BLANCHARD

Probability of Death: $\pi(t) = pe^{-pt}$;

Probabilty of living till $t : \Omega(t) = \int_t^\infty pe^{-pt}dt = e^{-pt}$

Probabilty of death given survival till $t : \frac{pe^{-pt}}{e^{-pt}} = p$

Expected life at $t : \int_t^\infty (s-t)\, pe^{-(s-t)p}ds = p^{-1}$

Population normalized to constant cohort size p. Cohort born at 0 has size pe^{-pt} : (prob of living till t times pop. size-deterministic)

Population: $\int_{-\infty}^t pe^{(s-t)p}ds = e^{(s-t)p} \mid_{-\infty}^t = 1$

Insurance exists. Agents die with prob. density p. So no net profits for insurance implies they receive p for each unit they leave. They plan to leave all, and receive pw if alive. On average insurance companies pay out pW and receive pW from estates where W is aggregate wealth.

$$Max \int_t^{\infty} \ell n \; c(v,s)\, e^{(\theta+p)(t-v)} dv$$

ST

$$\frac{dw(s,t)}{dt} = (r(t)+p)\, w(s,t) + y(s,t) - c(s,t)$$
$$0 = \lim_{v\to\infty} e^{-\int_t^v (r(u)+p)du} w(s,v)\, dv$$

Notes: If $r(u)$ is constant $\int_t^v (r(u)+p)\, du = (r+p)(v-t)$. To assure zero terminal wealth $r(u) > -p$ is assumed. Also $\Omega(v-t) = e^{p(t-v)}$ in Yaari's terms. $r > -p$ is assumed.

Hamiltonian:

$$H = \ell n \; c + \lambda((r+p)\, w + y - c)$$

FOC

$$c^{-1} = \lambda; \quad \frac{\dot{c}}{c} = -\frac{\dot{\lambda}}{\lambda}$$
$$\dot{\lambda} = \lambda(-r-p+p+\theta) = -\lambda(r-\theta)$$
$$\dot{c} = c(r-\theta)$$

Budget

$$\int_t^\infty c(s,v)\,e^{-\int_t^v (r(u)+p)du}dv = w(s,t) + h(s,t)$$

$$(r+p)(w+h) - c = \dot{w} + \dot{h}$$

Human capital is discounted future labor income:

$$h = \int_t^\infty y(s,v)\,e^{-\int_t^v (r(u)+p)du}dv$$

$$\dot{h} = (r+p)h - y$$

$$\dot{w} = (r+p)(w+h) - c - \dot{h}$$

$$\dot{w} = (r+p)\,w + y - c$$

Conjecture

$$
\begin{aligned}
c\,(s,t) &= (p+\theta)\,(w+h) \\
\dot{c} &= (p+\theta)\,((r+p)\,w+y-c+(r+p)h-y) \\
\dot{c} &= (p+\theta)\,((r+p)\,(w+h)-c) \\
\dot{c} &= (r+p)\,(p+\theta)\,(w+h)-(p+\theta)\,c \\
\dot{c} &= (r-\theta)\,c \qquad \textit{checks out}
\end{aligned}
$$

AGGREGATE CONSUMPTION

$$C = (p+\theta)(W+H)$$

Aggregate H

$$H(t) = \int_{-\infty}^{t} pe^{p(s-t)} \int_{t}^{\infty} y(s,v)\, e^{-\int_t^v (r(u)+p)du} dv ds$$

Assume common income or wage $y(v)$, and noting that population integrates to 1:

$$\begin{aligned} H(t) &= \int_{-\infty}^{t} pe^{p(s-t)} \int_{t}^{\infty} y(s,v)\, e^{-\int_t^v (r(u)+p)du} dv ds \\ &= \int_{t}^{\infty} \left[\int_{-\infty}^{t} y(s,v)\, e^{-\int_t^v (r(u)+p)du} pe^{p(s-t)} ds \right] dv \\ &= \int_{t}^{\infty} \left[Y(v)\, e^{-\int_t^v (r(u)+p)du} ds \right] dv \end{aligned}$$

$$\dot{H} = (r+p) H - Y(t)$$

Aggregate W

$$W(t) = \int_{-\infty}^{t} w(s,t)e^{p(s-t)}ds$$

$$\begin{aligned}
\dot{W} &= w(t,t) - pW + \int_{-\infty}^{t} \frac{dw(s,t)}{dt} pe^{p(s-t)}ds \\
&= -pW + \int_{-\infty}^{t} ((r+p)w + y - c)\, pe^{p(s-t)}ds \\
&= -pW + (r+p)W + Y - C = rW + Y - C
\end{aligned}$$

So while those alive have their wealth grow at $(r+p)$, aggregate wealth accumulates at rate r, since pW is a transfer from the dead.

Aggregate Dynamics

$$
\begin{aligned}
C &= (p+\theta)(H+W) \\
\dot{C} &= (p+\theta)\left(\dot{H}+\dot{W}\right) \\
\dot{C} &= (p+\theta)(rW+Y-C+(r+p)H-Y) \\
&= (p+\theta)((r+p)(W+H)-C-pW) \\
&= (r+p)C-(p+\theta)C-(p+\theta)pW \\
&= (r-\theta)C-(p+\theta)pW
\end{aligned}
$$

$$
\begin{aligned}
\dot{C} &= (r-\theta)C-(p+\theta)pW \\
\dot{W} &= rW+Y-C
\end{aligned}
$$

Steady State

$$
C = \frac{(p+\theta)pW}{r-\theta}
$$

OPEN ECONOMY

$$\begin{aligned}\dot{C} &= (r-\theta)\,C - (p+\theta)\,pF \\ \dot{F} &= rF + Y - C\end{aligned}$$

$$J = \begin{bmatrix} r-\theta & -(p+\theta)\,p \\ -1 & r \end{bmatrix}$$

Det:

$$(r-\theta)\,r - (p+\theta)\,p < 0 \quad ?$$

At Steady State:

$$C = \left(\frac{(p+\theta)\,p}{(r-\theta)\,r}\right) rF$$

Note: $r \gtrless \theta$. So F may be negative. If r is too big however the economy may explode: you need $r < p+\theta, or\ r-\theta < p$. This implies $r\,(r-\theta) < p\,(p+\theta)$. So $Det < 0$. Furthermore, If $(r-\theta) > 0$ this implies $\left(\frac{(p+\theta)p}{(r-\theta)r}\right) > 1$, or $C > rF$.

CLOSED ECONOMY:

$$
\begin{aligned}
F(K) &= \tilde{F}(K,1) - \delta K,\ r = F' \\
\dot{C} &= (r(K) - \theta) C - (p+\theta) pK \\
\dot{K} &= F(K) - C
\end{aligned}
$$

$$
J = \begin{bmatrix} r-\theta & -(p+\theta)p \\ -1 & F' \end{bmatrix}
$$

For $C > 0$,at steady state, $r(k^*) > \theta$.

$$
If\ r-\theta > p, \quad (r-\theta) C = (r-\theta) F(K) > pC
$$

$$
\begin{aligned}
(r-\theta) F(K) &= (p+\theta) pK > pC = pF(K) \\
(p+\theta) K &> F(K)
\end{aligned}
$$

The latter implies, if $r > p+\theta$, $rK > F(K)$,but output must exceed $MPK * K$, so $r(K) < (p+\theta)$. So $Det\ J < 0$.

Declining Labor Income:

$$y(s,v) = aY(v)e^{\alpha(s-v)}$$

Total income at :sum of cohort income weighted by pop .size

$$Y(v) = \int_{-\infty}^{v} y(s,v)pe^{p(s-v)}ds = Y(v)\frac{ap}{p+\alpha}$$
$$a = \frac{p+\alpha}{p}$$

$$h(s,t) = \int_{t}^{\infty} aY(v)\, e^{\alpha(s-v)} e^{-\int_t^v (r(u)+p)du} dv$$
$$= e^{\alpha(s-t)}\left(\frac{p+\alpha}{p}\right)\int_{t}^{\infty} Y(v)\, e^{-\int_t^v (r(u)+p+\alpha)du} dv$$

$$
\begin{aligned}
& H(t) \\
= & \int_{-\infty}^{t} p e^{p(s-t)} \\
& \left(e^{\alpha(s-t)} \left(\frac{p+\alpha}{p} \right) \int_{t}^{\infty} Y(v) e^{-\int_t^v (r(u)+p+\alpha) du} dv \right) ds \\
= & \int_{-\infty}^{t} (p+\alpha) e^{(p+\alpha)(s-t)} \\
& \left(\int_{t}^{\infty} Y(v) e^{-\int_t^v (r(u)+p+\alpha) du} dv \right) ds \\
= & e^{(p+\alpha)(s-t)} \mid_{-\infty}^{t} \left(\int_{t}^{\infty} Y(v) e^{-\int_t^v (r(u)+p+\alpha) du} dv \right) \\
= & \left(\int_{t}^{\infty} Y(v) e^{-\int_t^v (r(u)+p+\alpha) 1` du} dv \right)
\end{aligned}
$$

Dynamics:

$$
\begin{aligned}
C &= (\theta+p)(H+W) \\
\dot{H} &= (r+p+\alpha) H - Y \\
\dot{W} &= rW + Y - C
\end{aligned}
$$

Compute $\dot{C}$ using $\dot{W}$, $\dot{H}$, and substitue for H from first equation above:

$$\begin{aligned}\dot{C} &= (r(k)+\alpha-\theta)\,C-(p+\alpha)(p+\theta)\,K\\ \dot{K} &= F(K)-C\end{aligned}$$

Steady State:

$$\begin{aligned}C &= \frac{(p+\alpha)(p+\theta)\,K}{(r(K)+\alpha-\theta)}\\ F(K) &= \frac{(p+\alpha)(p+\theta)}{r(r(K)+\alpha-\theta)}rK\end{aligned}$$

Define $\hat{k}|r\left(\hat{k}\right)=\theta-\alpha$:Note that $\theta-\alpha$ can be negative, so $r\left(\hat{k}\right)<0.$ Plot steady state C as a function of K, which asymptotes at $\hat{k}$.

$$J=\begin{bmatrix}(r(k)+\alpha-\theta) & r'(K)\,C-(p+\alpha)(p+\theta)\\ -1 & F'(K)\end{bmatrix}$$

$DET=r(r(k)+\alpha-\theta)+r'(K)\,C-(p+\alpha)(p+\theta)$

It is negative as before because we assume $(r(k)+\alpha-\theta)<(p+\alpha)(p+\theta)$ and $r'(K)<0$.

If we allow Money:

$$W = K + m$$

Let $m = \frac{M}{P}$,

$$\frac{\dot{m}}{m} = \sigma - \pi$$

where σ,is the growth of nominal balances, π is inflation. Equilibrium condition: return on money equals return on capital, $r(k)$. If $\sigma = 0$, $r(k) = -\pi$, the return on holding money equals to the deflation rate (Compare with the case below where money enters utility, and has an additional utility return.) If money pays interest σ, in proportion to money holdings, then in equilibrium, $r(k) = \sigma - \pi$. If $\sigma = r(k)$, in equilibrium, $\pi = 0$. In any case, now equilibrium conditions are:

$$\begin{aligned} \dot{C} &= (r(K) - \theta) C - (p + \theta) p (K + m) \\ \dot{K} &= F(K) - C \\ \dot{m} &= r(k) m \end{aligned}$$

Steady states: $m = 0$, $r(k) = 0$

CRRA UTILITY

$$U(c) = (1 - \sigma)^{-1} c^{(1-\sigma)}$$

$$\frac{dc(s,t)}{dt} = \sigma^{-1} (r(t) - \theta) c(s,t)$$

$$c(s,t) = \Delta^{-1} (w(s,t) + h(s,t))$$

$$\Delta = \int_t^{\infty} e^{\sigma^{-1} \int_t^v [(1-\sigma)(r(u)+p)-(\theta+p)]du} dv$$

$$\frac{dc\,(s,t)}{dt} = \Delta^{-1}\left(\dot{w} + \dot{h}\right) - \Delta^{-2}\,(w+h)\,\dot{\Delta}$$

$$\begin{aligned}
\dot{C} &= \Delta^{-1}\left((r+p)\,(H+W) - pW - C\right) \\
&\quad -\Delta^{-1}\,(W+H)\,\frac{\dot{\Delta}}{\Delta} \\
&= \Delta^{-1}\,(r+p)\,C - \Delta^{-1}C - \Delta^{-1}pW \\
&\quad -(W+H)\,\Delta^{-1}\frac{\dot{\Delta}}{\Delta} \\
&= C\left(\begin{array}{c} (r+p) - \Delta^{-1} + \Delta^{-1} \\ +\sigma^{-1}\left((r+p)\,(1-\sigma) - (\theta+p)\right) \end{array}\right) - pW\Delta^{-1}
\end{aligned}$$

So

$$\begin{aligned}
\dot{C} &= \sigma^{-1}\,(r\,(t) - \theta)\,C - pW\Delta^{-1} \\
\dot{K} &= F\,(K) - \\
\dot{\Delta} &= -1 - \sigma^{-1}\left((1-\sigma)\,(r\,(k)+p) - (\theta+p)\right)\Delta
\end{aligned}$$

Note: Solve for steady state Δ as a function of r and plug into $\dot{C} = 0$,

$$C = pK\left(\frac{(\sigma-1)\,(r+p) + (\theta+p)}{r-\theta}\right)$$

as opposed to log utility, where $C = \left(\frac{p(p+\theta)}{r-\theta}\right) pK$.Consider cases $\frac{dC}{dK}$ according to $\sigma \lesseqgtr 1$. If $\sigma < 1$, $\frac{dC}{dk}$

may be negative: increasing K causes decreasing r.Denominator will decrease if r increases, as will numerator, but denominator can dominate.

Local Dynamics:

$$\begin{bmatrix} \frac{r-\theta+\alpha}{\sigma} & \frac{Cr'}{\sigma} - (p+\alpha)\Delta^{-1} & (p+\alpha)k\Delta^{-2} \\ -1 & F' & 0 \\ 0 & \frac{-(1-\sigma)}{\sigma}\Delta r' & \frac{(\sigma-1)(r+p+\alpha)+(\theta+p)}{\sigma} \end{bmatrix}$$

$DET < 0,\ TRACE > 0, ROOT\ STRUCTURE: + + -$

INTRODUCING MONEY IN UTILITY

$$Max \int_0^\infty (\ell n\ c + \ell n\ m)\, e^{-(\theta+p)t} dt$$

$$\dot{w} = (r+p)\,w + y - c - (\pi + r)\,m$$

$$H = \ell n\ c + \ell n\ m + \lambda\left((r+p)\,w + y - c - (\pi + r)\,m\right)$$

FOC

$$c^{-1} = \lambda,\ \frac{1}{m} = \lambda(\pi + r)\,,\ c = (\pi + r)\,m$$

$$\dot{c} = c\,(r-\theta)$$

Wealth

$$
\begin{aligned}
w+h &= \int_t^{\infty} \left(c\left(s,v\right)\right) e^{-\int_t^v (r+p)du} dv \\
h &= \int_t^{\infty} \left(y\left(s,v\right)-m\left(r+\pi\right)\right) e^{-\int_t^v (r+p)du} dv
\end{aligned}
$$

Differentiating $h, w+h$:

$$
\dot{h} = -y\left(s,v\right) + m\left(r+\pi\right) + (r+p)h
$$

$$
\begin{aligned}
-c+\left(r+p\right)w &= \dot{w}+\dot{h} = \dot{w}-c-y+\left(\pi+r\right)m \\
\dot{w} &= \left(r+p\right)w+y-c-\left(\pi+r\right)m
\end{aligned}
$$

Consumption function:

$$
c = \left(\theta+p\right)\left(w+h\right)
$$

$$
\begin{aligned}
\dot{c} &= \left(\left(\theta+p\right)\left[\left(r+p\right)w+y-c-\left(\pi+r\right)m-y+m\left(r\right.\right.\right. \\
&= \left[\left(r+p\right)\left(\theta+p\right)\left(w+h\right)-\left(\theta+p\right)c\right] \\
&= \left(r+p\right)c-\left(\theta+p\right)c = \left(r-\theta\right)c
\end{aligned}
$$

Aggregating

$$
\begin{aligned}
H &= \int_t^{\infty} \left(Y\left(v\right) - \left(\pi + r\right) M\left(v\right)\right) e^{-\int_t^v (r+p)du} dv \\
W &= \int_{-\infty}^{t} w\left(s,t\right) e^{p(s-t)} ds \\
\dot{H} &= \left(r+p\right) H - Y + \left(r+\pi\right) M \\
& \quad \textit{integrating by parts} \\
\dot{W} &= w\left(t,t\right) - pW + \int_{-\infty}^{t} \frac{dw\left(s,t\right)}{dt} p e^{p(s-t)} ds \\
&= -pW + \left(r+p\right) W + Y - C - \left(\pi + r\right) M
\end{aligned}
$$

$$
\begin{aligned}
C &= \left(\theta + p\right)\left(W + H\right) \\
\dot{C} &= \left(\theta + p\right) \left(\begin{array}{c} \left(r+p\right) H - Y + \left(r+\pi\right) M \\ -pW + \left(r+p\right) W + Y - C - \left(\pi + r\right) M \end{array} \right. \\
\dot{C} &= \left(\theta + p\right)\left(\left(r+p\right)\left(W+H\right) - C - pW\right) \\
\dot{C} &= \left(r+p\right)\left(\theta+p\right)\left(W+H\right) - \left(\theta+p\right) C - p\left(\theta+p\right) W \\
\dot{C} &= \left(r+p\right) C - \left(\theta+p\right) C - p\left(\theta+p\right) W \\
&= \left(r-\theta\right) C - \left(\theta+p\right) pW
\end{aligned}
$$

From FOC:

$$\begin{aligned}
(\pi + r) M &= C \\
(\pi + r) M &= (\theta + p)(W + H) \\
\pi M &= C - rM \\
\dot{M} &= -\pi M = -C + rM
\end{aligned}$$

Dynamics

$$\begin{aligned}
\dot{M} &= -\pi M = -C + rM \\
\dot{K} &= F(K) - C \\
\dot{C} &= (r - \theta) C - (p)(p + \theta)(K + M)
\end{aligned}$$

INTRODUCING ENDOGENOUS LABOR

$$Max \int_0^{\infty} \left(\ell n\ c + a\ell n\left(\bar{l} - l\right)\right) e^{-(\theta+p)t}$$

$$\dot{w} = (r + p) w + ly + c$$

FOC

$$\begin{aligned}
\frac{a}{\bar{l} - l} &= \lambda y,\ \bar{l}y - ac = yl \\
&\dot{c} - (r - \theta) c
\end{aligned}$$

$$
\begin{aligned}
h &= \int_t^\infty \frac{(p+\alpha)}{p} e^{\alpha(s-t)} \left(\bar{l}y - ca\right) e^{-\int_t^v (r+p+\alpha)du} dv \\
&= \frac{(p+\alpha)}{p} e^{\alpha(s-t)} \int_t^\infty (ly)\, e^{-\int_t^v (r+p+\alpha)du} dv
\end{aligned}
$$

$$
\begin{aligned}
c &= \left(\frac{(\theta+p)}{1+a}\right)(w+h) \\
\dot{c} &= \left(\frac{(\theta+p)}{1+a}\right)\left((r+p)\,w + ly + c + (r+p+\alpha)\,h - y\bar{l}\right. \\
\dot{c} &= \left(\frac{(\theta+p)}{1+a}\right)\left((r+p)\,w + c + (r+p+\alpha)\,h + ac\right) \\
&= c\,(r-\theta)
\end{aligned}
$$

$$
\begin{aligned}
C &= \left(\frac{(\theta+p)}{1+a}\right)(W+H) \\
\dot{C} &= \left(\frac{(\theta+p)}{1+a}\right)\left(\begin{array}{c} rW + LY - C \\ +\,(r+p+\alpha)\,H - \bar{L}Y \end{array}\right) \\
\dot{C} &= (r-\theta)\,C + \frac{(p+\theta)\,(p+\alpha)}{1+a} W
\end{aligned}
$$

Dynamics

$$\begin{aligned}
\dot{C} &= (r-\theta)\,C + \frac{(p+\theta)\,(p+\alpha)}{1+a}K \\
\dot{K} &= F\,(K,L) - C \\
L &= \frac{\bar{L}Y - BC}{Y} \\
Y &= F_L\,(K,L)
\end{aligned}$$