Howitt-McAffee QJE 1988
We postulate a model with a large number of identical firms, and an even larger number of identical households, all risk neutral and all with the same constant rate of pure time preference.

There are three tradable objects: output, homogeneous labor services, and money. The money is a pure accounting device.

A firm’s receipts are instantaneously transferred to its workers and owners and must be used for purchasing output from other firms during the current period.

No credit market is assumed to exist, but that is no restriction in a world of risk neutrality and identical rates of time-preference.

The output market is perfectly competitive, in that all firms and households perceive a perfectly elastic demand schedule, and aggregate demand always equal aggregate output.
Firms incur a transaction cost to operate in the market. This transaction cost takes the form of output used up in the selling process.

Thus, a firm employing $n$ units of labor will have a gross revenue of $f(n)$, where $f$ is its production function, and will pay a total transaction cost of $\sigma(n, \bar{n})$, where $\bar{n}$ is aggregate employment (per firm).

This transaction cost depends upon the firm’s own employment because the more it sells the greater the required cost. It depends upon aggregate employment (per firm) $\bar{n}$ because of the trade externality.

The larger is $\bar{n}$, the greater is the equilibrium level of aggregate demand, and by assumption, the less the cost of selling a given quantity, $\sigma_2 < 0$. (Aggregate demand externality)
The typical firm’s wage bill will be \( w(n, \bar{n}) \).

This function can be derived in a number of different ways: for example, if the labor market is perfectly competitive when \( w(n, \bar{n}) = w^S(\bar{n}).n \), where \( w^S \) is the supply price of \( \bar{n} \) units of labor (per firm), \( w_1 > 0, \ w_2 > 0 \).

Could be result of bargaining.

The firm also faces hiring-costs which it incurs in the form of output used up.

The costs are given by the function \( \gamma(\dot{n} + \delta n, \bar{n}) \), where \( \delta \) can be interpreted as either the death rate of workers or the exogenous rate at which job separations occur for noneconomic reasons.

As \( \bar{n} \) creases, the size of the pool from which the typical firm draws its new recruits is thereby reduced, so \( \gamma_2 > 0 \), and \( \gamma_{12} > 0 \).
Profit flows

\[ L = f(n) - \sigma(n, \bar{n}) - w(n, \bar{n}) - \gamma(\dot{n} + \delta n, \bar{n}) \equiv \Pi(n, \bar{n}) - \gamma(\dot{n} + \delta n, \bar{n}) \]

- Assume \( L \) is twice differentiable, and for any \( \bar{n} \) concave in \((n, \bar{n})\), \( L_{11} = -\gamma_{11} < 0 \).
- If \( r \) is the time discount, stationary (steady state) equilibria depend on
  \[ g(n) = L_2(0, n, n) - rL_1(0, n, n) \]
  \[ = f'(n) - \sigma_1(n, \bar{n}) - w_1(n, n) - (r + \delta)\gamma_1(\delta n, n) \]

- This function describes the steady state marginal profit of employment; i.e., the marginal contribution of a unit of labor to the stationary flow of profits: \( L_2 = \Pi_1 - -\delta \gamma_1 \), minus the interest cost of the initial recruiting outlay, \(-rL_1 = \gamma r_1\).
Assume $g(n) < 0$ for $n \in (0, n_1)$ and $n \in (n_2, 0)$, $g(n) > 0$ for $n \in (n_1, n_2)$. 

**Figure I** Multiple Equilibria
\[
\max \int_0^\infty e^{-rt} L(\dot{n}, n, \bar{n}) dt \\
L_2 (\dot{n}, n, \bar{n}) - rL_1 (\dot{n}, n, \bar{n}) = \frac{d}{dt} L_1 (\dot{n}, n, \bar{n})
\]
which is the optimal solution if transversality holds,
\[
\lim_{t \to \infty} e^{-rt} L_1 (\dot{n}, n, \bar{n}) = 0.
\]
Linearizing and evaluating at steady state \( n^* \):
\[
0 = L_{11} \ddot{n} + (L_{13} - rL_{11}) \dot{n} - (L_{22} + L_{23} + rL_{12} + rL_{13}) (n - n^*) \\
= -\gamma_{11} \ddot{n} + (r\gamma_{11} - \gamma_{12}) \dot{n} - g'(n^*) (n - n^*)
\]
The roots of this system, \((\lambda_1, \lambda_2)\) must satisfy.

\[
\begin{align*}
\lambda_1 + \lambda_2 &= r - \gamma_{12}/\gamma_{11} \\
\lambda_1 \lambda_2 &= g'(n^*)/\gamma_{11}
\end{align*}
\]

- **Note:** if \(\gamma_{12} = 0\), \(\lambda_1 + \lambda_2 = r > 0\); if \(g'(n^*) < 0\), \(\lambda_1 \lambda_2 < 0\) : no externality → saddle: Given initial \(n\), single choice of \(\dot{n}\) to converge to \(n^*\).

- **But if** \(\lambda_1 + \lambda_2 = r - \gamma_{12}/\gamma_{11}\) is negative (big enough \(\gamma_{12}/\gamma_{11}\) ) and \(g'(n^*) > 0\) both roots of the same sign: can be sink, or source surrounded by cycle (Diamond-Fudenberg).
Figure II
Local Stability
Figure III
Perfect Foresight Paths
$J$ is value of employee to firm, $u$ is unemployment. Initial $u_{-1}$ is given