Notes on Control Theory

$$\max \int_{t_0}^{t_1} f(t, x(t), u(t))dt \qquad \#$$

$$\dot{x} = g(t, x(t), u(t)) \qquad \#$$

$$t_0, t_1, x(t_0) = x_0 \;\; \textit{fixed}, \; t_1 \, \textit{can be} \, \infty.$$

$$x(t_1) \; \textit{may be free or fixed}$$

The choice variable is a function $u(t)$ which is piecewise continuous, that is we are allowed to choose $u(t)$ from the set of piecewise continuous functions. We assume f and g are differentiable. Further restrictions to define the feasible domain of $u(t)$ and $x(t)$ are possible, but we will not take them up. We must also assume that over feasible paths, $\int_{t_0}^{t_1} f(t, x(t), u(t))dt < \infty$, or a solution will not exist. When $g(t, x(t), u(t)) = u(t)$ this becomes a simpler calculus of variations problem.

Let $u^*(t)$ be the optimal solution, and define another function $u(t) = u^*(t) + ah(t)$ where $h(t)$is some arbitrary function defined on (t_0, t_1). Suppose we use $u(t)$ in equation ref: state defined for some a. Assume we obtain the solution to that differential equation $y(t, a)$ with initial condition $y(t_0, a) = x_0$. This means we must have $h(t_0) = 0$. Note that if $x(t_1)$ was fixed, we would have to restrict $h(t_1) = 0$ So $y(t, a)$ satisfies ref: state. Clearly for $a = 0$, we would have the optimal trajectory of x. We must show, that is find the conditions for which $u^*(t)$ is locally better that $u(t)$ for arbitrary $h(t)$ and small a. Define:

$$J(a) = \int_{t_0}^{t_1} f(t, y(t, a), u^*(t) + ah(t))dt \qquad \#$$

Store the above for a while.

Consider

$$\int_{t_0}^{t_1} f(t,x(t),u(t))dt = \int_{t_0}^{t_1}\left(\begin{array}{c} f(t,x(t),u(t)) \\ +\lambda(t)g(t,x,u(t)) - \lambda(t)\dot{x}(\end{array}\right.$$

where $\lambda(t)$ is assumed differentiable on $[t_0, t_1]$ and integrate by parts the last term:

$$-\int_{t_0}^{t_1} \lambda(t)\dot{x}(t)dt = \int_{t_0}^{t_1} \dot{\lambda}(t)x(t)dt$$
$$- \lambda(t_1)x(t_1) + \lambda(t_0)x(t_0)$$

Substituting this into the previous equation we get:

$$\int_{t_0}^{t_1} f(t,x(t),u(t))dt$$
$$= \int_{t_0}^{t_1} (f(t,x(t),u(t)) + \lambda(t)g(t,x,u(t)) - \dot{\lambda}(t)x(t))dt$$
$$- \lambda(t_1)x(t_1) + \lambda(t_0)x(t_0)$$

Now we go back and substitute for y instead of $x,$ because $y(t,a)$ is a feasible trajectory: ref: state holds for y as well by its definition. We obtain:

$$J(a) =$$

$$\int_{t_0}^{t_1} \left(\begin{array}{c} f(t, y(t,a), u^*(t) + ah(t)) \\ +\lambda(t) g(t, y(t,a), u^*(t) + ah(t)) + y(t,a)\dot{\lambda}(\end{array} \right.$$

$$-\lambda(t_1) y(t_1, a) + \lambda(t_0) y(t_0, a)$$

Now let us differentiate $J(a)$ with respect to a. Since $h(t)$ is arbitrary, this is checks the variaton in $J(a)$ with respect to arbitrary small variations in the control function $u(t)$ around $u^*(t)$. We areof course assuming interiority, that y, u do not bump into boundaries restricting x and u. To get a critical point we set the variations to zero and check the conditions implied, just like taking a derivative. We are of course abstracting here from Kuhn-Tucker type issues that would arise if $x(t)$ or $u(t)$ on the boundary of their feasable set.

Differentiating $J(a)$ with respect to a, **evaluated at $\mathbf{a = 0}$**, we get:

$$J'(0) = \int_{t_0}^{t_1} \begin{pmatrix} (f_x + \lambda g_x + \dot{\lambda}) y_a \\ +(f_u + \lambda g_u) h \end{pmatrix} dt - \lambda(t_1) y_a(t_1, 0)$$

$$= 0$$

Note that $y_a(t_0, 0) = 0$ because $y(t_0, a) \equiv x_0$. The condition above will be satisfied for arbitrary $h(t)$ if

$$\dot{\lambda} = -(f_x + \lambda g_x), \qquad f_u + \lambda g_u = 0 \qquad \#$$

and if, for finite $t_1, \lambda(t_1) = 0$. If $t_1 = \infty$, then we must have $\lim_{t \to \infty} \lambda(t) y_a(t, 0) = 0$, which is generally called the transversality condition. Since $y_a(t, 0)$ is the variation in $x(t)$ this condition will also appear as $\lim_{t \to \infty} \lambda(t)(x(t) - x^*(t)) = 0$, where $x^*(t)$ is the candidate optimal path and $x(t)$ is a small arbitrary variation around $x^*(t)$. Of course if we knew that all feasable paths must be bounded, then the condition would reduce to $\lim_{t \to \infty} \lambda(t) = 0$.

The first order conditions given by ref: foc can be written in “Hamiltonian” format. Let

$$H = f(t, x(t), u(t)) + \lambda(t) g(t, x, u(t))$$

Then we obtain the first oder conditions given by ref: foc if we “maximize” the Hamiltonian with respect to $u,$ and find a differentiable $\lambda(t)$ on (t_0, t_1) which satisfies transversality conditions and

$$\dot{\lambda} = -\frac{\partial H(x, \lambda, u)}{\partial x}$$

Why maximize H rather than minimize? After all we only took a derivative of J and set it to zero, so we may be minimizing. We only have necessary conditions for an internal critical point. More on this later.

First let us consider the special case where $f(t,x(t),u(t)) = e^{-rt}f(x(t),u(t))$, which is most common in economics because of discounting. Then we can write

$$\begin{aligned} H &= e^{-rt}f(x(t),u(t)) + \lambda(t)g(t,x,u(t)) \\ &= e^{-rt}(f(x(t),u(t)) + e^{rt}\lambda(t)g(t,x,u(t))) \\ &= e^{-rt}(f(x(t),u(t)) + \mu(t)g(t,x,u(t))) \end{aligned}$$

where $\mu(t) = e^{rt}\lambda(t)$. The standard first order conditions would correspond to:

$$H_u = e^{-rt}f_u + \lambda g_u = 0 \qquad \#$$

$$\dot{\lambda} = -\frac{\partial H(x,\lambda,u)}{\partial x} = -(e^{-rt}f_x + \lambda g_x) \qquad \#$$

If we write these in terms of μ,

$$H_u = e^{-rt}(f_u + \mu g_u) = 0 \qquad \#$$

we can ignore the term e^{-rt} above. Also , since

$$\dot{\mu} = re^{rt}\lambda + e^{rt}\dot{\lambda} = r\mu + e^{rt}\dot{\lambda}$$

we can write equation ref: Hx1 as:

$$e^{rt}\dot{\lambda} = -(f_x + e^{rt}\lambda g_x)$$

$$e^{rt}\dot{\lambda} = -(f_x + \mu g_x) = \dot{\mu} - r\mu$$

$$\dot{u} = -(f_x + \mu g_x) + r\mu \qquad \#$$

Now define the so called modified Hamiltonian $\tilde{H}$:

$$\tilde{H} = e^{rt}H = f + \mu g$$

Then the first order conditions in the form of ref: Huu and ref: Hxx can be written as:

$$\tilde{H}_u = f_u + \mu g_u = 0$$

$$\dot{\mu} = -\frac{\partial \tilde{H}}{\partial x} + r\mu$$

In economics this is the usual form. The transversality condition can be expressed as

$$\lim_{t\to\infty} \lambda(t)(x(t) - x^*(t)) = \lim_{t\to\infty} e^{-rt}\mu(t)(x(t) - x^*(t)) = 0$$

Often $\mu(t)$ is referred to as a current (shadow) price and $\lambda(t)$ as a (shadow) price discounted to the beginning.

Exercise: Differentiate J with respect to x_0 and show that the result is $\lambda(0)$ just like a standard Lagrange multiplier: it is a shadow price beacuse it is the marginal value to the program of having one more unit of the stock x_0.

Now back to sufficient conditions to assure that the original problem ref: prob is indeed a maximum. For this we need additional assumptions that f and g are concave (thus, note that we are indeed maximizing the Hamiltonian with respect to $u,$ not minimizing.)

Take a path (x^*, λ^*, u^*) that satisfies the first order and transversality conditions, and take any other path that satisfies ref: state, and starts from $x_0,$ denoted by $(x, \lambda, u).$ We must show that

$$D = \int_{t_0}^{t_1} (f^* - f) dt \geq 0$$

where f^* is evaluated at the candidate optimal path and f is the other path. Since f is concave we can expand it around the opimal path in a Taylor series and obtain, from concavity:

$$f^* - f \geq (x^* - x) f_x^* + (u^* - u) f_u^*$$

and thetrfore substituting from first order conditions:

$$D \geq \int_{t_0}^{t_1} ((x^* - x)f_x^* + (u^* - u)f_u^*)dt$$

$$= \int_{t_0}^{t_1} ((x^* - x)(-\lambda g_x^* - \dot{\lambda}) + (u^* - u)(-\lambda g_u^*))dt$$

Integrating by parts the terms involving $\dot{\lambda}$ above, we can now derive, noting that $\dot{x} = g,$

$$D \geq \int_{t_0}^{t_1} (\lambda(g^* - g) - (x^* - x)\lambda g_x^* + (u^* - u)(-\lambda g_u^*))dt$$

$$+ (x(t_1) - x^*(t_1))\lambda^*(t_1)$$

$$\geq 0$$

The above holds because the first line is a Taylor expansion of $g,$ which is concave, and the second line will hold because of the transversalitiy condition: if t_1 is finite $\lambda^*(t_1)$ is zero (at the end the value of unused stock is zero), or if $t_1 = \infty,$

$$\lim_{t \to \infty}(x(t) - x^*(t))\lambda^*(t) = 0 \qquad \#$$

In the equation above when integrating by parts the term $(x(t_0) - x^*(t_0))\lambda^*(t_0)$ was discarded because it must be zero by the requirement that any feasable path must start from $x_0,$ so that $(x(t_0) - x^*(t_0)) = 0.$

Note that the transversality condition ref: trcon, derived under concavity of f and $g,$ is global: the variations are not just local since the alterntive path in $x(t)$ in $(x(t) - x^*(t))$ can be **any other** feasible path.

We can also show that under strict concavity of f in (x, u) the solution under certain conditions is unique. Suppose there were two solutions, (x', u') and (x'', u'') yielding J' and J''respectively, with $J' = J''.$ Suppose now that $(x''', u''') = 0.5((x', u') + (x'', u''))$ **is also a feasable solution**. Then $f(x''', u''', t) > 0.5f(x', u', t) + 0.5f(x'', u'', t).$ If $(x', u') \neq (x'', u'')$ for some $t,$ then $(x', u') \neq (x'', u'')$ over some interval if (x, u) are continuous functions of $t.$ But then $\infty J'''(\mathbf{x}''', \mathbf{u}''') > 0.5J'(\mathbf{x}', \mathbf{u}') + 0.5J''(\mathbf{x}'', \mathbf{u}''),$ which is a contradiction.

Finally, note that

$$H(x,u,t) = f(t,x(t),u(t)) + \lambda(t)g(t,x,u(t))$$

$$\frac{dH(x,u,t)}{dt} = (f_x + \lambda g_x)\dot{x}$$

$$+ (f_u + \lambda g_u)\dot{u} + g\dot{\lambda} + \frac{\partial H}{\partial t}$$

$$= (f_x + \lambda g_x + \dot{\lambda})\dot{x} + (f_u + \lambda g_u)\dot{u} + g\dot{\lambda} + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial t}$$

So that if H is autonomous, $H(x,u,t) = H(x,u)$, then H is constant. However note that we have a discount factor in economics.

Relation to calculus of variations where

$\mu = \dot{x}$:

$$H = f(x(t), \dot{x}, t) + \lambda(t)\dot{x}$$

where $\mu(t) = e^{rt}\lambda(t)$. The standard first order conditions would correspond to:

$$H_u = f_{\dot{x}} + \lambda = 0 \qquad \#$$

or

$$\lambda = -f_{\dot{x}}$$

$$\dot{\lambda} = -\frac{\partial H(x, \lambda, u)}{\partial x} = -f_x \qquad \#$$

$$\frac{d(f_{\dot{x}})}{dt} = f_x$$

RAMSEY MODEL

$$Max_{\{c\}} \int_0^{\infty} u(c(t))e^{-\delta t}dt$$

subject to:

$$\dot{k} = f(k) - nk - c$$

$$k(0) \ given$$

Hamiltonian:

$$H = u(c) + \lambda(f(k) - nk - c)$$

FOC

$$u'(c) = \lambda$$

$$\dot{\lambda} = -\frac{\partial H}{\partial k} + \delta\lambda = \lambda(\delta - f'(k) + n)$$

$$\dot{k} = f(k) - nk - c$$

Let's get rid of λ :

$$u''\dot{c} = \dot{\lambda}$$

$$u''\dot{c} \;== u'(c)(\delta - f'(k) + n)$$

$$\dot{c} = \left(\frac{u'(c)}{cu''}\right)c(\delta - f'(k) + n)$$

$$\dot{c} = \left(\frac{-u'(c)}{cu''}\right)c(-\delta + f'(k) - n)$$

$$\dot{c} = \frac{c}{\sigma}(f' - (\delta + n))$$

So

$$\dot{k} = f(k) - nk - c$$

$$\dot{c} = \frac{c}{\sigma}(f' - (\delta + n))$$

Interpretation of

$$\frac{\dot{\lambda}}{\lambda} + f'(k) - n = \delta$$

(Rate of return equals discount rate) Why is λ a price?

Graph-Transversality

LINEARIZATION at

$$(k, c) = (\{k^* | f'(k) = \delta + n\}, \{c^* | c = f(k^*) - nk^*\})$$

Let $z = \begin{pmatrix} k - k^* \\ c - c^* \end{pmatrix}$, for small deviations

$$\dot{z} = \begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \begin{bmatrix} f'(k) - n & -1 \\ \frac{c}{\sigma} f''(k^*) & 0 \end{bmatrix} z = \begin{bmatrix} \delta \\ \frac{c}{\sigma} f''(k^*) \end{bmatrix}$$

$$= Jz : \qquad DET(J) < 0$$

Linear Analysis:

$$Let \quad Vx = z; \qquad V\dot{x} = \dot{z}$$

$$V\dot{x} = JVx, \quad \dot{x} = V^{-1}JVx = Dx$$

$$\dot{x}_1 = \lambda_1 x_1 \qquad \dot{x}_2 = \lambda_2 x_2$$

$$x_1 = x_1(0)e^{\lambda_1 t} \qquad x_2 = x_2(0)e^{\lambda_2 t}$$

Let $Q = V^{-1}$. If $\lambda_1 > 0$,
$x_1(0) = 0 \rightarrow Q_{11}z_1 + Q_{12}z_2 = 0. \rightarrow z_2 = \frac{Q_{11}}{Q_{12}} z_1.$
for the non-explosive unique solution, we get "The Policy Function"

$$dc = F(dk)$$

COMPETITIVE MARKET SOLUTION:

Let the representative consumer earn

wages and rent from capital ownership, w and r. Ramsey Model

$$Max_{\{c\}} \int_0^{\infty} u(c(t))e^{-\delta t}dt$$

subject to:

$$\dot{k} = w + rk - nk - c$$

$$k(0) \ \ given$$

Hamiltonian:

$$H = u(c) + \lambda(w + rk - nk - c)$$

FOC

$$u'(c) = \lambda$$

$$\dot{\lambda} = -\frac{\partial H}{\partial k} + \delta\lambda = \lambda(\delta - r + n)$$

$$\dot{k} = w + rk - nk - c$$

Now let the factor market be competitive:

$$r = f'(k)$$

$$w = f(k) - kf'(k)$$

so

$$w + rk = f(k) - kf'(k) + kkf'(k)$$

So we are back to

$$\dot{k} = f(k) - nk - c$$

$$\dot{c} = \frac{c}{\sigma}(f' - (\delta + n))$$

Endogenous labor:

$$Max_{\{c\}} \int_0^\infty u(c(t)) + v(1-L)e^{-\delta t} dt$$

subject to:

$$\dot{k} = F(K,L) - nk - c$$

$$K(0) \ given$$

Hamiltonian:

$$H = u(c) + v(1-L) + \lambda(F(K,L) - nK - c)$$

FOC

$$u'(c) = \lambda$$

$$v'(1-L) = \lambda F_L(K,L) = u'(c)F_L(K,L)$$

$$\dot{\lambda} = -\frac{\partial H}{\partial k} + \delta\lambda = \lambda(\delta - F_k(K,L) + n)$$

$$\dot{k} = F(K,L) - \delta K - c$$