace{\eta F_h(z)}_{\text{Jump Out}}$$

Equilibrium for $\kappa = 1$

$$F_\ell(z) = \frac{F_\ell'(0)}{(F_\ell'(0) - \eta/\gamma)(1+\hat{\lambda})}\left(1 - e^{-\alpha z}\right) \propto F_h(z)$$

$$v_l(z) = \frac{\bar{\lambda}}{\gamma(1+\nu)}\left(e^z + \frac{1}{\nu}e^{-\nu z} + \frac{\eta}{r-\gamma}\left(e^{\bar{z}} + \frac{1}{\nu}e^{-\nu \bar{z}}\right)\right)$$

Where α is the tail index of the asymptotic "power law"

$$\alpha \equiv (1+\hat{\lambda})(F_\ell'(0) - \eta/\gamma)$$

And ν the rate of diffusion option value decrease

$$\nu \equiv \frac{r-\gamma+\eta}{\gamma}\bar{\lambda}$$

And $\bar{z} < \infty$ where

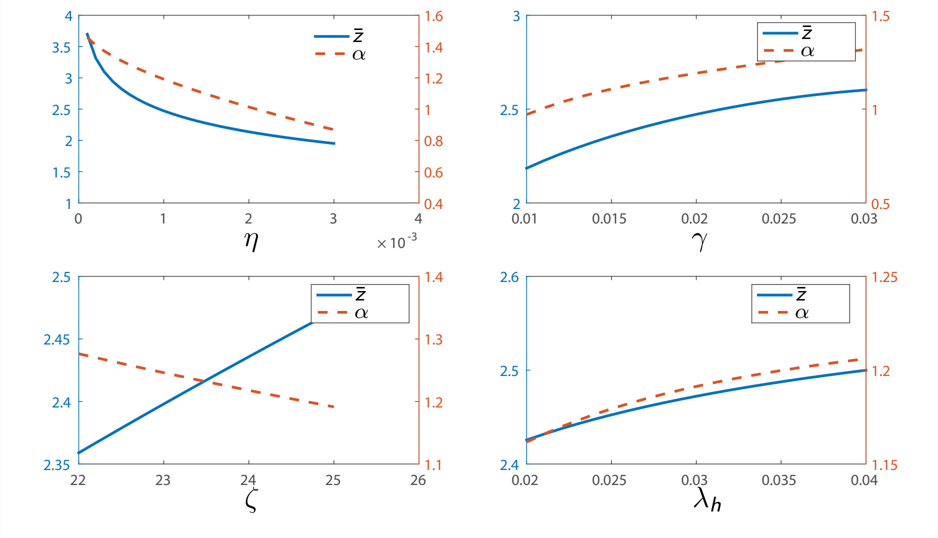
$$\bar{z} = \frac{\log(\gamma F_\ell'(0)/\eta)}{\alpha}$$

And an implicit equation for $F_\ell'(0)$. **Unique Equilibrium**

Summary of Diffusion and Innovation Interaction

- The distribution is endogenous (since α is the tail parameter)
- The frontier is endogenous (since $\bar{z}$ is the relative frontier)
- The option value of technology adoption determines incentives
  - Closer to the frontier = smaller option value
  - Due to discounting and time before executing option
  - Higher stochastic dispersion increases option value since volatility decreases expected execution time
- The unbounded case is different from the bounded case
  - Option value of adoption at frontier = 0 when unbounded
  - With endogenous innovation, innovation incentive would not depend on adoption parameters

Numerical Example: Bounded Eqm. Comparative Statics



Implications for Subsidy Policies (preliminary)

- Tensions with innovation and diffusion determine α and $\bar{z}$. E.g.,

1. $\uparrow \eta$ is more jumps to frontier. Maybe interpret as weaker patents?
   - Generates lower $\bar{z}$, but thicker tails as relative technology diffusion incentives decrease.
2. $\downarrow \zeta$ is lower adoption cost. Adoption subsidy/tax. Maybe interpret as relaxing financing frictions or market entry deterrence.
   - Shrinks the frontier and decreases productivity dispersion. When adoption is easy, hard to have big ratio of best to worst firms.
3. $\uparrow \gamma$ is increase in innovation rate. Innovation subsidy/tax.
   - Stretches out $\bar{z}$, but also decreases expected time to technology adoption (which increases option value). When innovation is easy, can have big dispersion between best and worst firms, but not as many best firms.

Endogenous Innovation

- Let innovating firms choose $\gamma(t, Z) \geq 0$.
  - Pay a proportional convex cost $\frac{1}{\chi}\gamma^2 Z$
- The h agents cross the diffusion threshold if $\gamma(0) < g$
- Add the endogenous γ choice to the HJBE and make stationary

$$(r-g)v_h(z) = \max_{\gamma \geq 0}\{e^z - \underbrace{\frac{1}{\chi}e^z\gamma^2}_{\text{R\&D cost}} - \underbrace{(g-\gamma)v_h'(z)}_{\text{Control Drift}}$$
$$+ \lambda_h(v_\ell(z) - v_h(z)) + \eta(v_\ell(\bar{z}) - v_h(z))\}$$

- The KFE depends on the endogenous $\gamma(z)$ for the h types,

$$0 = \underbrace{(g-\gamma(z))F_h'(z)}_{\text{Controlled Drift}} + \lambda_\ell F_\ell(z) - \lambda_h F_h(z) - \eta F_h(z) - S_h$$

Endogeneity and γ

- Can show that the optimal choice is,

$$\boxed{\gamma(z) = \frac{\chi}{2} e^{-z} v_h'(z) \qquad 0 \leq z \leq \bar{z}}$$

  - $\gamma(0) = 0$ and $g \equiv \gamma(\bar{z})$
  - γ increasing in z because of option value
- The value of adoption now includes jumps to the frontier,

$$\boxed{v_s \equiv \frac{1 + \eta v_\ell(\bar{z})}{r - g + \eta}}$$

- This creates feedback between endogenous $\bar{z}$ and v_s

Interaction of Decisions

- If firm is an innovator, with easier adoption, then innovate less
  - Only occurs if the option value is > 0
  - If $\bar{z} \to \infty$, then option value at frontier $\to 0$
- If near endogenous adoption threshold, then no reason to innovate at all since will execute option soon
- Both decisions effect the aggregate productivity distribution, creating spillovers
  - Not internalized
- Where to spend \$1 subsidy: innovation vs. adoption?

Extensions

1. Endogenous Jumps and Directed Technology Adoption
   - Adopting firms choose $\theta \in [0, 1)$ and $\kappa > 0$ (i.e., directed adoption)
   - Proportional quadratic costs are $\frac{1}{\varsigma}\theta^2$ and $\frac{1}{\vartheta}\kappa^2$
   - Value of technology adoption (i.e. value matching condition)

$$v_i(0) = \max_{0 \leq \theta < 1,\, \kappa > 0} \left\{ (1-\theta) \int_0^{\bar{z}} v_\ell(z) \mathrm{d}F(z)^\kappa + \theta v_\ell(\bar{z}) - \zeta - \frac{1}{\varsigma}\theta^2 - \frac{1}{\vartheta}\kappa^2 \right\}$$

2. Partial Excludability
   - Assume adopter of a z pays some up-front licensing cost
   - Nash bargaining: Adopter can recall, licensor can reject to transfer z
   - As γ investment depends on marginal profits, this creates a feedback between adoption and innovation–even in the $\bar{z} \to \infty$ case
     - Adoption decreases innovation by increasing the option value of adoption
     - Adoption increases innovation by increasing licensing revenue

Conclusion

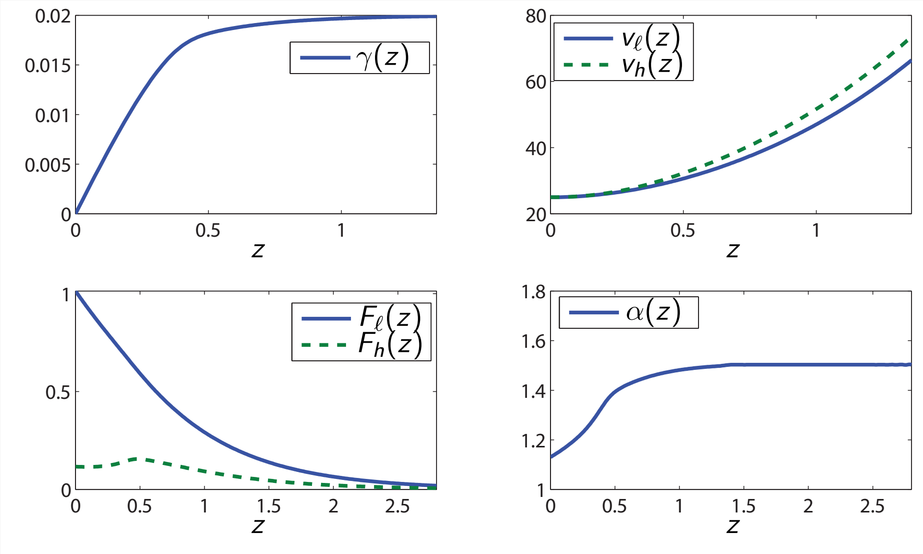
Success was answer + laboratory to analyze:

> *How do technology adoption and innovation* **jointly** *determine: (1) the* **aggregate growth rate***; (2) the* **shape of the productivity distribution***; and (3) the* **technology frontier***?*

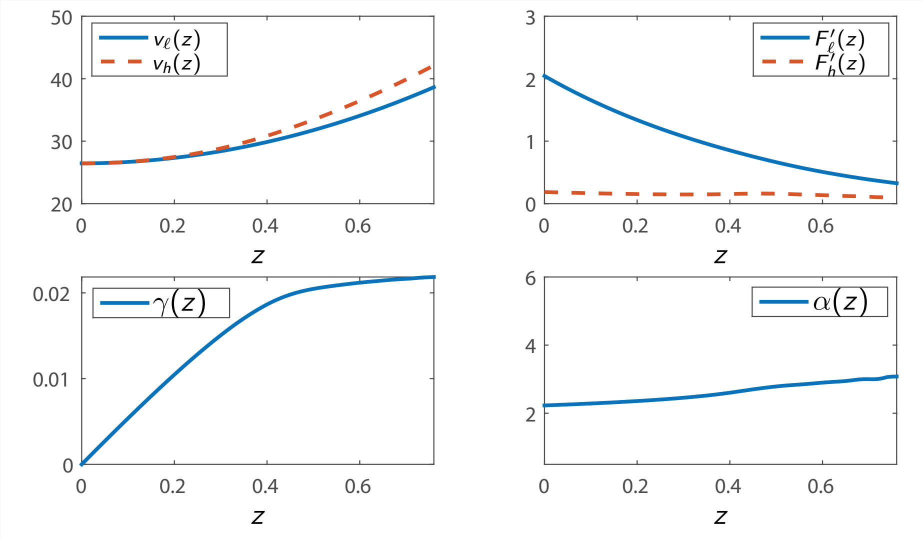
- To model innovation and technology diffusion with a frontier **need**:
  - Stochastic productivity + Markov-chain
  - If a bounded frontier is empirically necessary need jumps to the frontier
  - Endogenous shape: Innovation stretches, adoption compresses
- Asymptotically unbounded $\neq$ infinite support
- Feedback between option value of diffusion and technology frontier the key to innovation and diffusion interaction
  - Growth rate of frontier is that of the frontier agent
  - High or volatile growth = high productivity inequality
- TODO: policy analysis with endogenous innovation

Appendix

Example Unbounded Endogenous $\gamma(z)$ ($\eta = 0$ case)



Example Bounded Endogenous $\gamma(z)$



Endogenous Jumps and Directed Technology Adoption

- Adopting firms choose $\theta \in [0,1)$ and $\kappa > 0$ (i.e., directed adoption)
- Proportional quadratic costs are $\frac{1}{\varsigma}\theta^2$ and $\frac{1}{\vartheta}\kappa^2$
- Value of technology adoption (i.e. value matching condition)

$$v_i(0) = \max_{0\leq\theta<1,\, \kappa>0} \left\{(1-\theta)\int_0^{\bar{z}} v_\ell(z)\mathrm{d}F(z)^\kappa + \theta v_\ell(\bar{z}) - \zeta - \frac{1}{\varsigma}\theta^2 - \frac{1}{\vartheta}\kappa^2\right\}$$

- Can show optimal choices at adoption time are,

$$\theta = 1 - \sqrt{1 - \varsigma(v_\ell(\bar{z}) - v_\ell(0) - \zeta)}$$
$$\kappa = \frac{-\vartheta(1-\theta)}{2}\int_0^{\bar{z}} v_\ell'(z)\log(F(z))F(z)^\kappa \mathrm{d}z$$

Partial Excludability

- Assume adopter of a z pays some up-front licensing cost
  - The total surplus is $v_\ell(z)$
  - Threats: adopter can recall, $v(0)$, licensor can just reject
  - Let $\hat{v}$ be value of bargain to adopter
- Use Nash Bargaining with adopter bargaining power $\psi \in (0, 1]$

$$\arg\max_{\hat{v}} \left\{ (\hat{v} - v(0))^{\psi} (v_\ell(z) - \hat{v})^{1-\psi} \right\}$$

- Can show licensing simply increases adoption costs
- As γ investment depends on marginal profits, this creates a feedback between adoption and innovation–even in the $\bar{z} \to \infty$ case
  1. Adoption decreases innovation by increasing the option value of adoption
  2. Adoption increases innovation by increasing licensing revenue

Adoption Costs and Profits with Excludability

- Can show licensing simply increases adoption costs. Value matching:

$$v(0) = (1-\theta)\int_0^{\bar{z}} v_\ell(z)\mathrm{d}F(z)^\kappa + \theta v_\ell(\bar{z}) - \underbrace{\frac{1}{\psi}}_{\text{Increases}}\left(\zeta + \frac{1}{\varsigma}\theta^2 + \frac{1}{\vartheta}\kappa^2\right)$$

- And operating firms now have g dependent marginal profits

$$\pi'(z) = e^z + (1-\psi)gF'(0)v_\ell'(z)$$

- As γ investment depends on marginal profits, this creates a feedback between adoption and innovation–even in the $\bar{z} \to \infty$ case
  1. Adoption decreases innovation by increasing the option value of adoption
  2. Adoption increases innovation by increasing licensing revenue
- We speculate that there is an optimal ψ level for maximum growth

Uniqueness of Stationary Equilibrium

- If $\eta = 0$ or $\eta \to 0$, then $\bar{z}$ is unbounded, and equilibrium is **unique**
  - Option value of diffusion is asymptotically 0

$$g \equiv \lim_{\eta \to 0} g_{\max}(\eta) = \bar{\lambda} r \left[1 - \sqrt{1 - \frac{\chi}{\bar{\lambda} r^2}} \right]$$

- If $\eta > 0$, then $\bar{z}$ is bounded, and there is **hysteresis**
  - Continuum of $g \leq g_{\max}$ with accompanying $\bar{z}$ and $F_i(z)$
  - If $\bar{z} < \infty$, then option value of diffusion is always positive.
  - Different shapes and $\bar{z}$ provide different adoption incentives, which provide different γ choices, which can fulfill the shape

Geometric Brownian Motion

Back

- Start with a standard baseline: exogenous GBM
  - One i type, idiosyncratic Z
  - Will use to understand role of stochastics, decompose growth rates
- Exogenous GBM for innovation: $\mathrm{d}Z = (\gamma + \sigma^2/2)Z\mathrm{d}t + \sigma Z\mathrm{d}W$
  - W is standard Brownian motion
  - Mean growth rate of Z is γ
  - If $\sigma > 0$, then $B(t) = \infty$ due to Brownian shocks
- Solve as an optimal stopping problem for endogenous threshold $M(t)$
  - Continuation value of firm $V(t, Z)$
  - Value matching condition at $M(t)$, the value of technology diffusion
  - Smooth pasting conditions necessary at $M(t)$

Firm's Problem

Bellman Equation + Value Matching + Smooth Pasting

$$rV(t,Z) = \underbrace{Z}_{\text{Flow Profits}} + \underbrace{(\gamma + \sigma^2/2)Z\,\partial_Z V(t,Z) + \frac{\sigma^2}{2}Z^2\,\partial_{ZZ}V(t,Z)}_{\text{Exogenous Innovation}} + \underbrace{\partial_t V(t,Z)}_{\text{Capital Gains}}$$

$$\underbrace{V(t,M(t))}_{\text{Value at Threshold}} = \underbrace{\int V(t,Z')\mathrm{d}\Phi(t,Z')^{\kappa}}_{\text{Gross Adoption Value}} - \underbrace{\zeta M(t)}_{\text{Adoption Cost}}$$

$$\partial_Z V(t,M(t)) = \underbrace{0}_{\text{Cost} \perp Z}$$

Evolution of the Distribution

The Kolmogorov Forward Equation (for the CDF $\Phi(t, Z)$)

$$\partial_t \Phi(t, Z) = \underbrace{-(\gamma + \sigma^2/2)\partial_Z Z\Phi(t, Z)}_{\text{Deterministic Drift}} + \underbrace{\frac{\sigma^2}{2}\partial_{ZZ} Z^2\Phi(t, Z)}_{\text{Brownian Motion}}$$
$$+ \underbrace{S(t)\Phi(t, Z)^\kappa - S(t)}_{\text{Firm draws - Adopters}}, \quad \text{for } M(t) \leq Z \leq B(t)$$

$$\Phi(t, M(t)) = 0$$
$$\Phi(t, \infty) = 1$$

- $S(t)$ the endogenous flow of adopters crossing $M(t)$
- If $\sigma > 0$, frontier is infinite immediately
- If $\sigma = 0$ and $B(0) < \infty$, frontier grows at rate γ

Stationary Equilibrium for GBM (if $\kappa = 1$)

The **continuum of equilibria** are indexed by $\alpha > 1$, where

$$g = \gamma + \underbrace{\frac{1-(\alpha-1)\zeta(r-\gamma)}{(\alpha-1)^2\zeta}}_{\text{Catch-up Diffusion}} + \underbrace{\frac{\sigma^2}{2}\frac{\alpha\left(\alpha(\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta-2\right)+1}{(\alpha-1)\left((\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta-1\right)}}_{\text{Stochastic Diffusion}}$$

Where σ also affects expected time to execute diffusion "option"

$$F(z) = 1 - e^{-\alpha z}$$

$$v(z) = \frac{1}{r-\gamma}e^{z} + \frac{1}{\nu(r-\gamma)}e^{-\nu z}$$

$$\nu = \frac{g-\gamma}{\sigma^2} + \sqrt{\left(\frac{g-\gamma}{\sigma^2}\right)^2 + \frac{r-g}{\sigma^2/2}} > 0$$

Dynamics with Deterministic Innovation

Proposition (Dynamic Solution with Deterministic Innovation)

Define $\tilde{g}(M) = M'(M)/M$, the growth rate of M at state M. For an arbitrary initial condition $\Phi(0, Z)$, $\kappa = 1$, and $r > \gamma$

$$\tilde{g}(M) = \gamma + \frac{\frac{1}{M}\int_M^\infty Z \mathrm{d}\Phi_M(Z) - (1 + \zeta(r - \gamma))}{\zeta M \Phi'_M(Z)} \quad (1)$$

where $\Phi_M(Z) \equiv \frac{\Phi(0,Z) - \Phi(0,M)}{1 - \Phi(0,M)}$, is a truncation of the initial condition.

Note: given $M(0)$ can use $\tilde{g}(M)$ to construct $M(t)$

Asymptotics with Deterministic Innovation

Proposition

Let $g(t) \equiv M'(t)/M(t) =$ *output growth on a BGP*

1. $\Phi(0,Z)$ *not power law* $\implies \lim_{t\to\infty} g(t) = \gamma$.
2. $\Phi(0,Z)$ *not power law* $\implies \lim_{t\to\infty} \frac{B(t)}{M(t)} > 1$ *(i.e., doesn't collapse)*
3. $\Phi(0,Z)$ *power law with tail parameter* $\alpha > 0$ *and* $\kappa = 1$

$$\lim_{t\to\infty} g(t) = \gamma + \frac{1-\zeta(r-\gamma)(\alpha-1)}{\zeta\alpha(\alpha-1)} \tag{2}$$

$$\lim_{t\to\infty} V(t,Z) = \underbrace{\frac{Z}{r-\gamma}}_{\text{Perpetuity}} + \overbrace{C_1 \underbrace{e^{(r-\gamma)t}}_{\nearrow M(t)} \underbrace{Z^{-\frac{r-g}{g-\gamma}}}_{\searrow Z \text{ given } t}}^{\text{Option Value of Diffusion}} \tag{3}$$

BGP Summary: not power law $\implies$ no "catchup diffusion"
$\implies$ require stochastic innovation for asymptotic technology diffusion

References I