Age, Luck, and Inheritance

Jess Benhabib Shenghao Zhu*
New York University

June 2008, Revised March 2009

Abstract

We introduce the investment risk into a heterogeneous agents model and present a mechanism to analytically generate a double Pareto distribution of wealth. We replicate the distribution of the U.S. wealth and especially the three prominent features: a high Gini coefficient, skewness to the right, and Pareto tails. We disentangle the contribution of inheritance, age and stochastic rates of capital return to wealth inequality, in particular to the Gini coefficient. Finally, we investigate the effects of the fiscal and redistributive policies on wealth inequality and social welfare.

*We would like to thank Alberto Bisin, Andrew Caplin, Raquel Fernandez, Liutang Gong, Leslie Greengard, Frank C. Hoppensteadt, Matthias Kredler, Wei Liao, Henry P. McKean, Jun Nie, Thomas Sargent, Sam Schulhofer-Wohl, Robert Shimer, Gianluca Violante, Pengfei Wang, Edward N. Wolff, Zheng Yang, Atilla Yilmaz, and seminar participants at The Third Federal Reserve Bank of New York/Philadelphia Workshop on Quantitative Macroeconomics for their helpful comments and suggestions.

1. Introduction

Age, luck, and inheritance all play a significant role in the accumulation of wealth. Households accumulate wealth as they age. Rates of return vary across households and over time depending on the luck of the draw. And some agents start their economic life with large inheritances.[1] We develop a dynamic model incorporating these three factors with utility-optimizing agents to match the U.S. wealth distribution.

The U.S. empirical wealth distribution displays three well-known features: a high Gini coefficient, skewness to the right and a Pareto upper tail. Investigating the data of Survey of Consumer Finances (SCF) in the U.S. in 2001, Wolff (2004) finds that the Gini coefficient of wealth is 0.826 and the top 1% of the richest households holds 33.4% of the wealth in the U.S.. Using the richest sample of the U.S., the Forbes 400, during 1988-2003 Klass et al. (2006) find that the top end of the wealth distribution obeys a Pareto law with an average exponent of 1.49. We try to generate these three features in our model.

A large literature on incomplete markets studies the wealth distribution in heterogeneous agents model. Agents face uncertain labor income and hold precautionary savings against uninsurable labor earnings. Yet the standard models of Aiyagari (1994) and Huggett (1993) do not produce either high inequality or heavy tails. The reason is that the constant or bounded relative risk aversion utility functions in these models imply that the stationary distribution of wealth has bounded support, as pointed out by Schechtman and Escudero (1977).[2]

A number of authors such as Huggett (1996), Krusell and Smith (1998), Castaneda, Diaz-Gimenez and Rios-Rull (2003) and De Nardi (2004), have introduced new features to the basic incomplete market models to try to replicate the U.S. distribution of wealth, with high Gini coefficients and right skewness. For example,

[1] Intergenerational transfers also play an important role in the aggregate capital accumulation. Kotlikoff and Summers (1981) find that intergenerational transfers account for sustaining the vast majority, up to 70%, of the aggregate U.S. capital formation. See also Gale and Scholtz (1994) for more moderate findings on this topic. For an account of the the role of inhertance on the Forbes 400 see Elwood et al. (1997) and Burris (2000).

[2] The mechanism that generates a non-degenarate stationary distribution for such additive shock (stochastic labor income) models requires the gross interest rate to be no greater than the reciprocal of the time discount rate. This is also the feature that generates a stationary distribution in the calibrated models, for example of Aiyagari (1994), Huggett (1993), and Castaneda, Diaz-Gimenez and Rios-Rull (2003). In our model the product of the interest rate and the time discount rate exceeds unity so we have growth.

Castaneda, Diaz-Gimenez and Rios-Rull (2003) incorporate life cycle features, a social security system, progressive income, estate taxes and intergenerational transmission of stochastic earnings ability into their model. They find through simulations that the labor efficiency shock helps to account for the U.S. earnings and wealth inequality almost exactly.

We depart from these authors by introducing idiosyncratic investment risk. Bertaut and Starr-McCluer (2002), using the Survey of Consumer Finances (SCF) in 1998, document that 19.2% of households have direct investment in stocks as a financial asset, and 89.9% of households have nonfinancial assets most of which with highly nondiversified risk. The 2004 SCF shows that about 49.77% of all households have assets in quasi-liquid[3] pension funds, constituting 13.45% of household wealth. According to Bertaut and Starr-McCluer (2002), the share of household sector financial assets invested in corporate equities, either directly or through a mutual fund, retirement account, other managed asset or DC pension fund was 35% in 1998, involving 49% of households. The catalogue of nonfinancial assets consists of primary residence, investment real estate, business equity, and other nonfinancial assets. 66.3% of households choose to own rather than rent their primary residence. 18.6% of households have investments in real estate and 11.5% of households have business equity. These financial and non-financial assets are subject to considerable idiosyncratic risk. Case and Shiller (1989), using the data for four U.S. cities from 1970 to 1986, document that individual housing price changes have large standard deviations of annual percentage change, close to 15% a year for individual housing prices. Flavin and Yamashita (2002), using the data from the Panel Study of Income Dynamics (PSID) demonstrate that the owner-occupied housing prices have a large idiosyncratic component, and the standard deviation of the return to housing, at the level of the individual house, is about 14%. Moskowitz and Vissing-Jorgensen (2002) document that entrepreneurs have poor diversification of private equity holdings and that private firms are subject to dramatic idiosyncratic risk. Even conditional on survival, the distribution of equity returns across entrepreneurs is wide. Bertaut and Starr-McCluer (2002) also document that the sum of primary residence (home), private business equity and investment real estate account for more than half the total asset of U.S. households in 1998. Household wealth therefore, as measured by the SCF, is

[3] These consist of employer sponsored IRAs, Keoghs, thrift type accounts, and future and current account type pensions, but exclude both Social Security benefits and employer sponsored defined benefit (DB) plans.

subject to considerable idiosyncratic risk.

Some recent papers introduce idiosyncratic investment returns into simulated models of wealth accumulation. Quadrini (2000), introduces entrepreneurial wealth subject to idiosyncratic shocks, and generates a concentration of wealth with a heavy top tail similar to the U.S. economy. Cagetti and De Nardi (2006) permit the entrepreneurial entry and exit in a model with idiosyncratic entrepreneurial ability shocks, and their simulations match the fraction of entrepreneurs in the U.S. population. Angeletos (2007) examines the impact of idiosyncratic investment risk on aggregate saving and income in the heterogenous agents model but does not study the implications of idiosyncratic investment risk on wealth distribution.[4]

In the econophysics literature, some authors try to use statistical models to replicate the Pareto upper tail of wealth and income distribution. Levy (2003) uses a stochastic multiplicative process with lower reflecting barrier to generate the Pareto upper tail of the wealth distribution. Reed (2001) proves that if the cross section age distribution in a society is an exponential distribution and the agents' incomes follow identical Geometric Brownian Motion with the same initial condition, the cross section of income is a double Pareto distribution.[5] Mizuno et al. (2002) employ a Kesten process to describe the income evolution of a company and replicate the Pareto tail of the income distribution of Japanese companies. None of these models however have rational utility-optimizing agents.

We incorporate the investment risk into an incomplete market model with utility-optimizing agents to study the distribution of wealth. Unlike the simulated models, we provide analytic solutions of the stationary wealth distribution.[6] We try to replicate the three prominent features of the U.S. wealth distribution: a

[4]While Angeletos (2007) finds that the idiosyncratic production risk causes the underaccumulation of capital, Covas (2006)'s simulations show that the idiosyncratic production risk causes overaccumlation of capital. Another difference between them is that the individual production in Angeletos (2007) displays constant returns to scale, while that in Covas (2006) displays decreasing returns to scale. Angeletos and Calvet (2006) also investigate the implication of idiosyncratic production risk on the propagation mechanism of business cycle.

[5]Reed (2001) also explains the power law behaviour in the upper tail of city size distributions. The plots of Reed (2001) reveal power law behaviour in both the upper tail and lower tail of 1998 U.S. male earnings distributions and 1998 U.S. settlement sizes distribution. Huberman and Adamic (1999) use the same argument to explain the distribution of the number of web pages per site of the World Wide Web.

[6]For a recent paper giving closed forms in an Aiyagari model with stochastic labor income rather than stochastic returns, see Wang (2007).

high Gini coefficient, skewness to the right, and Pareto tails. We obtain a wealth distribution with upper and lower Pareto tails. We calibrate our model to match the U.S. distribution and we obtain a Gini coefficient of 63.7%. The top 1% of households hold 34.33% of the wealth. We then decompose the Gini coefficient to isolate the effects of age, luck and inheritance. We find that under our calibration luck or the stochastic rate of return captures about 31% of wealth inequality in terms of the Gini coefficient, while life-cycle accumulation or age contributes about 37%.

In contrast to other results in literature, we show that government redistributive policies have important consequences for wealth inequality through their effects on the growth rates of wealth. Our simulations demonstrate that, due to the presence of idiosyncratic returns on investments, increases in capital taxes and to some extent estate taxes can significantly decrease the Gini coefficient of the economy. (See Table 4.9 of section 5.1 in Appendix). Finally we show that fiscal policies can have an impact on social welfare, defined as the sum of the discounted utility streams of those who are alive. (See section 5.2).

A further feature that differentiates our model from the existing heterogenous agents models is that we not only match the prominent characteristics and the Lorenz curve of the wealth distribution, but we also generate a double Pareto distribution that display almost the same shape as the U.S. wealth distribution.

Our analysis is based on Benhabib and Bisin (2006) who also investigate the impact of intergenerational transmission and redistributive policies on the wealth inequality. In a model with one riskless asset Benhabib and Bisin (2006) generate the Pareto upper tail and find that wealth inequality induced by inheritance accounts for just a little less than a third of the size of the Gini coefficient in the U.S. in 1992. We introduce a risky asset into the model to generate the high Gini coefficient and to better identify the contributions of the stochastic rate of return, of age, and of inheritance to the inequality of wealth.

Our model is a continuous time overlapping generations model with a continuum of agents as in Yaari (1965) and Blanchard (1985), with optimizing agents. There are three kinds of financial assets: a risk-free asset, a home-specific risky asset and life insurance or annuities. For each agent the return to the idiosyncratic risky asset is stochastic and follows the same Geometric Brownian Motion. Under optimal consumption and investment behavior, the wealth of agents also follows a Geometric Brownian Motion.

A standard mechanism to generate Pareto upper fat tail in stationary distribution is to construct a stochastic process with negative drift and a lower reflecting barrier. Champernowne (1953) was the first to employ a Markov process for income with an arbitrary reflecting barrier to generate a Pareto distribution. Wold and Whittle (1957) introduced a birth and death process with exogenous exponential wealth accumulation and bequests to generate a stationary wealth distribution with a Pareto tail. (See also Vaughan (1988)). More recently Gabaix (1999) used this mechanism to study the distribution of city sizes. In Benhabib and Bisin (2006) the wealth accumulation process is based on optimizing behavior of agents and has positive deterministic growth. But as in Wold and Whittle (1957), the geometrically distributed death and inheritance processes, together with estate taxes, result in a stationary distribution. We dispense with the need for a positive lower bound of the stochastic process: bad luck does in fact drive wealth down. In our model, Geometric Brownian Motion of wealth yields a stationary distribution of wealth that follows a Pareto law in both tails (double Pareto distribution).

The rest of this paper is organized as follows. In section 2, we present the basic structure of our continuous time OLG economy. We investigate the cross-sectional wealth distribution of the economy in section 3. We present a calibrated economy and disentangle the contribution of inheritance, age, and stochastic rates of capital return to wealth inequality in section 4. Section 5 contains the analysis of the effect of redistributive policy on wealth inequality and social welfare. In section 6 we briefly present an alternative economy with across the board lump-sum subsidies; these subsidies can also be interpreted as the value of non-stochastic lifetime labor income or human capital. We conclude with a discussion in section 7 and leave the proofs to the Appendix.

2. An OLG economy

There is a continuum of agents in the economy who invest their wealth in a riskless asset and an idiosyncratic risky asset. Thus there is a continuum of risky assets in the economy. We can view the risky asset as the composite of household-owned housing, private business and stocks. The stochastic processes for the risky assets held by the agents are independent, but they follow the same Geometric Brownian Motion process. For the agent who born at time s, the value of his idiosyncratic risky asset at time t, $S(s,t)$, follows[7]

$$dS(s,t) = \alpha S(s,t)dt + \sigma S(s,t)dB(s,t)$$

[7] Of course $t \geq s$.

where $B(s,t)$ is the standard Brownian motion, α is the instantaneous conditional expected percentage change in value per unit time and σ is the instantaneous conditional standard deviation per unit time. The Geometric Brownian Motion process implies that the value of the risky asset is log-normally distributed and the rate of return does not depend on the risky asset value. Every agent invests his/her wealth in his/her own risky asset. Risk sharing on the return of risky asset is not permitted.[8]

The value of the riskless asset, $Q(t)$, follows

$$dQ(t) = rQ(t)dt$$

where r is the rate of return of the riskless asset and $r < \alpha$. The rate of return of the riskless asset is identical for all agents in the economy.

The agent allocates individual wealth among current consumption, investment in a risky asset, a riskless asset and the purchase of life insurance. Negative life insurance purchases, allowed in our model, corresponds to the purchase of annuities. $Z(s,t)$ denotes the bequest that the agent born at time s leaves at time t if the agent dies. The bequest consists of two parts: the agent's wealth invested in riskless asset and risky asset, and the life insurance/annuities that the agent purchases. The price of the life insurance is μ. The agent pays $P(s,t)dt$ to buy the life insurance. Should the agent die in the next short period dt, the life insurance company pays $\frac{P(s,t)}{\mu}$. Let $W(s,t)$ be wealth at time t of an agent born at time s. The relation between the bequest, $Z(s,t)$, wealth, $W(s,t)$, and the payment from the insurance company, $\frac{P(s,t)}{\mu}$, is given by

$$Z(s,t) = W(s,t) + \frac{P(s,t)}{\mu}$$

If $P(s,t) < 0$, the life insurance company is an annuity company, paying $P(s,t)dt$ to those alive and receiving $\frac{P(s,t)}{\mu}$ from the estate of the agents who die.

There is an uncertainty about the duration of the agent's life: it follows an

[8] Home ownership is of course fully undiversified. For the private business, Moskowitz and Vissing-Jorgensen (2002) document that entrepreneurs have poor diversification of private equity holdings. Bitler et al. (2005) show that why the entrepreneurs concentrate large fractions of their wealth in a single private firm may be due to the agency costs caused by the moral hazard. Aghion et al. (2007) document that the mean institutuional ownership of a sample of over 1,000 U.S. stocks between 1991 and 2004 is only 42.4%.

exponential distribution with rate parameter of p. Each agent will die at a time $t \in [0, +\infty)$ by a probability density function $\pi(t) = pe^{-pt}$. In a small time Δt, the agent has probability of $p\Delta t$ to die, conditioning on the event that the agent is sill alive. When the agent dies, the agent's child is born. Each agent has one child.

Life insurance or annuity companies operate competitively and earn zero profits. They effectively act as clearing houses. In a short period of length Δt payments and disbursements are equal: $p\frac{P(s,t)}{\mu}\Delta t = P(s,t)\Delta t$, so that $\mu = p$.[9]

While the agents have uncertain lifetimes in our model, in the presence of the perfect life insurance markets there are no accidental bequests. This feature is in contrast to some of the literature on precautionary saving, such as Abel (1985) and Fuster (2000), who investigate how the lack of annuity markets affects saving behavior and the intergenerational transfer of wealth under uncertain lifetimes.

We assume that the bequest motive takes the form of "joy of giving": The bequest enters parents' utility function but parents do not care about children's utility directly. Utility from bequests is given by $\chi\phi((1-\zeta)Z(s,t))$ where ζ is the estate tax rate, χ represents the strength of the bequest motive, and $\phi(\cdot)$ is the bequest utility function. We choose CRRA functions for both the consumption and the bequest utilities.

We assume that there are two groups of agents in the economy. A fraction $\frac{q}{p} \leq 1$ of the people have a bequest motive,with bequest motive parameter $\chi > 0$, and a fraction $1 - \frac{q}{p}$ of people do not have a bequest motive with bequest motive parameter $\chi = 0$.[10]

2.1. The individual problem

Let $J(W(s,t))$ be the optimal value function of agents. The agent's utility maximization problem is

$$J(W(s,t)) = \max_{C,\omega,P} E_t \int_t^{+\infty} e^{-(\theta+p)(v-t)}[\frac{C^{1-\gamma}(s,v)}{1-\gamma} + p\chi\frac{((1-\zeta)Z(s,v))^{1-\gamma}}{1-\gamma}]dv \quad (1)$$

[9] In fact collections and disbursements occur every instant in continuous time as $\Delta t \to 0$.

[10] The assumption that agents have heterogneous bequest motives is not nessary for our model to generate the Pareto tails. We can allow $q = p$.

subject to

$$dW(s,t) = [(r-\tau)W(s,t)+(\alpha-r)\omega(s,t)W(s,t)-C(s,t)-P(s,t)]dt+\sigma\omega(s,t)W(s,t)dB(s,t) \quad (2)$$

where θ is the time discount rate. $C(s,t)$ is the consumption at time t of an agent born at time s and $\omega(s,t)$ is the share of wealth the agent invests in risky asset. τ is the capital tax on wealth[11]. The transversality condition for the agent's problem is[12]

$$\lim_{t\to+\infty} Ee^{-(\theta+p)(t-s)}J(W(s,t)) = 0$$

The set-up of the agent's problem is that of Richard (1975). We add a capital tax and an estate tax to Richard's (1975) model. The agent's optimal policy is same as that of Richard (1975), except that we have to take into account the influence of taxes.

Proposition 1. *The agent's optimal policies are characterized by*

$$\begin{aligned} C(s,t) &= A^{-\frac{1}{\gamma}}W(s,t),\ \omega(s,t) = \frac{\alpha-r}{\gamma\sigma^2}, \\ Z(s,t) &= (\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}W(s,t) \end{aligned}$$

with $A = \left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1+(p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma}$ *and*

$$dW(s,t) = gW(s,t)dt + \kappa W(s,t)dB(s,t).$$

with $g = \frac{r-\tau+\mu-\theta-p}{\gamma} + \frac{1+\gamma}{2\gamma}\frac{(\alpha-r)^2}{\gamma\sigma^2}$ *and* $\kappa = \frac{\alpha-r}{\gamma\sigma}$.

Note that $P(s,t)$ may be positive or negative depending on the sign of $Z(s,t)-W(s,t)$, and determines whether the agent buys annuities or life insurance.[13]

The mean growth rate of the agent's wealth, g, is independent of the bequest parameter χ. This is due to the specific form of the utility function and the

[11] Of course τ can be redefined so it is a tax on capital income. We also use the term of capital income tax for τ in this paper.

[12] For the transversality condition of the continuous-time stochastic dynamic programming problem, see Merton (1992).

[13] If $\mu = p$, then the sign of $Z(s,t) - W(s,t)$ is determined by $(\frac{\chi}{A})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} - 1$.

completeness of the life insurance market. And g depends on the income tax rate τ, but not the estate tax rate ζ.

The share of risky asset, $\omega(s,t)$, is only influenced by the risk premium of the risky asset, the degree of risk aversion, and the volatility of the return of the risky asset. This is the same result as that of Merton (1971). And $\omega(s,t)$ does not depend on "joy of giving" parameter, χ, or the government tax policies.

The volatility of the growth rate of the agent's wealth, κ, does not depend on the bequest motive parameter, χ. Note that κ is negatively related to the standard deviation of the price of the risky asset, σ, even though $\kappa = \sigma\omega(s,t)$. This is because $\omega(s,t)$ is negatively related to σ^2. The aversion to the risk causes the agent to overreact to risk such that the volatility of wealth is negatively related to the volatility of the risky asset. Government policy has no impact on κ.

The agent's wealth evolves as a Geometric Brownian Motion

$$dW(s,t) = gW(s,t)dt + \kappa W(s,t)dB(s,t) \tag{3}$$

Equation (3) also implies that wealth growth displays Gibrat's Law. The growth rate of the wealth is independent of the level of wealth. In many of the mechanisms generating a power law distribution, Gibrat's Law plays a fundamental role, and this is also true in our model.

2.2. The aggregate economy

The age cohorts are large enough such that the law of large numbers holds whenever we try to use it. This assumption implies: 1) Even though each agent faces uncertainty about the duration of life, the size of the cohort born at s is $pe^{-p(t-s)}$ at time t. The size of the population at any time t is $\int_{-\infty}^{t} pe^{p(s-t)}ds = 1$. 2) Although different agents within a cohort have different wealth levels, the aggregate wealth level of a cohort depends on the age of the cohort, but not on the wealth distribution within the cohort.

At time t, conditional on the event that the agents born at time s are still alive, the mean wealth of the cohort is denoted by $E_sW(s,t)$, where the expectation is calculated with respect to the cross-section distribution of wealth of agents born at time s who are still alive at time t. Similarly, let $E_sW(s,s)$ be the mean starting wealth of the agents born at time s, where the expectation is computed with respect to the cross-section distribution of wealth of the newborns at time s.

Then[14]

$$E_sW(s,t) = E_sW(s,s)e^{g(t-s)} \tag{4}$$

Following Benhabib and Bisin (2006), we derive the aggregate wealth growth rate. Let $W(t)$ be the aggregate wealth of the economy. Integrating the mean wealth of all age cohorts with respect to the stationary population distribution, we obtain the aggregate wealth

$$W(t) = \int_{-\infty}^{t} E_sW(s,t)pe^{p(s-t)}ds \tag{5}$$

Plugging formula (4) into formula (5), we have

$$W(t) = \int_{-\infty}^{t} E_sW(s,s)pe^{(g-p)(t-s)}ds \tag{6}$$

Differentiating equation (6) with respect to t, we have the aggregate wealth growth equation

$$\frac{dW(t)}{dt} = pE_tW(t,t) + (g-p)W(t) \tag{7}$$

We need to compute $pE_tW(t,t)$ in formula (7). Since $E_tW(t,t)$ is the mean starting wealth of the agents born at time t, $pE_tW(t,t)$ represents the aggregate starting wealth of the newborns at time t. The aggregate starting wealth $pE_tW(t,t)$ consists of two parts: the private bequest and the public subsidy.

The newborn whose parents have a bequest motive receives an inheritance. By the bequest function in Proposition 1, we find that the aggregate inheritance which the newborns receive from their parents after estate tax is $q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}(1-\zeta)W(t)$.

The public subsidy is determined by the government's budget. The government collects capital and estate taxes, finances government expenditures proportional to the aggregate wealth, and provides subsidies to qualifying newborns. The overall government budget is balanced at any time.

Let $\Upsilon(t)$ denote the aggregate subsidy from the government, and let η denote the ratio of government expenditure to aggregate wealth. The government budget

[14] For this point we benefit from the discussion with Zheng Yang. The convinience of aggregation here is from two facts: (1) The stochastic return shock is idiosyncratic, and (2) The stochastic growth rate is independent of the individual wealth level (see the equation (3)). The same facts ease the difficulty of aggregation in Angeletos (2007).

constraint yields

$$\Upsilon(t) = q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta W(t) + \tau W(t) - \eta W(t)$$

where $q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta W(t)$ is the estate tax revenue, which is derived by using the expression of the bequest function in Proposition 1, and $\tau W(t)$ is capital tax revenue. We assume that the government tax revenue is greater than government expenditure $\eta W(t)$, so that $\Upsilon(t) > 0$. Note that the total subsidies to newborns do not depend on how they subsidies are distributed. Summing the aggregate inheritance and the aggregate subsidy, we have the aggregate starting wealth of newborns:

$$pE_tW(t,t) = (q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} + \tau - \eta)W(t) \tag{8}$$

Substituting equation (8) into equation (7), we obtain:

$$\frac{dW(t)}{dt} = (q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} + g - p + \tau - \eta)W(t) \tag{9}$$

Then formula (9) plus the initial aggregate wealth, $W(0)$, determines the evolution of the aggregate wealth of the economy.

Let $\tilde{g}$ denote the growth rate of the aggregate wealth. From equation (9), we know that $\tilde{g} = q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} + g - p + \tau - \eta$. Thus the relative growth rate of individual wealth to aggregate wealth is

$$g - \tilde{g} = p - q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} - (\tau - \eta) \tag{10}$$

3. Wealth distribution and inequality

We now investigate the cross-sectional wealth distribution. Since the aggregate wealth level is growing, one way to study the cross-sectional wealth distribution is to investigate the distribution of the ratio of individual wealth to aggregate wealth.[15] Following Gabaix (1999), we define $X(s,t)$ as the the ratio of the indi-

[15] This is equivalent to discounting the individual wealth level by the aggregate wealth growth rate, $\tilde{g}$, plus the normalization, $W(0) = 1$, as in Benhabib and Bisin (2006).

vidual wealth to the aggregate wealth.

$$X(s,t) = \frac{W(s,t)}{W(0)e^{\tilde{g}t}} \tag{11}$$

By equations (3) and (11), $X(s,t)$ also generates Geometric Brownian Motion.

$$dX(s,t) = (g - \tilde{g})X(s,t)dt + \kappa X(s,t)dB(s,t) \tag{12}$$

Then $X(s,t)$ is lognormally distributed, and

$$X(s,t) = X(s,s)\exp[(g - \tilde{g} - \frac{1}{2}\kappa^2)(t-s) + \kappa(B(s,t) - B(s,s))]$$

where we assume that $g - \tilde{g} - \frac{1}{2}\kappa^2 \geq 0$.[16]

To investigate the cross-sectional distribution of $X(s,t)$, we need to know not only the evolution function of $X(s,t)$ during an agent's the lifetime, but also the change of $X(s,t)$ between two consecutive generations. The evolution of wealth during an agent's the lifetime reflects the impact of age and of the stochastic rates of return on capital. The change of $X(s,t)$ between two consecutive generations reflects the role of inheritance and government subsidies.

Government subsidies are distributed according to the following rule. Government only subsidizes the newborns. If a newborn's inheritance is lower than a threshold level that is proportional to the aggregate wealth, the government gives the newborn a subsidy that brings their starting wealth to the threshold level.[17] If the newborn's inheritance is higher than the threshold level, the newborn does not receive a wealth subsidy from the government. The newborn whose parents do not have a bequest motive receives a wealth subsidy which is equal to the threshold level, and starts life at the threshold level of wealth.[18]

[16]This is a technical assumption. Otherwise $X(\cdot,t)$ converges to 0 almost surely. Note that this assumption implies that $g - \tilde{g} \geq \frac{1}{2}\kappa^2 > 0$.

[17]We may think of part of these subsidies as the discounted value of lifetime transfers, or consider initial wealth at birth to be the discounted value of lifetime earnings. See also section 6 below.

[18]Benhabib and Bisin (2006) also use a welfare policy where the subsidy is designed to top up all bequests to newborns, including zero bequests, that fall short of a minimum wealth level that grows at the rate of growth of the economy. Benhabib and Bisin (2006) show that if the parents cared not about the "joy of giving" bequests net of taxes, but the bequests including top-up subsidies, the optimal bequests would be different due to the induced non-convexity in the

Since we assume that the government tax revenue is greater than government expenditure, government can always guarantee a positive threshold level. The redistributive policy then implies that the newborn without inheritance has positive starting wealth. This specific wealth level is endogenous and will be determined below.

Let $x^*W(t)$ be the threshold level of wealth below which newborns qualify for the government wealth subsidy. For the agents who have bequest motives, suppose that a parent with wealth $W(e,s)$ leaves bequest $Z(e,s)$ to his child.[19] If $(1-\zeta)Z(e,s) \geq x^*W(s)$, the child's starting wealth is determined by $W(s,s) = (1-\zeta)Z(e,s)$. Since, by the optimal policy, $Z(e,s) = (\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}W(e,s)$, we have

$$W(s,s) = (1-\zeta)Z(e,s) = (\frac{p\chi(1-\zeta)}{A\mu})^{\frac{1}{\gamma}}W(e,s) \tag{13}$$

Dividing both sides of equation (13) by $W(0)e^{\tilde{g}s}$, and applying the definition of $X(s,t)$, equation (11), we have $X(s,s) = (\frac{p\chi(1-\zeta)}{A\mu})^{\frac{1}{\gamma}}X(e,s)$. Let

$$\rho = (\frac{p\chi(1-\zeta)}{A\mu})^{\frac{1}{\gamma}} \tag{14}$$

The transfer wealth ratio between two consecutive generations when the inherited wealth of the newborn is above the threshold for a government subsidy then is $X(s,s) = \rho X(e,s)$.

If on the other hand the parents have a bequest motive but their wealth level $W(e,s) < \frac{x^*}{\rho}W(s)$, or if the parents do not have a bequest motive, then the government subsidizes their children. The children start their lives at the wealth level of $x^*W(s)$.

We now characterize the cross-sectional distribution of $X(\cdot,t)$ at time t, $f(x,t)$.

agent's optimization problem. Using standard smooth pasting arguments Benhabib and Bisin (2006) show in their appendix that if parents cared about bequests with the top-up subsidies included, that the optimal net bequest would be zero until a wealth threshold level that exceeds the minimum wealth for the newborn, and then revert exactly to the level prescribed by the model above that wealth threshold. As the prescribed minimum wealth subsidy to newborns goes to zero, the optimal bequest and consumption functions of the two models where parents care about topped up bequests and where parents care about "joy of giving" bequests converge. In section 6 below we also consider a policy where all newborns, irrespective of inheritance, receive the same subsidy, which may also be interpreted, after adjusting the fiscal policies, as the discounted value of lifetime labor earnings.

[19] By our notation, e in $W(e,s)$ and $Z(e,s)$ means that the parent is born at time e and $e \leq s$.

In section 8.2 of the Appendix we derive the forward Kolmogorov equation of $f(x,t)$

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\kappa^2 x^2 f(x,t)) - \frac{\partial}{\partial x}((g-\tilde{g})xf(x,t)) - pf(x,t) + qf(\frac{x}{\rho},t)\frac{1}{\rho}, \quad x > x^* \tag{15}$$

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\kappa^2 x^2 f(x,t)) - \frac{\partial}{\partial x}((g-\tilde{g})xf(x,t)) - pf(x,t), \qquad x < x^* \tag{16}$$

The partial differential equations do not hold at $x = x^*$.[20] By the definition of $X(s,t)$, equation (11), we know that

$$\int_0^{+\infty} xf(x,t)dx = 1, \forall t \geq 0 \tag{17}$$

It is difficult to solve the partial differential equations with an arbitrary initial distribution. Instead, we investigate the behavior of the equations in the long run, the stationary distribution of the wealth.[21]

Proposition 2. *The stochastic process, $X(\cdot,t)$, is ergodic.*

Proposition 2 guarantees the ergodicity of the stochastic process, $X(\cdot,t)$: starting from any initial distribution, the stochastic process converges to the unique stationary distribution. In the proof of the Proposition 2 we use the existence of the stationary distribution which we derive from the Proposition 5 and Proposition 6 below.

In the stationary distribution, we have $\frac{\partial f(x,t)}{\partial t} = 0$. We deduce, from the partial differential equations, that the stationary distribution $f(x)$ satisfies the following ordinary differential equations:

$$\frac{1}{2}\kappa^2 x^2 f''(x) + (2\kappa^2 - (g-\tilde{g}))xf'(x) + (\kappa^2 - (g-\tilde{g}) - p)f(x) + qf(\frac{x}{\rho})\frac{1}{\rho} = 0, \quad x > x^* \tag{18}$$

[20]For this point, we greatly benefited from the discussion with Matthias Kredler and Henry P. McKean.

[21]Benhabib and Bisin (2006) do study the transition dynamics and convergence of the PDE above, but their setting is simpler because it does not involve stochastic returns.

and

$$\frac{1}{2}\kappa^2 x^2 f''(x) + (2\kappa^2 - (g-\tilde{g}))xf'(x) + (\kappa^2 - (g-\tilde{g}) - p)f(x) = 0, \quad x < x^* \quad (19)$$

Proposition 5 gives us the specific functional form of the stationary distribution. Proposition 2 guarantees the uniqueness of the solution of equations (18) and (19) with the boundary conditions

$$\int_0^{+\infty} f(x)dx = 1 \quad (20)$$

and

$$\int_0^{+\infty} xf(x)dx = 1 \quad (21)$$

Condition (20) follows from the normalization of population size. Condition (21), which is the mean preservation condition, is from equation (17).

The endogenous value x^* is determined by government's subsidy policy:

$$(p-q)x^* + q\int_0^{\frac{x^*}{\rho}} (x^* - \rho x)f(x)dx = q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta + \tau - \eta. \quad (22)$$

On the left hand side of equation (22), the term $(p-q)x^*$ is the government subsidy to newborns whose parents do not have a bequest motive, and the term $q\int_0^{\frac{x^*}{\rho}} (x^* - \rho x)f(x)dx$ is the government subsidy to the newborns who receive inheritance lower than x^*. The term on the right hand side of equation (22), $q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta+\tau-\eta$, is the subsidy to all the newborns from the government's budget.

3.1. Pareto distribution

In this subsection, we first discuss the special case of no inheritance and then proceed to the general case of inheritance. In both cases the stationary distribution turns out to be a double Pareto distribution.

3.1.1. No inheritance

If agents do not have bequest motive, they leave no bequest to their children. The starting wealth of the newborn is the government subsidy. This closes one of the channels for the intergenerational transmission of inequality in wealth distribution, and corresponds to the special case of $q = 0$ in the general model. From equations (18) and (19), we know that the density function of the stationary distribution, $f(x)$, solves

$$\frac{1}{2}\kappa^2 x^2 f''(x) + (2\kappa^2 - (g - \tilde{g}))x f'(x) + (\kappa^2 - (g - \tilde{g}) - p) f(x) = 0, x \neq x^* \quad (23)$$

All of the newborns are injected into the economy through the discounted wealth level $x^* = \frac{\tau - \eta}{p} > 0$.

Proposition 3. *The stationary distribution in the no inheritance case has the following kernel*

$$f(x) = \begin{cases} C_1 x^{-\beta_1} & \text{when } x \leq x^* \\ C_2 x^{-\beta_2} & \text{when } x \geq x^* \end{cases}$$

where β_1 and β_2 are the two roots of

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g - \tilde{g}))\beta + \kappa^2 - p - (g - \tilde{g}) = 0.$$

Then $\beta_1 = \frac{\frac{3}{2}\kappa^2 - (g-\tilde{g}) - \sqrt{(\frac{1}{2}\kappa^2 - (g-\tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2}$ and $\beta_2 = \frac{\frac{3}{2}\kappa^2 - (g-\tilde{g}) + \sqrt{(\frac{1}{2}\kappa^2 - (g-\tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2}$.

When people die, there is a shift of the wealth level.[22] By assumption, the individual wealth growth rate is higher than the aggregate growth rate, $g - \tilde{g} = p - (\tau - \eta) > 0$. The following proposition shows that $f(x)$ is integrable on $(0, +\infty)$ and therefore is a distribution function. Furthermore the proposition shows that $x f(x)$ is also integrable on $(0, +\infty)$.

Proposition 4. *$\beta_1 < 1$ and $\beta_2 > 2$.*

The normalization condition, equation (20), gives us $\int_0^{x^*} C_1 x^{-\beta_1} dx + \int_{x^*}^{+\infty} C_2 x^{-\beta_2} dx = 1$, and the mean preservation condition, equation (21), gives us $\int_0^{x^*} C_1 x^{1-\beta_1} dx + \int_{x^*}^{+\infty} C_2 x^{1-\beta_2} dx = 1$. Combining these two conditions, we can determine C_1 and

[22] For the dying people whose wealth levels are higher than $x^* = \frac{\tau - \eta}{p}$, the wealth levels of heirs shift down. For the dying people whose wealth levels are lower than $x^* = \frac{\tau - \eta}{p}$, the wealth levels of heirs shift up.

C_2.[23] The special no inheritance case treated here is an alternative way to derive the same distribution that Reed (2001) obtains.[24]

3.1.2. The general case

Now we set $p \geq q \geq 0$ so that some agents may have a bequest motive.

Proposition 5. *The stationary distribution has the following kernel*

$$f(x) = \begin{cases} C_1 x^{-\beta_1} & \text{when } x \leq x^* \\ C_2 x^{-\beta_2} & \text{when } x \geq x^* \end{cases}$$

where β_1 is the smaller root of the characteristic equation

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g - \tilde{g}))\beta + \kappa^2 - p - (g - \tilde{g}) = 0 \tag{24}$$

and β_2 is the larger solution of the characteristic equation

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g - \tilde{g}))\beta + \kappa^2 - p - (g - \tilde{g}) + q\rho^{\beta-1} = 0. \tag{25}$$

The characteristic equations represent the forces that influence wealth inequality in the economy. Note that $g - \tilde{g} = p - q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} - (\tau - \eta)$ is the relative growth rate of individual wealth to the aggregate wealth. Through this term the parameter of preference for bequests, χ, the volatility of risky asset, σ, risk aversion, γ, the capital tax rate, τ, and the estate tax, ζ, influence the Pareto coefficients β_1 and β_2.

The volatility of the price of the risky asset is reflected by the term $\kappa = \frac{\alpha - r}{\gamma\sigma}$. The estate tax rate, ζ, is reflected in the intergenerational transmission term, $\rho = (\frac{p\chi(1-\zeta)}{A\mu})^{\frac{1}{\gamma}}$. The strength of bequest motive χ also influences ρ. Note that the capital tax rate, τ, has an impact on the relative growth rate, but has no impact on the volatility of wealth growth, κ.

---

[23] Of course, we need to check whether the kernel given in Proposition 3 satisfies some boundary conditions of $f(x)$ at x^*. See the discussion of boundary conditions in Section 4.

[24] The simpler techniques, moment generating functions, used by Reed (2001) that avoid PDEs however cannot be applied directly to our model with inheritance. For an even simpler model that still generates the double Pareto distribution without inheritance and without government taxes and transfers, and where all agents are born with the same positive initial wealth, see Appendix 8.13.

The following proposition characterizes the two solutions of the characteristic equations (24) and (25).

Proposition 6. $\beta_1 < 1$ *and* $\beta_2 > 2$.

The proposition implies the integrability of $f(x)$ and $xf(x)$ on $(0, +\infty)$. The integrability of $f(x)$assures that it is a distribution function. But this does not imply that the variance necessarily exists.

From the normalization condition, $\int_0^{x^*} C_1 x^{-\beta_1} dx + \int_{x^*}^{+\infty} C_2 x^{-\beta_2} dx = 1$, and the mean preservation condition, $\int_0^{x^*} C_1 x^{1-\beta_1} dx + \int_{x^*}^{+\infty} C_2 x^{1-\beta_2} dx = 1$, we can determine the terms C_1 and C_2 of the stationary distribution density function, $C_1 = (1 - \frac{1-\beta_2}{2-\beta_2} x^*)(x^*)^{\beta_1 - 2} \frac{(2-\beta_1)(2-\beta_2)(1-\beta_1)}{\beta_2 - \beta_1}$and $C_2 = (1 - \frac{1-\beta_1}{2-\beta_1} x^*)(x^*)^{\beta_2 - 2} \frac{(2-\beta_1)(2-\beta_2)(1-\beta_2)}{\beta_2 - \beta_1}$.[25]

With the explicit form of $f(x)$, we can find the endogenous x^* by equation (22).

$$x^* = \frac{q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta + \tau - \eta + \rho q - \frac{\rho^{\beta_2 - 1}(2-\beta_1)}{\beta_2 - \beta_1} q}{p - \frac{\rho^{\beta_2 - 1}(1-\beta_1)}{\beta_2 - \beta_1} q}$$

Both in cases of no inheritance and inheritance, the relative wealth ratios follow double Pareto distributions. In the Appendix 8.8 we derive the Lorenz curve and the Gini coefficient of the double Pareto distribution.

4. The calibrated economy

We calibrate parameters to simulate our highly stylized and abstract model economy. We explore the numerical relationship between the Gini coefficient and the fundamental parameters. We choose the annual time discount rate, $\theta = 0.03$, the preference parameter for bequests, $\chi = 15$, the coefficient of relative risk aversion, $\gamma = 3$, the annual risk-free interest rate, $r = 1.8\%$,[26] the annual average return on the risky asset, $\alpha = 8.8\%$, which implies that the risk premium of the risky asset is $\alpha - r = 7\%$, and the volatility of return of the risky asset, $\sigma = 0.26$.[27] As in

[25] Of course, we need to check whether the kernel given in Proposition 5 satisfies some boundary conditions of $f(x)$ at x^*. See the discussion of boundary conditions in Section 4.

[26] This number is between the two estimates of mean real interest rates obtained using different data sets by Campbell and Viceira (2002).

[27] For the standard deviation of the return to risky asset, several works have estimaties for home equity, private business, and individual stocks. Case and Shiller (1989)'s estimate of the

Benhabib and Bisin (2006), we pick $p = 0.016$, which implies that agents have an expected working life of $\frac{1}{p} = 62.5$ years. As noted earlier, under the fair insurance market we set the life insurance price $\mu = p = 0.016$. Kopczuk and Lupton (2007), using the panel data of the 1995, 1998, and 2000 waves of the Asset and Health Dynamics Among the Oldest Old (AHEAD) survey, find that roughly 3/4 of the elderly single population has a bequest motive. Setting $\frac{q}{p} = 0.75$ implies that $q = 0.012$. Following Friedman and Carlitz (2005) we calibrate the effective estate tax rate at $\zeta = 0.19$. Since only net government expenditures affect our results and play a role in our analysis, we set government expenditures $\eta = 0$. This leaves the calibration of the capital tax on wealth at τ. Since we have $\eta = 0$ we have to consider τ as net of capital and income taxes that are collected for purposes other than redistribution. We have to set τ such that, together with estate taxes, it will generate the revenue to subsidize transfer payments. These transfers, in discounted value, correspond to the expected government wealth transfer to the young. At about $9 - 10\%$ of GDP in U.S., transfers amount to about a trillion dollars or about $\$9,000$ per household.[28] Discounted over working life at an interest rate of 6.5%, this corresponds to an initial wealth of about $\$130,000$. Thus we set $\tau = 0.004$ so that together with estate taxes, the capital income taxes can finance the redistributive transfers.[29] Of course if we had positive government expenditures, $\eta > 0$, we would have to raise τ to finance not just transfers but

standard deviation of annual percentage change of individual housing prices is close to 15% although events in 2008 may require revising this volatility upwards. Flavin and Yamashita (2002)'s estimation of the standard deviation of the return to house ownership is about 0.14. Moskowitz and Vissing-Jorgensen (2002) set the standard deviation of the return to private equity as 30% per year and claim that it is likely fairly low for a typical single private firm. Using the transaction prices of the 30 stocks in the Dow Jones Industrial Average from January 2, 1993 until May 29, 1998, Andersen et al. (2001) examine the realized return volatility of individual stocks. They find that an annualized standard deviation of the daily return for the typical stock is around 28%. Our calibration means that the volatility of the return of the risky asset is 3 times of the mean of the return of the risky asset. However, in Table 4.6 we also explore the sentivity of the Gini coefficient of the stationary wealth distribution to lower volatility levels.

[28] If we add state subsidies to public education, transfers would be even higher.

[29] From the U.S. 2004 Survey of Consumer Finances (SCF) average household wealth is about \$448,000, so that total household wealth is about 50 trillion. At the calibrated linear estate tax of 19% our model would produce a fraction $q\left(\frac{p\chi}{A\mu}\right)^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta$ of household wealth in estate taxes, amounting to 13.5 billion, a little less than half of the actual estate tax collections. The shortfall is due to the linearity of the estate tax in the model, missing the progressive higher rates on larger estates. In our model, relative to capital taxes collected through τ to finance transfers for new households, estate tax collections are quite small. In the U.S. economy, as a

also these expenditures.

The following table reports the numerical results of the calibrated economy:

	A	ω	ρ	g	$\tilde{g}$	$g-\tilde{g}$	κ
Results	13988.6	0.345168	0.0954114	0.0107745	0.000187993	0.0105865	0.0897436

Table 4.1: Numerical Results

From the simulation results, the portfolio share that the agent invests in the risky asset is $\omega = 0.345168$. The bequest function is[30] $Z(s,t) = (\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}W(s,t) = 0.117792W(s,t)$ so that purchased life insurance is given by $P(s,t) = p(Z(s,t) - W(s,t)) = -0.0141153W(s,t)$. Negative life insurance corresponds precisely to annuities. Given, τ, and ω the average return on wealth, including annuity receipts, is 5.18% and the individual consumption function is $C(s,t) = A^{-\frac{1}{\gamma}}W(s,t) = 0.0415026W(s,t)$.

The growth rate of the wealth of agents is $g = 0.0107745$, while the growth rate of the aggregate economy, $\tilde{g}$, is almost zero. The reason for the low growth rates is simple and can be explained to a rough approximation as follows. We calibrate the riskless rate of return at 1.8% and the mean of the risky rate at 8.8%,to match the 7% premium for risky asset. In the optimal portfolio 34% of wealth is held in the risky asset, so that together with annuity receipts agents receive a return on wealth of 5.18%. The Euler equation relates the consumption growth rate to the difference between the mean return on wealth and the discount rate, multiplied by the intertemporal elasticity of substitution $\gamma^{-1} = \frac{1}{3}$. Thus the small difference between the mean return on wealth and the discount rate necessarily results in low growth rates. In Benhabib and Zhu (2008) we provide a sensitivity analysis by raising both the riskless and the mean risky returns while maintaining the 7% risk premium, and there we obtain higher growth rates of wealth.

We can now plot the stationary distribution of wealth for this calibration in Figure 4.1. It demonstrates the double Pareto distribution.

In the simulated economy, x^*, the threshold level of wealth below which newborns qualify for the government wealth subsidy, is equal to 0.313596, correspond-

fraction of tax revenues or GDP, they are similarly insignificant.

[30]The bequest function, abstracting from inter-vivos transfers, is only for the agents who do leave bequests. Therefore for the pre-tax bequest flow we must multiply the right side by q, so bequest flows are $0.0014W$.

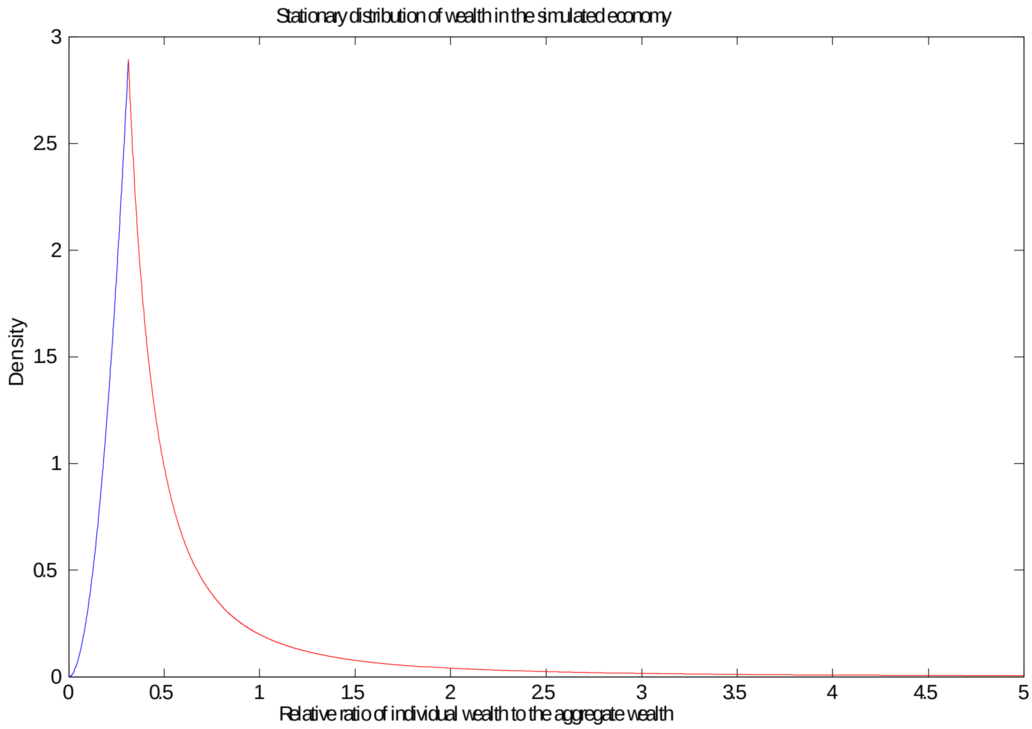

Figure 4.1: Model Data

ing to \$129, 000 or 31% of mean household wealth of \$448, 000 which is normalized to unity in our model. This is very close to the discounted value of lifetime transfers of roughly \$130, 000 that we used in order to calibrate τ above. Of course calibrating a lower (larger) government subsidy and therefore a lower (higher) capital tax will shift the simulated distribution to the left (right), bringing the mode closer to (further from) zero.[31,32] We note that bad luck, that is low returns, can drive the wealth of an agent below the mode x^* during his or her lifetime.

We denote by F^* the percentage of the population whose wealth level is lower

[31] When $x < x^*$, the density of the distribution is governed by C_1 and β_1 (the increasing part of the density in Figure 4.1). When $x > x^*$, the density is governed by C_2 and β_2 (the decreasing part of the density in Figure 4.1).

[32] We show that the boundary conditions at $x = x^*$ are satified for the calibrated model in the Appendix 8.10.

than x^*. Table 4.2 lists the parameters of the wealth distribution in Figure 4.1 above:

	x^*	F^*	β_1	C_1	β_2	C_2
Distribution	0.313596	0.30558	−1.96772	28.3256	2.30647	0.199409

Table 4.2: Parameters of the Wealth Distribution

Empirical distribution

We use the data of net worth of wealth in the U.S. 2004 Survey of Consumer Finances (SCF) to draw the empirical wealth distribution in Figure 4.2. The net worth of households in SCF is obtained by subtracting debt from asset. The assets of households consists of stock, primary residence, real estate investments, and business equity etc.. The empirical wealth distribution displays the two power-law-like tails albeit with some jagged wiggles. Our model replicates the double power-tail distribution in the data, but not the zero and negative wealth levels held by 8.9% of the population. The mode of the empirical distribution is around the zero wealth level, since unlike our model, it excludes the discounted value of government transfers to households. However adding transfer wealth to the data would shift the empirical distribution to the right, closely matching the double Pareto distribution that we obtain in our model.

Besides the Pareto upper tail, the other two prominent features of wealth distribution are the high Gini and the upper skewness. Like many augmented incomplete market models, such as Castaneda, Diaz-Gimenez and Rios-Rull (2003) and De Nardi (2004), our model generates the high Gini, the skewness to the right and the Lorenz curve. The following table shows the Gini coefficient and quintiles of the wealth distribution in U.S. and in our model economy.[33]

Economy	*Gini*	*First*	*Second*	*Third*	*Fourth*	*Fifth*
United States	0.78	−0.39	1.74	5.72	13.43	79.49
Model	0.64	4.07	6.21	8.16	12.24	69.32

Table 4.3: Gini Coefficient and Quintiles of the Wealth Distribution

[33] The data of the U.S. economy in Table 4.3 and Table 4.4 are from Castaneda, Diaz-Gimenez and Rios-Rull (2003) who calculate these tables from 1992 Survey of Consumer Finances (SCF).

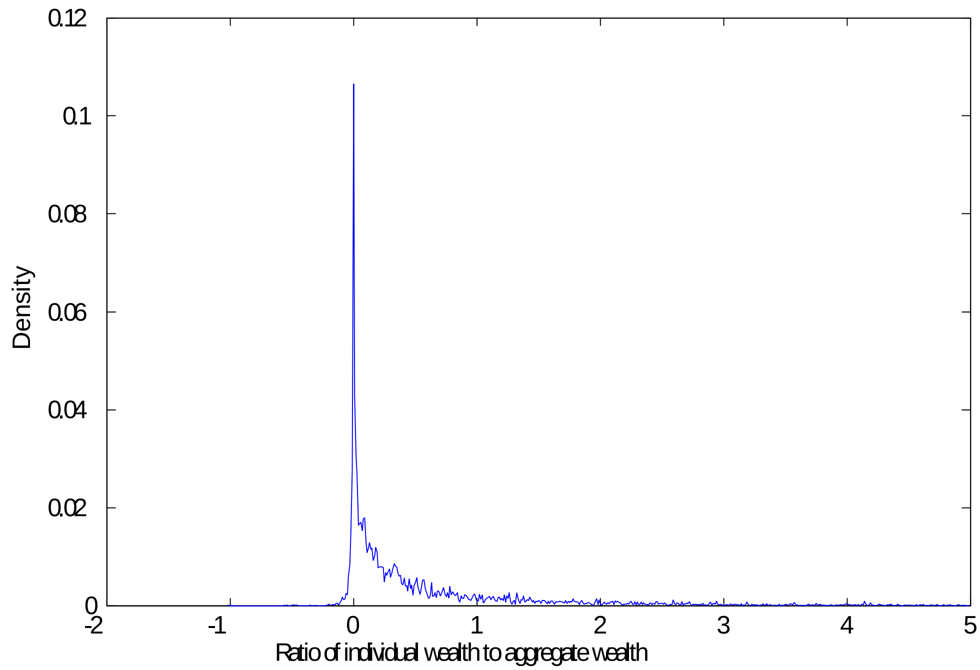

Figure 4.2: U.S. Data

To highlight the skewness to the right and heavy top tail, we further disaggregate the top groups, and compare the percentiles of the wealth distribution for U.S. and the benchmark model economy.

$Economy$	$90th - 95th$	$95th - 99th$	$99th - 100th$
$United\ States$	12.62	23.95	29.55
$Model$	8.84	15.75	34.33

Table 4.4: Top Tail of the Wealth Distribution

Our model overpredicts the wealth share of the upper 1% wealth group. Our prediction is 34.33%, while in the U.S. data the number is 29.55%. Our model underpredicts the wealth share of the $90\% - 95\%$ group and $95\% - 99\%$ group. The predicted shares are, respectively, 8.84% and 15.75% while in the data the corresponding numbers are 12.62% and 23.95%.

For the group of top 20%, the predicted number is 69.32% which is lower than the number of data, 79.49%. The model overpredicts the wealth shares held by the first, second and third quintile groups. Our model predicts a Gini coefficient lower than that of data: Our prediction is 0.64, while in the data Gini is 0.78. Much of the wealth inequality in our model is due to the heavy upper tail.

Our model predicts the heavy tail, but underpredicts the Gini of the wealth distribution as a whole. This in part is because we do not capture the inequalities in wealth induced by disparities in labor earnings. The idiosyncratic investment risk helps us to generate the high inequality and right skewness. We will discuss the role of risky asset in our model to generate the high Gini in section 4.5.

4.1. Wealth distribution conditional on age

Within the same cohort, even though the agents have the same age, they differ in their starting wealth levels at age 0 due to inheritance, and their accumulation of wealth depends on differing realizations of the stochastic rate of return during their life. These two factors generate wealth inequality within an age cohort.[34]

The wealth distribution within the age 0 group has a mass point at x^*, which has a positive probability. All the newborns, including those who do not receive inheritance (if their parents may not have bequest motive) and those whose inheritance is lower than the threshold level x^* (even though their parents have bequest motive), have a starting wealth x^*. For older cohorts, the distribution of wealth also reflects the element of luck coming from stochastic returns.

In Appendix 8.11 we derive the distribution of wealth within age t group, $f_t(x)$, and prove that $f_t(x) \sim x^{-\beta_2}$ as $x \to +\infty$. The distribution of wealth within age groups displays the same limiting Pareto exponent as that of the stationary distribution of the wealth in the whole cross section economy. We plot the wealth distribution conditional on age of our simulated model in Figure 4.3.

To investigate the relationship between wealth inequality within the cohort and the age, we plot the Gini coefficient of the wealth distribution within the age group from ages 1 to age 80.[35] We find that the wealth inequality decreases as age goes up.[36]

[34]For this subsection, we benefited from the discussion with Gianluca Violante.

[35]We do not plot the Gini coefficient of wealth distribution within age 0 (newborn) cohort, even though it is an interesting distribution.

[36]Huggett (1996) also studies the wealth inequality within age groups and notes a U shape: the Gini coeffienct declines to about agee 50 and then picks up again. See also Hendricks (2007)

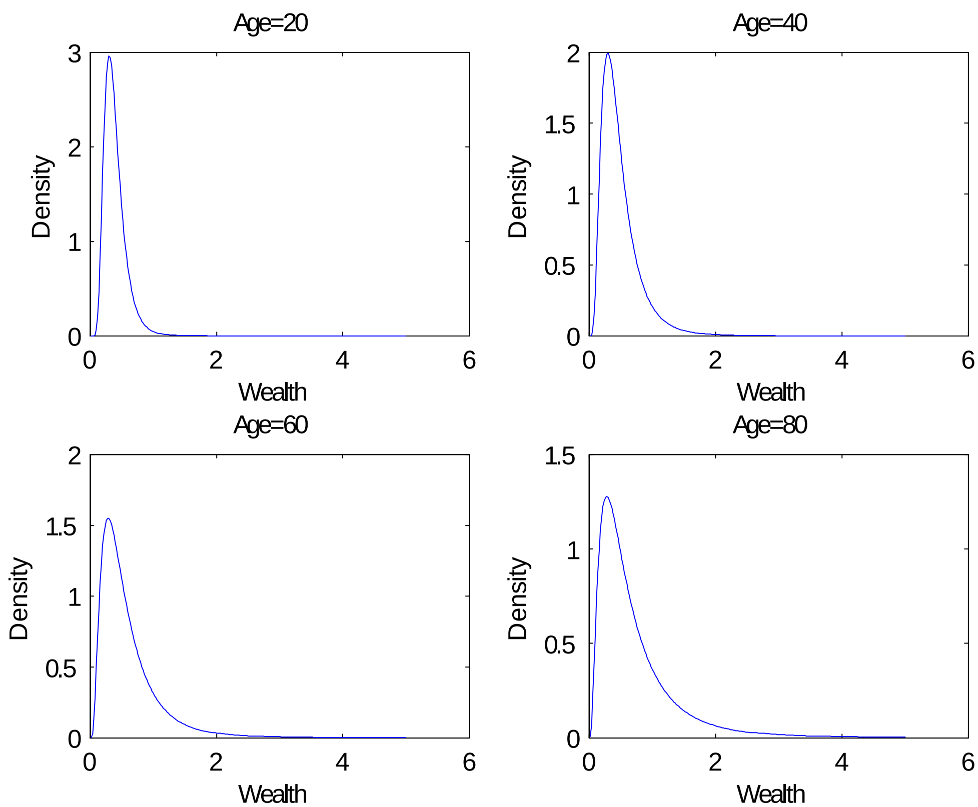

Figure 4.3: Wealth Distribution by Age Cohorts

4.2. Inequality and bequests

Wealth inequality decreases as the parameter of bequest motive, χ, increases. On the one hand, when people have stronger bequest motives and leave higher bequests, the wealth process becomes more persistent across generations. More wealth inequality is inherited. On the other hand, if people purchase more life insurance or buy fewer annuities, the aggregate growth rate of wealth increases and the relative growth rate of individual wealth decreases. The lower relative

for empirical findings that the Gini coefficient declines with age of cohorts.

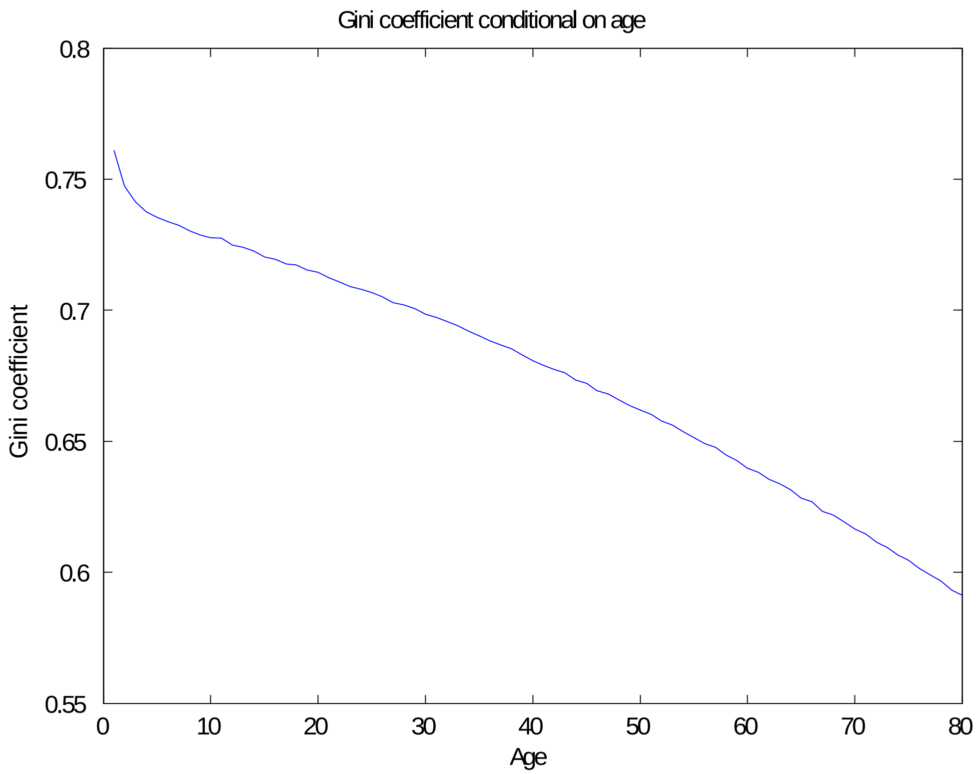

Figure 4.4: Gini by Age Cohorts

growth rate of wealth causes the wealth distribution to become more equal. Our simulated results, for $\sigma = 0.26$, show that the relative growth effect dominates the inheritance effect as in Table 4.5. Therefore a higher bequest motive, χ, implies a lower Gini coefficient.

χ	β_1	β_2	$Gini$
14	-1.97299	2.30493	0.637806
15	-1.96772	2.30647	0.636731
16	-1.9627	2.30795	0.635706
17	-1.95789	2.30936	0.634729

Table 4.5: Inequality and Bequests

4.3. Inequality and the volatility of the risky asset

Wealth inequality decreases as the volatility of the risky asset increases. This counter-intuitive result is mainly due to the endogenous choice of the risky asset. When the volatility of risky asset increases, people hold a smaller share of risky asset, and in effect the volatility of the overall portfolio declines. The net effect of the portfolio reallocation and lower volatility of the wealth portfolio in turn lowers inequality. Setting $\chi = 15$, we show the relationship between inequality and the volatility of the risky asset in Table 4.6.

σ	β_1	β_2	$Gini$
0.2	-0.820429	2.25666	0.690649
0.21	-0.992948	2.26628	0.67958
0.22	-1.17291	2.27534	0.669451
0.23	-1.36038	2.28385	0.66018
0.24	-1.55528	2.29185	0.651688
0.25	-1.75772	2.29942	0.643865
0.26	-1.96772	2.30647	0.636731

Table 4.6: Inequality and the Volatility of the Risky Asset

4.4. Inequality and risk aversion

Risk aversion affects wealth inequality through all the three channels: growth, volatility, and inheritance. We set $\chi = 15$ and $\sigma = 0.26$ to simulate the economy for $\gamma = 2$, $\gamma = 2.5$ and $\gamma = 3$. The Gini coefficient decreases with the increase of the coefficient of relative risk aversion in Table 4.7.

γ	β_1	β_2	$Gini$
2	-0.366113	2.24326	0.714494
2.5	-1.09162	2.28013	0.667223
3	-1.96772	2.30647	0.636731

Table 4.7: Inequality and Risk Aversion

4.5. Inheritance, stochastic return, and the age effect

We now explore the roles of inheritance, age, and stochastic rates of the capital return on wealth inequality. We investigate special cases to isolate the effect of each of these factors. To identify the effect of these three factors we construct two schemes.

In scheme I, we first eliminate the investment opportunity of agents in the risky asset. Then our model reduces to that of Benhabib and Bisin (2006).[37] After we close the channel of stochastic returns, we find that the Gini coefficient of the economy decreases. Comparing the Gini coefficient of this special economy with the general case, we isolate the effect of the luck on wealth inequality. We then eliminate the bequest motive by setting $\chi = 0$ for all the agents, and study an economy without luck or inheritance.

In scheme II, we first limit the age effect by setting the wealth growth rate of the agent relative to the growth rate of the economy to be as low as possible. Comparing the Gini coefficient of this special economy with that of the general case, we estimate the age effect. We then close the inheritance channel while keeping the relative growth rate as low as possible to identify the effect of inheritance on wealth inequality.

4.5.1. Scheme I

Stochastic rates of capital return We disentangle the contribution of stochastic rates of capital return to wealth inequality by shutting down the investment opportunity in the risky asset. When the agents have no access to the investment opportunity of the risky asset, our model reduces to that of Benhabib and Bisin (2006).

In order to maintain the growth rate of the economy, we set the annual risk-free interest rate here as the mean return of the risky asset in our benchmark economy, $r = 8.8\%$. For the calibration of other parameters as in our benchmark model, the Gini coefficient without risky asset is 0.439633. Comparing this Gini coefficient with that of the economy with risky asset, 0.636731, we find that the Gini coefficient decreases by about 31% when we close the investment opportunity for the risky asset. We can view this number as the contribution of luck to wealth inequality.

Intergenerational transmission and age effect After we close the channel for stochastic returns our model reduces to that of Benhabib and Bisin (2006). In

[37] To eliminate the investment opportunity of agents in the risky asset is not to decrease σ to 0. Setting $\sigma = 0$ would induce arbitrage in the model, since, then there would be two risk-free assets in the model and they had the different rates of return.

order to close the intergenerational transmission channel, we now also set $\chi = 0$. Note that in the calibration of our benchmark model, $\chi = 15$.

The wealth process is more persistent across generations with inheritance than without. When people have bequest motives and leave higher bequests, more of the wealth inequality is inherited. On the other hand if people leave bequests, they receive smaller annuities and consume less. The growth rate of the aggregate economy increases because the initial wealth of agents at birth is higher due to higher bequests. As shown in Proposition 1 however, for our CRRA preferences the individual agent's growth rate is not influenced by the bequest motive parameter, χ. When the aggregate economy grows faster, the lucky agents who earn high returns relative to the economy will not break away as easily and leave others behind by as much. The lower relative wealth growth rate, $g - \tilde{g}$, therefore causes the wealth distribution to become more equal.

The simulated results in Table 4.8 show that in fact the growth effect dominates the inheritance effect. Therefore we can have a higher Gini coefficient after we close the intergenerational transmission channel. Surprisingly, a stronger bequest motive and higher inheritance rates may decrease wealth inequality.

χ	g	$\tilde{g}$	$g - \tilde{g}$	β_2	x^*	$Gini$
0	0.018	0.006	0.012	2.33333	0.25	0.6
15	0.018	0.00867146	0.00932854	2.63731	0.389242	0.439633

Table 4.8: The Economies with and without Inheritance

4.5.2. Scheme II

Age effect In this experiment we allow the stochastic return to remain and we pick $\tau = 0.0107$ so that the relative growth rate is $g - \tilde{g} \approx \frac{1}{2}\kappa^2$. Note that with stochastic returns $\frac{1}{2}\kappa^2$ is the lowest bound for $g - \tilde{g}$ that yields a non-degenerate stationary distribution. In this economy, the Gini coefficient is 0.402667 whereas in our benchmark model economy the Gini coefficient is 0.636731. After we close the age effect, the Gini coefficient decreases by about 37%. We can view this number as the lower bound for the contribution of the age effect to wealth inequality.

Inheritance We now also set $\chi = 0$ in order to close the inheritance channel and we set $\tau = 0.011973$ so that the economy-wide relative growth rate is $g - \tilde{g} = \frac{1}{2}\kappa^2$ so that the age effect is also minimized. In this economy the Gini coefficient becomes 0.401526. Compared to the previous case with inheritance but with no age effect, the Gini is only marginally lower: inheritance has a small effect on inequality.

5. Redistributive policies

We now investigate the impact of redistributive government policy on wealth inequality and aggregate welfare in our calibrated model.

5.1. Taxes and wealth inequality

Capital taxes and and to a lesser extent estate taxes influence wealth inequality through the relative growth rate of wealth, through their effects on the intergenerational transmission of wealth, and through their redistributive effects. In contrast to models without idiosyncratic returns on capital, in economies with stochastic capital returns a streak of lucky runs can generate significant wealth accumulation with heavy tails. Taxes on capital and estates can effectively dampen such accumulation, especially in the heavy right tails of the wealth distribution. We use our benchmark calibration and vary the capital and estate taxes τ and ζ to illustrate their effect on the Gini coefficient. in the table below:

$\zeta \backslash \tau$	0.004	0.005	0.006	0.007	0.008	0.009	0.0099
0	0.650234	0.594185	0.546586	0.506215	0.472115	0.443518	0.421965
0.1	0.643547	0.588788	0.542218	0.502678	0.469257	0.441223	0.420096
0.2	0.635948	0.582625	0.537207	0.498605	0.465956	0.438565	0.417929
0.3	0.627151	0.575452	0.531349	0.493824	0.462069	0.435427	0.415367
0.4	0.616726	0.566905	0.524334	0.488076	0.457382	0.431636	0.412267
0.5	0.603986	0.556398	0.515668	0.480947	0.451551	0.426912	0.408404
0.6	0.587734	0.542913	0.504491	0.471718	0.443984	0.420775	0.408404

Table 4.9: Gini on Taxes

The higher is the capital tax, τ, the lower is the relative growth rate, $g - \tilde{g}$. A lower $g - \tilde{g}$ makes the wealth distribution more equal. The higher is the capital tax, τ, the lower is the intergenerational transmission parameter, ρ. Lower ρ implies that the wealth process between two consecutive generations becomes less persistent and the wealth distribution becomes more equal. Furthermore when τ

increases, x^* increases, implying more redistribution. Therefore, on all accounts, a higher τ implies a lower Gini coefficient.

The higher the estate tax, ζ, the lower is the relative growth rate, $g - \tilde{g}$. At the same time the higher the estate tax, ζ, the lower is the intergenerational transmission parameter, ρ. The effect of ζ on $g - \tilde{g}$ and ρ can be obtained by analyzing equations (10) and (14). However, the higher is the estate tax, ζ, the higher is the bequest that the agent leaves to his children to partly offset the higher estate taxes. Furthermore x^* increases as ζ increases. The overall effect of a higher ζ is a lower Gini coefficient.[38]

These results on the effects of taxes are consistent with our intuition: redistributive policies tend to reduce wealth inequality.

5.2. Taxes and welfare

Government policies influence both the individual utilities and the wealth distribution in the economy. As in Benhabib and Bisin (2006), we take aggregate welfare to be the integral of the individual utilities with respect to the cross-sectional wealth distribution. Thus we add the utilities of those currently alive. We compute the aggregate welfare of the economy and find the optimal government fiscal policies.

There are two kinds of people in the economy. People with a bequest motive account for $\frac{q}{p}$ in the population. And people without a bequest motive account for $1-\frac{q}{p}$ in the population. In section8.12 of the Appendix we the derive the value function of people with bequest motives:

$$U(w) = \frac{1}{1-\gamma}\left(\frac{\theta + p - (1-\gamma)(r - \tau + \mu + \frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1 + (p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma} w^{1-\gamma}$$

and the value function of people with no bequest motive:

$$U_0(w) = \frac{1}{1-\gamma}\left(\frac{\theta + p - (1-\gamma)(r - \tau + \mu + \frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma}\right)^{-\gamma} w^{1-\gamma}$$

[38] We can also calculate the minimum Gini coefficient for combinations of the capital tax, τ and the estate tax, ζ, in a parameter region such that $g > 0$, $\tilde{g} > 0$ and $g - \tilde{g} - \frac{1}{2}\kappa^2 \geq 0$. The minimum value of the Gini coefficient is 0.4020, obtained for $\zeta = 0.95$ and $\tau = 0.0047$, where both ζ and τ are on the boundary of the parameter region and cannot be further increased.

The aggregate welfare of the economy is the weighted sum of the individual utilities with weights according to the cross-sectional wealth distribution of the two groups of agents.

$$\Omega(\tau,\zeta) = \frac{q}{p}\int_0^{+\infty} U(w)f(w)dw + \frac{p-q}{p}\int_0^{+\infty} U_0(w)f(w)dw$$

In section 8.12 of the Appendix we derive the aggregate welfare function:

$$\begin{aligned}\Omega(\tau,\zeta) \quad = \quad & [\frac{q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1+(p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma} \\ & +\frac{p-q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma}\right)^{-\gamma}]\times \\ & [\frac{C_1}{2-\gamma-\beta_1}(x^*)^{2-\gamma-\beta_1}-\frac{C_2}{2-\gamma-\beta_2}(x^*)^{2-\gamma-\beta_2}]\end{aligned}$$

Note that β_1 and β_2 are functions of capital tax rate, τ, and estate tax rate, ζ. Note also that β_2 has a non-linear relationship with τ, and ζ. In section 8.12 of the Appendix, we show that the aggregate welfare function is well-defined when $\beta_1 < -1$.

For the set of combinations of the capital tax, τ and the estate tax, ζ, such that $g > 0$, $\tilde{g} > 0$, $g - \tilde{g} - \frac{1}{2}\kappa^2 \geq 0$ and $\beta_1 < -1$, we calculate the aggregate welfare for our calibrated economy. The maximum of the welfare function is obtained for $\zeta = 0.18$ and $\tau = 0.0063$. The estate tax is very close to our calibration in the benchmark economy, but maximizing social welfare requires a high capital tax because our social welfare function, weighting only generations currently alive, puts a high emphasis on equality. Setting $\tau = 0.0063$ decreases the relative growth rate so that the Gini coefficient for the tax rates maximizing social welfare is now 0.5260.

Of course a different welfare specification that puts more weight on future generations by including the utilities of those not yet born would put a higher weight on growth, and shift the optimal taxes from τ which affects individual growth rates to ζ which does not.

6. A lump-sum subsidy policy

Previously we discussed a welfare policy for which only those newborns whose inheritance is lower than x^* receive a subsidy. Here we discuss the alternative policy where all the newborns receive a subsidy.[39] Note that this subsidy, after adjusting the redistributive taxes for a balanced government budget, can also be interpreted as the discounted value of lifetime labor earnings received by all newborns.

As we point out in section 2, the total subsidies to the newborns do not depend on how the subsidies are distributed to the newborns. Equation (8) gives us the total subsidies to the newborns. Under the lump-sum subsidy policy, each newborn with or without inheritance receives the same subsidy, which we denote by $b(t)$.

$$b(t) = \frac{q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta + \tau - \eta}{p} W(t) \tag{26}$$

Changing the subsidy policy from that in section 3 to the lump-sum subsidy policy only influences the transmission of wealth between two consecutive generations. The individual wealth accumulation equation and the aggregate wealth growth rate are the same as those in section 2.

Let

$$x^* = \frac{q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta + \tau - \eta}{p}$$

Note that by equation (26) newborns without any inheritance are injected into the economy with the wealth level $x^*W(t)$, and that x^* here is different from that in equation (22).

We still investigate the stochastic process of the ratio of individual wealth to aggregate wealth. Even though the transmission of wealth between two consecutive generations here is different from that in section 3, the stochastic process during the agent's lifetime is the same as that in section 3.

[39] See Huggett (1996) for a calibrated model where accidental bequests are distributed equally to everyone, not just newborns. De Nardi (2004) also has some specifications of calibrated models where accidental bequests are equally distributed to the population.

The stationary distribution $f(x)$ now satisfies

$$\frac{1}{2}\kappa^2x^2f''(x)+(2\kappa^2-(g-\tilde{g}))xf'(x)+(\kappa^2-(g-\tilde{g})-p)f(x)+qf(\frac{x-x^*}{\rho})\frac{1}{\rho}=0 \text{ when } x>\frac{x^*}{1-\rho} \tag{27}$$

Equation (27) differs from equation (18) only by x^* in the last term: $qf(\frac{x-x^*}{\rho})\frac{1}{\rho}$ as opposed to $qf(\frac{x}{\rho})\frac{1}{\rho}$. For large x, the influence of the shift term, x^*, can be ignored in $qf(\frac{x-x^*}{\rho})\frac{1}{\rho}$. Therefore the stationary wealth distribution under the lump-sum subsidy policy has an approximately Pareto upper tail, which, for large x, admits the same Pareto exponent as that of the general case in section 3.

Proposition 7. *The stationary distribution $f(x) \sim x^{-\beta_2}$ as $x \to +\infty$, where β_2 is the larger solution of the characteristic equation*

$$\frac{\kappa^2}{2}\beta^2-(\frac{3}{2}\kappa^2-(g-\tilde{g}))\beta+\kappa^2-p-(g-\tilde{g})+q\rho^{\beta-1}=0. \tag{28}$$

Equation (28) of Proposition 7 is the same as equation (25) of Proposition 5. This verifies our intuition that, as wealth goes to infinity, the stationary distribution under the lump-sum subsidy policy approaches to a Pareto distribution with the same exponent as that in Proposition 5.[40]

7. Conclusions

We incorporate investment risk into a heterogeneous agent model to replicate the distribution of U.S. wealth and the three of it's prominent features: a high Gini coefficient, skewness to the right, and Pareto tails. Stochastic returns, coupled with a birth and death process and with government subsidies, despite the complications introduced by inheritance, generates a stationary distribution of wealth. We analytically solve our model and obtain a double Pareto distribution that provides a close match to the data.

There are three basic forces in our model that cause wealth inequality and skewness: stochastic rates of return, age, and inheritance. Calibrating our model, we estimate that luck, or the stochastic returns contribute about 31% to wealth

[40] Though the above approximation analysis is intuitive, a formal proof using Kesten processes is available from the authors on request.

inequality in terms of the Gini coefficient, and wealth accumulation while aging contributes about 37%.

Finally, we show that government redistributive policies have important consequences for wealth inequality and welfare through their effects on the relative growth rates of wealth, on bequests, and on the size of government subsidies. In particular, because accumulation of wealth is subject to stochastic idiosyncratic returns, we find that capital taxes and to some extent estate taxes can significantly affect wealth inequality and the Gini coefficient.

References

[1] Abel, A. (1985), "Precautionary Saving and Accidental Bequests", *American Economic Review*, Vol. 75, 777-791.

[2] Aghion, P., J. Reenen and L. Zingales (2007), "Innovation and Institutional Ownership", AEAweb.

[3] Aiyagari, S. R. (1994), "Uninsured Idiosyncratic Risk and Aggregate Saving", *Quarterly Journal of Economics*, Vol. 109, 659-684.

[4] Andersen, T., T. Bollerslev, F. Diebold and H. Ebens (2001), "The Distribution of Realized Stock Return Volatility", *Journal of Financial Economics*, Vol. 61, 43-76.

[5] Angeletos, G. (2007), "Uninsured Idiosyncratic Investment Risk and Aggregate Saving", *Review of Economic Dynamics*, Vol. 10, 1-30.

[6] Angeletos, G. and L. Calvet (2006), "Idiosyncratic Production Risk, Growth and the Business Cycle", *Journal of Monetary Economics*, Vol. 53, 1095-1115.

[7] Benhabib, J. and A. Bisin (2006), "The Distribution of Wealth and Redistributive Policies", Manuscript, New York University.

[8] Benhabib, J. and S. Zhu (2008), "Age, Luck, and Inheritance", NBER Working Paper 14128.

[9] Bertaut, C. and M. Starr-McCluer (2002), "Household Portfolios in the United States", in L. Guiso, M. Haliassos, and T. Jappelli, Editor, *Household Portfolios*, MIT Press, Cambridge, MA.

[10] Bitler, M., T. Moskowitz and A. Vissing-Jorgensen (2005), "Testing Agency Theory with Entrepreneur Effort and Wealth", *Journal of Finance*, Vol. 60, 539-576.

[11] Blanchard, O. J. (1985), "Debt, Deficits, and Finite Horizons", *Journal of Political Economy*, Vol. 93, 223-247.

[12] Burris, V. (2000), "The Myth of Old Money Liberalism: The Politics of the "Forbes" 400 Richest Americans", *Social Problems*, Vol. 47, 360-378.

[13] Cagetti, M. and M. De Nardi (2006), "Entrepreneurship, Frictions, and Wealth", *Journal of Political Economy*, Vol. 114, 835-870.

[14] Campbell, J. and L. Viceira (2002), "Strategic Asset Allocation: Portfolio Choice for Long-Term Investors", Oxford University Press Inc., New York.

[15] Case, K. and R. Shiller (1989), "The Efficiency of the Market for Single-Family Homes", *American Economic Review*, Vol. 79, 125-137.

[16] Castaneda, A., J. Diaz-Gimenez and J.V. Rios-Rull (2003), "Accounting for the U.S. Earnings and Wealth Inequality", *Journal of Political Economy*, Vol. 111, 818-857.

[17] Champernowne, D. (1953), "A Model of Income Distribution", *Economic Journal*, Vol. 63, 318-351.

[18] Covas, F. (2006), "Uninsured Idiosyncratic Production Risk with Borrowing Constraints", *Journal of Economic Dynamics and Control,* Vol. 30, 2167-2190.

[19] De Nardi, M. (2004), "Wealth Inequality and Intergenerational Links", *Review of Economic Studies*, Vol. 71, 743-768.

[20] Elwood, Paul, S. M. Miller, Marc Bayard, Tara Watson, Charles Collins and Chris Hartman (1997), "Born on Third Base: The Sources of Wealth of the 1996 Forbes 400," Boston: Unified for a Fair Economy. See also http://www.faireconomy.org/press/archive/Pre_1999/forbes_400_study_1997.html

[21] Flavin, M. and T. Yamashita (2002), "Owner-Occupied Housing and the Composition of the Household Portfolio", *American Economic Review*, Vol. 92, 345-362.

[22] Friedman, J. and R. Carlitz (2005), "Estate Tax Reform Could Raise Much-Needed Revenue, Center on Budget and Policy Priorities", Washington, available at http://www.cbpp.org/3-16-05tax.htm

[23] Fuster, L. (2000), "Capital Accumulation in an Economy with Dynasties and Uncertain Lifetimes", *Review of Economic Dynamics*, Vol. 3, 650-674.

[24] Gabaix, X. (1999), "Zipf's Law for Cities: An Explanation", *Quarterly Journal of Economics*, Vol. 114, 739-767.

[25] Gale, W. G. and J. K. Scholz (1994): "Intergenerational Transfers and the Accumulation of Wealth", *Journal of Economic Perspectives*, Vol. 8, 145-160.

[26] Gastwirth, J. L. (1971), "A General Definition of the Lorenz Curve", *Econometrica*, Vol. 39, 1037-1039.

[27] Hendricks, L. (2007), "How important is discount rate heterogeneity for wealth inequality?", *Journal of Economic Dynamics and Control,* Vol. 31, 3042-3068.

[28] Huberman, B. and L. Adamic (1999), "Growth Dynamics of the World-Wide Web", *Nature*, Vol. 399, 131.

[29] Huggett, M. (1993), "The Risk-Free Rate in Heterogeneous-Agent Incomplete Insurance Economies", *Journal of Economic Dynamics and Control*, Vol. 17, 953-969.

[30] Huggett, M. (1996), "Wealth Distribution in Life-cycle Economies", *Journal of Monetary Economics*, Vol. 38, 469-494.

[31] Kamien, M. and N. Schwartz (1991), "Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management", Second Edition, Elsevier Science Publishing Co., New York.

[32] Karlin, S. and H. Taylor (1981), "A Second Course in Stochastic Process", Academic Press, San Diego.

[33] Klass, O.; O. Biham; M. Levy; O. Malcai and S. Solomon (2006), "The Forbes 400 and the Pareto Wealth Distribution", *Economics Letters*, Vol. 90, 290-295.

[34] Kopczuk, W. and J. Lupton (2007), "To Leave or Not to Leave: The Distribution of Bequest Motives", *Review of Economic Studies*, Vol. 74, 207-235.

[35] Kotlikoff, L. and L. Summers (1981), "The Role of Intergenerational Transfers in Aggregate Capital Accumulation", *Journal of Political Economy*, Vol. 89, 706-732.

[36] Krusell, P. and A. Smith (1998), "Income and Wealth Heterogeneity in the Macroeconomy", *Journal of Political Economy*, Vol. 106, 867-896.

[37] Levy, M. (2003), "Are Rich People Smarter?", *Journal of Economic Theory*, Vol. 110, 42-64.

[38] Merton, R. (1971), "Optimum Consumption and Portfolio Rules in A Continuous-Time Model", *Journal of Economic Theory*, Vol. 3, 373-413.

[39] Merton, R. (1992), "Continuous-Time Finance", Blackwell Publishers, Cambridge, MA.

[40] Meyn, S. P. and R. L. Tweedie (1993), "Markov Chains and Stochastic Stability", Springer, Berlin.

[41] Mizuno, T., M. Katori, H. Takayasu and M. Takayasu (2002), "Statistical Laws in the Income of Japanese Companies", in H. Takayasu, Editor, *Empirical Science of Financial Fluctuations: The Advent of Econophysics*, Springer, Tokyo.

[42] Moskowitz T. and A. Vissing-Jorgensen (2002), "The Returns to Entrepreneurial Investment: A Private Equity Premium Puzzle?", *American Economic Review*, Vol. 92, 745-778.

[43] Nygard, F. and A. Sandstrom (1981), "Measuring Income Inequality", Almqvist & Wiksell International, Stockholm, Sweden.

[44] Oksendal, B. (1995), "Stochastic Differential Equations", Fourth Edition, Springer, Berlin.

[45] Quadrini, V. (2000), "Entrepreneurship, Saving, and Social Mobility", *Review of Economic Dynamics*, Vol. 3, 1-40.

[46] Reed, W. (2001), "The Pareto, Zipf and Other Power Laws", *Economics Letters*, Vol. 74, 15-19.

[47] Reed, W. (2003), "The Pareto Law of Incomes-an Explanation and an Extension", *Physica A*, Vol. 319, 469-486.

[48] Richard, S. (1975), "Optimal Consumption, Portfolio and Life Insurance Rules for An Uncertain Lived Individual in a Continuous Time Model", *Journal of Financial Economics*, Vol. 2, 187-203.

[49] Ross, S. M. (1996), "Stochastic Processes", Second Edition, John Wiley & Sons, Inc., New York.

[50] Schechtman, J. and V. L. Escudero (1977), "Some Results on "An Income Fluctuation Problem"", *Journal of Economic Theory*, Vol. 16, 151-166.

[51] Vaughan, R. (1988), "Distributional Aspects of the Life Cycle Theory of Saving," in Kessler, D., and Masson, A. (eds.), *Modelling the Accumulation and Distribution of Wealth*, Oxford University Press, New York.

[52] Wang, N. (2007), "An Equilibrium Model of Wealth Distribution," *Journal of Monetary Economics*, Vol. 54, 1882-1904

[53] Wold, H. and P. Whittle (1957), "A Model Explaining the Pareto Distribution of Wealth", *Econometrica*, Vol. 25, 591-595.

[54] Wolff, E. (2004), "Changes in Household Wealth in the 1980s and 1990s in the U.S." in Edward N. Wolff, Editor, *International Perspectives on Household Wealth*, Elgar Publishing Ltd., forthcoming.

[55] Yaari, M. (1965), "Uncertain Lifetime, Life Insurance, and the Theory of the Consumer", *Review of Economic Studies*, Vol. 32, 137-150.

[56] Yosida, K. (1971), *Functional Analysis*, Third Edition, Springer, New York.

8. Appendix

8.1. Proof of Proposition 1.

Proof: Let $J(W(s,t))$ be the optimal value of the agent with wealth $W(s,t)$. Following Merton (1992) and Kamien and Schwartz (1991), we set up the Hamilton-

Jacobi-Bellman equation of the maximization problem

$$\begin{aligned}
&(\theta + p)J(W(s,t)) \\
= \;& \max_{C,\omega,P}\{\frac{C(s,t)^{1-\gamma}}{1-\gamma} + p\chi\frac{((1-\zeta)Z(s,t))^{1-\gamma}}{1-\gamma} \\
&+J_W(W(s,t))[(r-\tau)W(s,t) + (\alpha - r)\omega(s,t)W(s,t) - C(s,t) - P(s,t)] \\
&+\frac{1}{2}J_{WW}(W(s,t))\sigma^2\omega^2(s,t)W^2(s,t)\}
\end{aligned}$$

Using the relationship

$$Z(s,t) = W(s,t) + \frac{P(s,t)}{\mu}$$

we find the first order conditions:

$$C(s,t)^{-\gamma} = J_W$$

$$p\chi(1-\zeta)^{1-\gamma}Z(s,t)^{-\gamma}\frac{1}{\mu} = J_W$$

$$(\alpha - r)J_W W(s,t) = -J_{WW}\sigma^2\omega(s,t)W^2(s,t)$$

We guess the value function

$$J(W(s,t)) = \frac{A}{1-\gamma}W(s,t)^{1-\gamma}$$

where A is the undetermined constant. Then we find the expressions of $C(s,t)$, $Z(s,t)$, $P(s,t)$, and $\omega(s,t)$ from the first order conditions

$$C(s,t) = A^{-\frac{1}{\gamma}}W(s,t)$$

$$Z(s,t) = (\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}W(s,t)$$

$$P(s,t) = (\mu^{1-\frac{1}{\gamma}}(\frac{p\chi}{A})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} - \mu)W(s,t)$$

$$\omega(s,t) = \frac{\alpha - r}{\gamma\sigma^2}$$

Plugging these equations into the Hamilton-Jacobi-Bellman equation, we can determine the constant A:

$$A = \left(\frac{\theta + p - (1-\gamma)(r - \tau + \mu + \frac{(\alpha - r)^2}{2\gamma\sigma^2})}{\gamma(1 + (p\chi)^{\frac{1}{\gamma}} \mu^{\frac{\gamma-1}{\gamma}} (1-\zeta)^{\frac{1-\gamma}{\gamma}})} \right)^{-\gamma}$$

From the budget constraint we obtain the wealth accumulation equation

$$dW(s,t) = [\frac{r - \tau + \mu - \theta - p}{\gamma} + \frac{1+\gamma}{2\gamma} \frac{(\alpha - r)^2}{\gamma\sigma^2}] W(s,t)dt + \frac{\alpha - r}{\gamma\sigma} W(s,t) dB(s,t).$$

8.2. Derivation of the forward Kolmogorov equation

Following Ross (1996), we heuristically derive the forward Kolmogorov equations (15) and (16).

Let $f(x,t;y)$ be the probability density of $X(t)$, given $X(0) = y$. By the Markovian property of the process

$$\Pr\{X(t) = x | X(0) = y, X(t - \Delta t) = a\} = \Pr\{X(\Delta t) = x | X(0) = a\} = \Pr\{DB = \log x - \log a\}$$

where DB is a normal distribution with mean of $(g - \tilde{g} - \frac{1}{2}\kappa^2)\Delta t$, and variance of $\kappa^2 \Delta t$. Let $f_{DB}(\cdot)$ be the density function of the normal distribution with mean of $(g - \tilde{g} - \frac{1}{2}\kappa^2)\Delta t$, and variance of $\kappa^2 \Delta t$.

When $x > x^*$, we have

$$\begin{aligned}
f(x,t;y) &= (1 - p\Delta t) \int_0^{+\infty} f(a, t - \Delta t; y) f_{DB}(\log x - \log a) \frac{1}{x} da + q\Delta t \cdot f(\frac{x}{\rho}, t - \Delta t; y) \frac{1}{\rho} \\
&= (1 - p\Delta t) \int_0^{+\infty} [f(x,t;y) + (a - x) \frac{\partial}{\partial x} f(x,t;y) - \Delta t \frac{\partial}{\partial t} f(x,t;y) \\
&\quad + \frac{(a-x)^2}{2} \frac{\partial^2}{\partial x^2} f(x,t;y)] f_{DB}(\log x - \log a) \frac{a}{x} d\log a \\
&\quad + q\Delta t \cdot f(\frac{x}{\rho}, t - \Delta t; y) \frac{1}{\rho} + o(\Delta t) \\
&= (1 - p\Delta t)(1 - (g - \tilde{g})\Delta t + \kappa^2 \Delta t) f(x,t;y) + (1 - p\Delta t)(2\kappa^2 - (g - \tilde{g}))\Delta t x \frac{\partial}{\partial x} f(x,t;y) \\
&\quad - (1 - p\Delta t)\Delta t \frac{\partial}{\partial t} f(x,t;y) + (1 - p\Delta t) \frac{\kappa^2}{2} x^2 \Delta t \frac{\partial^2}{\partial x^2} f(x,t;y) + q\Delta t \cdot f(\frac{x}{\rho}, t - \Delta t; y) \frac{1}{\rho} \\
&\quad + o(\Delta t)
\end{aligned}$$

where we use the Taylor expansion in the second and third equality. Divide by Δt on both sides and let $\Delta t \to 0$

$$\begin{aligned} \frac{\partial}{\partial t} f(x,t;y) &= (\kappa^2 - p - (g-\tilde{g}))f(x,t;y) + (2\kappa^2 - (g-\tilde{g}))x\frac{\partial}{\partial x} f(x,t;y) \\ &\quad + \frac{1}{2}\kappa^2 x^2 \frac{\partial^2}{\partial x^2} f(x,t;y) + qf(\frac{x}{\rho},t;y)\frac{1}{\rho}, \quad x > x^* \end{aligned}$$

Then

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\kappa^2 x^2 f(x,t)) - \frac{\partial}{\partial x}((g-\tilde{g})xf(x,t)) - pf(x,t) + qf(\frac{x}{\rho},t)\frac{1}{\rho}, \; x > x^*.$$

Similarly, we have

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\kappa^2 x^2 f(x,t)) - \frac{\partial}{\partial x}((g-\tilde{g})xf(x,t)) - pf(x,t), x < x^*$$

8.3. Proof of Proposition 2

Proof: Following Benhabib and Bisin (2006), we use the Embedded Markov Chain method to establish the ergodicity of the wealth distribution of newborns, which then implies the ergodicity of the wealth distribution of the whole economy.

As in Karlin and Taylor (1981), we construct the embedded Markov Chain from the continuous time process, $X(\cdot,t)$. Let t_1, t_2, t_3, $\cdots$, denote the birthday of the generation 1, generation 2, generation 3,$\cdots$. By our notation, their starting wealth is $X(t_1,t_1)$, $X(t_2,t_2)$, $X(t_3,t_3)$, $\cdots$.

Let

$$\Phi_0 = X(\cdot,0), \quad \Phi_n = X(t_n,t_n), \quad n = 1,2,3,\cdots$$

Thus Φ_n is the newborns's starting wealth. Note that the state space for Φ_n is $S = [x^*, +\infty)$ by the subsidy policy of the government. The stochastic process Φ_n is a Markov Chain. Note that the duration of the life follows an exponential distribution with parameter p. When the agent is alive, his wealth follows a Geometrical Brownian Motion as in equation (12) in the text. Given the government subsidy policy for the newborns, the transition probability of Φ_n is

$$\begin{aligned} P(\Phi_{n+1} &= x^* \mid \Phi_n = x) = \frac{p-q}{p} \\ &+ \int_0^{x^*}\int_0^{+\infty} qe^{-pt}\frac{1}{y}\frac{1}{\sqrt{2\pi t\kappa^2}}\exp[-\frac{(\log(\frac{y}{\rho}) - \log(x) - (g-\tilde{g}-\frac{1}{2}\kappa^2)t)^2}{2t\kappa^2}]dtdy \end{aligned}$$

and

$$\begin{aligned} P(\Phi_{n+1} &= y \mid \Phi_n = x) \\ &= \int_0^{+\infty} qe^{-pt}\frac{1}{y}\frac{1}{\sqrt{2\pi t\kappa^2}}\exp[-\frac{(\log(\frac{y}{\rho}) - \log(x) - (g-\tilde{g}-\frac{1}{2}\kappa^2)t)^2}{2t\kappa^2}]dt \quad \text{for} \quad y > x^* \end{aligned}$$

By Theorem 13.3.3 of Meyn and Tweedie (1993), $\{\Phi_n\}_{n=0}^{\infty}$ will be ergodic whenever it is positive Harris and aperiodic.

We need to show the following conditions to draw the conclusion of the Proposition 2.

(1) Φ_n is ψ-irreducible.
(2) Φ_n admits an invariant probability measure.
(3) Φ_n is Harris recurrent.
(4) Φ_n is aperiodic.

It is easy to prove (1), due to the special lower bound of x^*. (2) is also true due to Proposition (5) and Proposition (6) in the text. The existence of the stationary wealth distribution, $f(x)$, implies the existence of the invariant probability measure of Φ_n. (3) is true since the government subsidy policy guarantees that, starting from any place in S, Φ_n visits x^* almost surely. (4) is obviously true. Actually, Φ_n is strongly aperiodic because of the special state x^*. (For these mathematical concepts, see Meyn and Tweedie (1993)).

8.4. Proof of Proposition 3

Proof: Plugging $f(x) = Cx^{-\beta}$ into the ordinary differential equation

$$\frac{1}{2}\kappa^2 x^2 f''(x) + (2\kappa^2 - (g-\tilde{g}))xf'(x) + (\kappa^2 - (g-\tilde{g}) - p)f(x) = 0, x \neq x^*$$

we have the characteristic equation

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g - \tilde{g}))\beta + \kappa^2 - p - (g - \tilde{g}) = 0$$

This quadratic equation has two roots

$$\beta_1 = \frac{\frac{3}{2}\kappa^2 - (g - \tilde{g}) - \sqrt{(\frac{1}{2}\kappa^2 - (g - \tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2}$$

and

$$\beta_2 = \frac{\frac{3}{2}\kappa^2 - (g - \tilde{g}) + \sqrt{(\frac{1}{2}\kappa^2 - (g - \tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2}.$$

8.5. Proof of Proposition 4

Proof: From Proposition 3, we have

$$\begin{aligned} \beta_1 &< 1 \Leftrightarrow \frac{\frac{3}{2}\kappa^2 - (g - \tilde{g}) - \sqrt{(\frac{1}{2}\kappa^2 - (g - \tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2} < 1 \\ &\Leftrightarrow \frac{1}{2}\kappa^2 - (g - \tilde{g}) < \sqrt{(\frac{1}{2}\kappa^2 - (g - \tilde{g}))^2 + 2\kappa^2 p} \end{aligned}$$

The last inequality obviously holds. Then $\beta_1 < 1$. Similarly,

$$\begin{aligned} \beta_2 &> 2 \Leftrightarrow \frac{\frac{3}{2}\kappa^2 - (g - \tilde{g}) + \sqrt{(\frac{1}{2}\kappa^2 - (g - \tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2} > 2 \\ &\Leftrightarrow \sqrt{(\frac{1}{2}\kappa^2 - (g - \tilde{g}))^2 + 2\kappa^2 p} > \frac{1}{2}\kappa^2 + (g - \tilde{g}) \\ &\Leftrightarrow p > g - \tilde{g} \end{aligned}$$

The last inequality holds since our assumption that government revenue is greater than the government expenditure, implies that $g - \tilde{g} = p - (\tau - \eta) < p$. Then $\beta_2 > 2$.

8.6. Proof of Proposition 5

Proof: Plugging $f(x) = Cx^{-\beta}$ into the ordinary differential equation

$$\frac{1}{2}\kappa^2 x^2 f''(x) + (2\kappa^2 - (g - \tilde{g}))xf'(x) + (\kappa^2 - (g - \tilde{g}) - p)f(x) = 0,\ x < x^*$$

we have the characteristic equation

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g - \tilde{g}))\beta + \kappa^2 - p - (g - \tilde{g}) = 0.$$

Plugging $f(x) = Cx^{-\beta}$ into the ordinary differential equation

$$\frac{1}{2}\kappa^2 x^2 f''(x) + (2\kappa^2 - (g - \tilde{g}))xf'(x) + (\kappa^2 - (g - \tilde{g}) - p)f(x) + qf(\frac{x}{\rho})\frac{1}{\rho} = 0,\ x > x^*$$

we have the characteristic equation

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g - \tilde{g}))\beta + \kappa^2 - p - (g - \tilde{g}) + q\rho^{\beta-1} = 0.$$

8.7. Proof of Proposition 6

Proof[41]: We first prove that $\beta_1 < 1$. From Proposition 5, we know

$$\beta_1 = \frac{\frac{3}{2}\kappa^2 - (g - \tilde{g}) - \sqrt{(\frac{1}{2}\kappa^2 - (g - \tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2}$$

Thus

$$\begin{aligned} \beta_1 &< 1 \Leftrightarrow \frac{\frac{3}{2}\kappa^2 - (g - \tilde{g}) - \sqrt{(\frac{1}{2}\kappa^2 - (g - \tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2} < 1 \\ &\Leftrightarrow \frac{1}{2}\kappa^2 - (g - \tilde{g}) < \sqrt{(\frac{1}{2}\kappa^2 - (g - \tilde{g}))^2 + 2\kappa^2 p} \end{aligned}$$

The last inequality obviously holds. Then $\beta_1 < 1$.

We then prove that $\beta_2 > 2$. Let $\Gamma(\beta) = \frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g - \tilde{g}))\beta + \kappa^2 - p -$

[41] For this proof, we benifit from the discussion with Henry P. McKean.

$(g-\tilde{g})+q\rho^{\beta-1}$. Since $\frac{\kappa^2}{2}>0$, we know that

$$\lim_{\beta\to-\infty}\Gamma(\beta)=+\infty \quad \text{and} \quad \lim_{\beta\to+\infty}\Gamma(\beta)=+\infty.$$

Note that $\Gamma(1)=q-p\leq 0$. And

$$\begin{aligned}\Gamma(2) &= g-\tilde{g}-p+q\rho \\ &= p-q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}-(\tau-\eta)-p+q(\frac{p\chi(1-\zeta)}{A\mu})^{\frac{1}{\gamma}} \\ &= -(q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta+\tau-\eta)<0\end{aligned}$$

The last inequality is from the assumption that government revenue is greater than the government expenditure. By the continuity of $\Gamma(\beta)$, we know that there exists $\beta\leq 1$, such that $\Gamma(\beta)=0$, and there exists $\beta>2$ such that $\Gamma(\beta)=0$. Since the function $\Gamma(\beta)$ is convex, it can have at most two roots. Then the unique $\beta_2>2$.

8.8. Gini coefficient of a double Pareto distribution

Following Nygard and Sandstrom (1981) and Gastwirth (1971), we derive the Lorenz curve, and Gini coefficient of a double Pareto distribution.[42]

The cumulative distribution function (CDF) of a double Pareto distribution is

$$F(x)=\int_0^x C_1 v^{-\beta_1}dv=\frac{C_1}{1-\beta_1}x^{1-\beta_1} \quad \text{When} \;\; x\leq x^*$$

and

$$F(x)=\int_0^{x^*} C_1 v^{-\beta_1}dv+\int_{x^*}^x C_2 v^{-\beta_2}dv=1-\frac{C_2}{\beta_2-1}x^{1-\beta_2} \quad \text{When} \;\; x\geq x^*$$

Let $x(F)$ be the inverse of the CDF.

$$x=(\frac{1-\beta_1}{C_1}F)^{\frac{1}{1-\beta_1}} \quad \text{When} \;\; x\leq x^*$$

[42] This is also an extension of the derivation of Lorenz curve and Gini coefficient for a Pareto distribution from Wikipedia (http://en.wikipedia.org/wiki/Pareto_distribution) to a double Pareto distribution.

and

$$x = [\frac{\beta_2 - 1}{C_2}(1-F)]^{\frac{1}{1-\beta_2}} \quad \text{When} \;\; x \geq x^*$$

The function of Lorenz curve is

$$L(F) = \frac{\int_0^{x(F)} xf(x)dx}{\int_0^{+\infty} xf(x)dx} = \frac{\int_0^F x(F')dF'}{\int_0^1 x(F')dF'} = \int_0^F x(F')dF'$$

where $x(F)$ is the inverse of the CDF.

Let $F^* = F(x^*) = \frac{C_1}{1-\beta_1}(x^*)^{1-\beta_1}$. When $F \leq F^*$

$$L(F) = \int_0^F x(F')dF' = (\frac{1-\beta_1}{C_1})^{\frac{1}{1-\beta_1}} \frac{1-\beta_1}{2-\beta_1} F^{\frac{2-\beta_1}{1-\beta_1}}$$

When $F \geq F^*$

$$\begin{aligned} L(F) &= \int_0^F x(F')dF' \\ &= \int_0^{F^*} x(F')dF' + \int_{F^*}^F x(F')dF' \\ &= \frac{C_1}{2-\beta_1}(x^*)^{2-\beta_1} + \int_{F^*}^F x(F')dF' \\ &= \frac{C_1}{2-\beta_1}(x^*)^{2-\beta_1} + (\frac{\beta_2-1}{C_2})^{\frac{1}{1-\beta_2}} \frac{1-\beta_2}{2-\beta_2}\{[1-F^*]^{\frac{2-\beta_2}{1-\beta_2}} - (1-F)^{\frac{2-\beta_2}{1-\beta_2}}\} \end{aligned}$$

The Gini coefficient of a double Pareto distribution is

$$\begin{aligned} G &= 1 - 2\int_0^1 L(F)dF \\ &= 1 - 2\int_0^{F^*} L(F)dF - 2\int_{F^*}^1 L(F)dF \\ &= 1 - 2(\frac{1-\beta_1}{C_1})^{\frac{1}{1-\beta_1}} \frac{1-\beta_1}{2-\beta_1}\frac{1-\beta_1}{3-2\beta_1}(F^*)^{\frac{3-2\beta_1}{1-\beta_1}} - 2\frac{C_1}{2-\beta_1}(x^*)^{2-\beta_1}(1-F^*) \\ &\quad -2(\frac{\beta_2-1}{C_2})^{\frac{1}{1-\beta_2}} \frac{1-\beta_2}{3-2\beta_2}(1-F^*)^{\frac{3-2\beta_2}{1-\beta_2}}. \end{aligned}$$

8.9. Two boundary conditions of the stationary distribution, $f(x)$, at $x = x^*$

Proof[43]: First, we see the infinitesimal generator of the stochastic process, $\mathcal{L}$, acting on h in the domain of the operator $\mathcal{L}$, $D(\mathcal{L}) \subset C_0^2[0, +\infty)$.[44]

$$(\mathcal{L}h)(x) := \lim_{t \to 0} \frac{E_x h(X_t) - h(x)}{t} \tag{A.1}$$

for $\forall h \in D(\mathcal{L})$. The domain of the operator $\mathcal{L}$, $D(\mathcal{L})$, is specified as

$$D(\mathcal{L}) = \{h(x) = \int_0^{+\infty} e^{-\alpha t} E_x \phi(X_t) dt : \phi \in C[0, +\infty)\}$$

Here, $\alpha \in R$ and $\alpha > 0$. But the set $D(\mathcal{L})$ is independent of the choosing of positive α.[45](See Yosida (1971), Chapter IX, Analytical Theory of Semi-groups)

Applying Ito formula to equation (A.1), we have

$$(\mathcal{L}h)(x) = [h'(x)(g - \tilde{g})x + \frac{1}{2}h''(x)\kappa^2 x^2] + q[h(\rho x) - h(x)] + (p - q)[h(x^*) - h(x)] \quad \text{if} \quad x \geq \frac{x^*}{\rho} \tag{A.2}$$

and

$$(\mathcal{L}h)(x) = [h'(x)(g - \tilde{g})x + \frac{1}{2}h''(x)\kappa^2 x^2] + p[h(x^*) - h(x)] \quad \text{if} \quad x < \frac{x^*}{\rho} \tag{A.3}$$

For the density function of the stationary distribution, $f(x)$, by the definition of the infinitesimal generator $\mathcal{L}$, we have

$$\int_0^{+\infty} (\mathcal{L}h)(x) f(x) dx = 0 \tag{A.4}$$

for $\forall h \in D(\mathcal{L})$.

The density function $f(x)$ may not be differentiable at $x = x^*$. Thus we write

[43]Without the help from Henry P. McKean, we could not write this rigorous proof of the boundary conditons.

[44]$C_0^2[0, +\infty)$ is the set of functions in $C^2[0, +\infty)$ with compact support.

[45]It can be proved that $\{f = \int_0^{+\infty} e^{-\alpha t} E_x g(X_t) dt : g \in C[0, +\infty)\}$
$= \{f = \int_0^{+\infty} e^{-\beta t} E_x g(X_t) dt : g \in C[0, +\infty)\}$ for $\forall \alpha, \beta > 0$.

equation (A.3) as

$$\int_0^{x^*} (\mathcal{L}h)(x)f(x)dx + \int_{x^*}^{\frac{x^*}{\rho}} (\mathcal{L}h)(x)f(x)dx + \int_{\frac{x^*}{\rho}}^{+\infty} (\mathcal{L}h)(x)f(x)dx = 0 \qquad \text{(A.5)}$$

Plugging equations (A.2) and (A.3) into equation (A.5), we have

$$\begin{aligned} & \int_0^{x^*} \{[h'(x)(g-\tilde{g})x + \frac{1}{2}h''(x)\kappa^2 x^2] + p[h(x^*) - h(x)]\ \}f(x)dx \\ & + \int_{x^*}^{\frac{x^*}{\rho}} \{[h'(x)(g-\tilde{g})x + \frac{1}{2}h''(x)\kappa^2 x^2] + p[h(x^*) - h(x)]\ \}f(x)dx \\ & + \int_{\frac{x^*}{\rho}}^{+\infty} \{[h'(x)(g-\tilde{g})x + \frac{1}{2}h''(x)\kappa^2 x^2] + q[h(\rho x) - h(x)] + (p-q)[h(x^*) - h(x)]\ \}f(x)dx \\ = & \ 0 \qquad \text{(A.6)} \end{aligned}$$

Applying integration by parts to equation (A.6) and taking into account the boundary at $x = x^*$, we obtain

$$\begin{aligned}
& (g-\tilde{g})x^* f_-(x^*)h(x^*) - (g-\tilde{g})x^* f_+(x^*)h(x^*) \\
& -(g-\tilde{g})\int_0^{x^*} h(x)[f(x)+xf'(x)]dx - (g-\tilde{g})\int_{x^*}^{+\infty} h(x)[f(x)+xf'(x)]dx \\
& +\frac{1}{2}\kappa^2 (x^*)^2 f_-(x^*)h'(x^*) - \frac{1}{2}\kappa^2 (x^*)^2 f_+(x^*)h'(x^*) \\
& -\frac{1}{2}\kappa^2[2x^* f_-(x^*) + (x^*)^2 f'_-(x^*)]h(x^*) + \frac{1}{2}\kappa^2[2x^* f_+(x^*) + (x^*)^2 f'_+(x^*)]h(x^*) \\
& +\frac{1}{2}\kappa^2 \int_0^{x^*} h(x)[2f(x)+4xf'(x)+x^2 f''(x)]dx \\
& +\frac{1}{2}\kappa^2 \int_{x^*}^{+\infty} h(x)[2f(x)+4xf'(x)+x^2 f''(x)]dx \\
& +ph(x^*)\int_0^{\frac{x^*}{\rho}} f(x)dx + (p-q)h(x^*)\int_{\frac{x^*}{\rho}}^{+\infty} f(x)dx \\
& +\int_{x^*}^{+\infty} qh(x)f(\frac{x}{\rho})\frac{1}{\rho}dx \\
& -\int_0^{+\infty} ph(x)f(x)dx \\
= \;& 0 \qquad \text{(A.7)}
\end{aligned}$$

where $f_-(x^*)$ and $f_+(x^*)$ are the left and right limit of the density function $f(x)$ at x^*. $f'_-(x^*)$ and $f'_+(x^*)$ are the left and right limit of $f'(x)$ at x^*.[46]

We can pick $h \in D(\mathcal{L})$ where $h(x) = 0$, when $x \geq x^*$. Thus we know $h'(x^*) = 0$, since $h \in C_0^2[0, +\infty)$. By equation (A.7), such $h(x)$ satisfies

$$\int_0^{x^*} h(x)\{\frac{1}{2}\kappa^2[2f(x)+4xf'(x)+x^2 f''(x)] - (g-\tilde{g})[f(x)+xf'(x)] - pf(x)\}dx = 0 \qquad \text{(A.8)}$$

[46] This integration by parts technique is also used in the derivation of the Kolmogorov's forward equation in Chapter VIII of Oksendal (1995).

Equation (A.8) holds for $\forall h \in D(\mathcal{L})$ where $h(x) = 0$, when $x \geq x^*$. Thus we have

$$\frac{1}{2}\kappa^2[2f(x) + 4xf'(x) + x^2 f''(x)] - (g - \tilde{g})[f(x) + xf'(x)] - pf(x) = 0 \quad \text{if} \quad x < x^* \tag{A.9}$$

Similarly, we can pick $h \in D(\mathcal{L})$ where $h(x) = 0$, when $x \leq x^*$. Thus we know $h'(x^*) = 0$, since $h \in C_0^2[0, +\infty)$. By equation (A.7), such $h(x)$ satisfies

$$\int_{x^*}^{+\infty} h(x)\{\frac{1}{2}\kappa^2[2f(x)+4xf'(x)+x^2 f''(x)]-(g-\tilde{g})[f(x)+xf'(x)]-pf(x)+qf(\frac{x}{\rho})\frac{1}{\rho}\}dx = 0 \tag{A.10}$$

Equation (A.10) holds for $\forall h \in D(\mathcal{L})$ where $h(x) = 0$, when $x \leq x^*$. Thus we have

$$\frac{1}{2}\kappa^2[2f(x)+4xf'(x)+x^2 f''(x)]-(g-\tilde{g})[f(x)+xf'(x)]-pf(x)+qf(\frac{x}{\rho})\frac{1}{\rho} = 0 \quad \text{if} \quad x > x^* \tag{A.11}$$

Plugging equations (A.9) and (A.11) into equation (A.7), we have

$$\begin{aligned}
&\{(g - \tilde{g})x^*[f_(x^*) - f_+(x^*)] - \kappa^2 x^*[f_(x^*) - f_+(x^*)] - \frac{1}{2}\kappa^2 (x^*)^2[f'_(x^*) - f'_+(x^*)] \\
&+p\int_0^{\frac{x^*}{\rho}} f(x)dx + (p - q)\int_{\frac{x^*}{\rho}}^{+\infty} f(x)dx\}h(x^*) \\
&+\frac{1}{2}\kappa^2 \left(x^*\right)^2 [f_(x^*) - f_+(x^*)]h'(x^*) \\
= &\ 0
\end{aligned} \tag{A.12}$$

Equation (A.12) holds for $\forall h \in D(\mathcal{L})$. By the coefficient before $h'(x^*)$, we have one of the two boundary conditions of $f(x)$ at $x = x^*$.

$$f\_(x^*) - f_+(x^*) = 0 \tag{A.13}$$

By the coefficient before $h(x^*)$ and equation (A.13), we have the other boundary condition of $f(x)$ at $x = x^*$.

$$\frac{1}{2}\kappa^2(x^*)^2[f'\_(x^*) - f'_+(x^*)] = p\int_0^{\frac{x^*}{\rho}} f(x)dx + (p - q)\int_{\frac{x^*}{\rho}}^{+\infty} f(x)dx \tag{A.14}$$

Equation (A.13) means that the density function $f(x)$ is continuous at $x = x^*$. Note that the right hand side of equation (A.13) is exactly the injection of the newborns at $x = x^*$. Thus equation (A.14) is the relationship about the injection and the difference between the left derivative and the right derivative of $f(x)$ at $x = x^*$.

8.10. Boundary conditions at $x = x^*$

Plugging the numbers of C_1, β_1, C_2, β_2, and x^* into the density function

$$f(x) = \begin{cases} C_1 x^{-\beta_1} & \text{when } x \leq x^* \\ C_2 x^{-\beta_2} & \text{when } x \geq x^* \end{cases}$$

we find that

$$f_-(x^*) - f_+(x^*) = -0.0011666 \tag{A.15}$$

where $f_-(x^*)$ and $f_+(x^*)$ are the left and right limit of the density function $f(x)$ at x^*. We expect that $f(x)$ will be continuous at $x = x^*$. In Appendix 8.9, we prove this result by finding two boundary conditions of the stationary distribution $f(x)$ at $x = x^*$:

$$f_-(x^*) - f_+(x^*) = 0 \tag{A.16}$$

and

$$\frac{1}{2}\kappa^2 (x^*)^2 [f'_-(x^*) - f'_+(x^*)] = p \int_0^{\frac{x^*}{\rho}} f(x)dx + (p-q) \int_{\frac{x^*}{\rho}}^{+\infty} f(x)dx \tag{A.17}$$

where $f'_-(x^*)$ and $f'_+(x^*)$ are the left and right limit of $f'(x)$ at x^*. Equation (A.16) means that the density function $f(x)$ is continuous at $x = x^*$. Note that the right hand side of equation (A.17) is exactly the injection of the newborns at $x = x^*$. Thus equation (A.17) relates the injection to the difference between the left derivative and the right derivative of $f(x)$ at $x = x^*$. For the two boundary conditions at $x = x^*$, equations (A.16) and (A.17), we can not verify them explicitly in the general case, since the equation (25) has no explicit solution. However, we can explicitly verify that equations (A.16) and (A.17) are satisfied in the no inheritance case. For the general case, we employ numerical methods to check whether the boundary conditions are satisfied. Equation (A.15) and equation (A.18) below in the simulated results show that both boundary conditions are satisfied in our calibrated economy.

$$\frac{1}{2}\kappa^2(x^*)^2[f'_-(x^*)-f'_+(x^*)]-p\int_0^{\frac{x^*}{\rho}} f(x)dx-(p-q)\int_{\frac{x^*}{\rho}}^{+\infty} f(x)dx = -4.74491\times 10^{-7} \tag{A.18}$$

8.11. Wealth distribution conditional on age

The distribution of the starting wealth at age 0 consists of two parts, one of which is a mass point. At x^*, the distribution has a positive probability, $\frac{q}{p}(C_1\int_0^{x^*} x^{-\beta_1}dx + C_2\int_{x^*}^{\frac{x^*}{\rho}} x^{-\beta_2}dx) + \frac{p-q}{p}$. For wealth levels higher than x^*, the density of the distribution is

$$v(y) = \frac{q}{p}C_2(\frac{y}{\rho})^{-\beta_2}\frac{1}{\rho}, \qquad y > x^*$$

Conditional on age t, the density of wealth distribution is the sum of two components: 1). the probability of reaching x at age t when the age 0 starting wealth is $y > x^*$, and 2). the probability of reaching x at age t when the age 0 starting wealth is x^*.

By equation (12), $X(s,t)$ is lognormally distributed. Thus following Reed (2003), we compute the wealth distribution at age t, $f_t(x)$, and prove that $f_t(x) \sim$

$x^{-\beta_2}$ as $x \to +\infty$.

$$
\begin{aligned}
f_t(x) &= \int_{x^*}^{+\infty} \frac{1}{x}\frac{1}{\sqrt{2\pi t\kappa^2}} \exp[-\frac{(\log(x)-\log(y)-(g-\tilde{g}-\frac{1}{2}\kappa^2)t)^2}{2t\kappa^2}]v(y)dy \\
&\quad +\frac{1}{x}\frac{1}{\sqrt{2\pi t\kappa^2}} \exp[-\frac{(\log(x)-\log(x^*)-(g-\tilde{g}-\frac{1}{2}\kappa^2)t)^2}{2t\kappa^2}] \times \\
&\quad [\frac{q}{p}(C_1\int_0^{x^*} x^{-\beta_1}dx + C_2\int_{x^*}^{\frac{x^*}{\rho}} x^{-\beta_2}dx) + \frac{p-q}{p}] \\
&= \int_{x^*}^{+\infty} \frac{1}{x}\frac{1}{\sqrt{2\pi t\kappa^2}} \exp[-\frac{(\log(x)-\log(y)-(g-\tilde{g}-\frac{1}{2}\kappa^2)t)^2}{2t\kappa^2}]\frac{q}{p}C_2(\frac{y}{\rho})^{-\beta_2}\frac{1}{\rho}dy \\
&\quad +\frac{1}{x}\frac{1}{\sqrt{2\pi t\kappa^2}} \exp[-\frac{(\log(x)-\log(x^*)-(g-\tilde{g}-\frac{1}{2}\kappa^2)t)^2}{2t\kappa^2}] \times \\
&\quad [\frac{q}{p}(C_1\int_0^{x^*} x^{-\beta_1}dx + C_2\int_{x^*}^{\frac{x^*}{\rho}} x^{-\beta_2}dx) + \frac{p-q}{p}] \\
&= \frac{1}{x}\frac{1}{\sqrt{2\pi t\kappa^2}} \exp[-\frac{(\log(x)-\log(x^*)-(g-\tilde{g}-\frac{1}{2}\kappa^2)t)^2}{2t\kappa^2}] \times \\
&\quad [\frac{q}{p}(C_1\int_0^{x^*} x^{-\beta_1}dx + C_2\int_{x^*}^{\frac{x^*}{\rho}} x^{-\beta_2}dx) + \frac{p-q}{p}] \\
&\quad +\frac{q}{p}C_2\rho^{\beta_2-1}\exp[\frac{(1-\beta_2)^2}{2}t\kappa^2-(1-\beta_2)(g-\tilde{g}-\frac{1}{2}\kappa^2)t] \times \\
&\quad \left(1-\Phi\left(\frac{\log(x^*)-\log(x)-(1-\beta_2)t\kappa^2+(g-\tilde{g}-\frac{1}{2}\kappa^2)t}{\sqrt{t}\kappa}\right)\right) \times \\
&\quad x^{-\beta_2} \\
&\sim x^{-\beta_2} \quad (x \to +\infty)
\end{aligned}
$$

where Φ is the cumulative distribution function of the standard normal distribution.

8.12. Derivation of Ω

To derive the aggregate welfare of the economy, we first note that there are two kinds of people in the economy. $\frac{q}{p}$ fraction of the people have a bequest motive and $1-\frac{q}{p}$ fraction of people do not have a bequest motive. The aggregate welfare of the economy is the weighted sum of the individual utilities with weights according to the cross-sectional wealth distribution of the two groups of agents.

$$\Omega(\tau,\zeta)=\frac{q}{p}\int_0^{+\infty}U(w)f(w)dw+\frac{p-q}{p}\int_0^{+\infty}U_0(w)f(w)dw$$

where $U(w)$ is the optimal value of the people with bequest motives, and $U_0(w)$ is the optimal value of the people without bequest motives. From Proposition 1, we know that people with bequest motives have the following value function:

$$\begin{aligned}U(w) &= \frac{A}{1-\gamma}W(s,t)^{1-\gamma} \\ &= \frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1+(p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma}w^{1-\gamma}\end{aligned}$$

Similarly, for people with no bequest motive, the value function is:

$$U_0(w)=\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma}\right)^{-\gamma}w^{1-\gamma}$$

Thus the aggregate welfare of the economy is

$$\begin{aligned}\Omega(\tau,\zeta) &= \frac{q}{p}\int_0^{+\infty}U(w)f(w)dw+\frac{p-q}{p}\int_0^{+\infty}U_0(w)f(w)dw \\ &= \frac{q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1+(p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma}\int_0^{+\infty}w^{1-\gamma}f(w)dw \\ &\quad +\frac{p-q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma}\right)^{-\gamma}\int_0^{+\infty}w^{1-\gamma}f(w)dw \\ &= [\frac{q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1+(p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma} \\ &\quad +\frac{p-q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma}\right)^{-\gamma}]\times \\ &\quad \int_0^{+\infty}w^{1-\gamma}f(w)dw\end{aligned}$$

We then compute $\int_0^{+\infty} w^{1-\gamma} f(w)dw$

$$\begin{aligned}
& \int_0^{+\infty} w^{1-\gamma} f(w)dw \\
= & \; C_1 \int_0^{x^*} x^{1-\gamma} x^{-\beta_1} dx + C_2 \int_{x^*}^{+\infty} x^{1-\gamma} x^{-\beta_2} dx \\
= & \; \frac{C_1}{2-\gamma-\beta_1}(x^*)^{2-\gamma-\beta_1} - \frac{C_2}{2-\gamma-\beta_2}(x^*)^{2-\gamma-\beta_2}
\end{aligned}$$

The last step is valid when $\beta_1 < -1$ since $\gamma = 3$ in our calibration. Plugging the above result of the integral, $\int_0^{+\infty} w^{1-\gamma} f(w)dw$, into the formula of $\Omega(\tau, \zeta)$, we have

$$\begin{aligned}
\Omega(\tau,\zeta) = & \; \frac{q}{p}\int_0^{+\infty} U(z)f(z)dz + \frac{p-q}{p}\int_0^{+\infty} U_0(z)f(z)dz \\
= & \; [\frac{q}{p}\frac{1}{1-\gamma}\left(\frac{\theta + p - (1-\gamma)(r - \tau + \mu + \frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1+(p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma} \\
& + \frac{p-q}{p}\frac{1}{1-\gamma}\left(\frac{\theta + p - (1-\gamma)(r - \tau + \mu + \frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma}\right)^{-\gamma}] \int_0^{+\infty} w^{1-\gamma} f(w)dw \\
= & \; [\frac{q}{p}\frac{1}{1-\gamma}\left(\frac{\theta + p - (1-\gamma)(r - \tau + \mu + \frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1+(p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma} \\
& + \frac{p-q}{p}\frac{1}{1-\gamma}\left(\frac{\theta + p - (1-\gamma)(r - \tau + \mu + \frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma}\right)^{-\gamma}] \times \\
& [\frac{C_1}{2-\gamma-\beta_1}(x^*)^{2-\gamma-\beta_1} - \frac{C_2}{2-\gamma-\beta_2}(x^*)^{2-\gamma-\beta_2}].
\end{aligned}$$

8.13. A simple mechanism underlying a double Pareto distribution of wealth without inheritance

Suppose that $X(s,t)$ is a geometric Brownian motion. We have

$$dX(s,t) = gX(s,t)dt + \kappa X(s,t)dB(s,t)$$

and

$$X(s,t) = X(s,s)\exp[(g-\frac{1}{2}\kappa^2)(t-s)+\kappa(B(s,t)-B(s,s))]$$

For simplicity, let $X(s,s)=1$, so initial wealth is fixed (no inheritance). Note that $X(s,t)$ is log-normal

$$\log(X(s,t)) = (g-\frac{1}{2}\kappa^2)(t-s)+\kappa(B(s,t)-B(s,s))$$

Now integrating over the population, we have the density function of the stationary distribution of $X(s,t)$:

$$\begin{aligned} f(x) &= \int_{-\infty}^{t} pe^{-p(t-s)}\frac{1}{x}\frac{1}{\sqrt{2\pi(t-s)\kappa^2}}\exp[-\frac{(\log(x)-(g-\frac{1}{2}\kappa^2)(t-s))^2}{2(t-s)\kappa^2}]ds \\ &= \int_{0}^{+\infty} pe^{-pv}\frac{1}{x}\frac{1}{\sqrt{2\pi v\kappa^2}}\exp[-\frac{(\log(x)-(g-\frac{1}{2}\kappa^2)v)^2}{2v\kappa^2}]dv \end{aligned}$$

Let

$$w(x,v) = \frac{1}{x}\frac{1}{\sqrt{2\pi v\kappa^2}}\exp[-\frac{(\log(x)-(g-\frac{1}{2}\kappa^2)v)^2}{2v\kappa^2}]$$

Thus

$$f(x) = \int_{0}^{+\infty} pe^{-pv}w(x,v)dv$$

$$f'(x) = \int_{0}^{+\infty} pe^{-pv}\frac{\partial w(x,v)}{\partial x}dv$$

$$f''(x) = \int_{0}^{+\infty} pe^{-pv}\frac{\partial^2 w(x,v)}{\partial x^2}dv$$

Note that

$$\frac{\partial w(x,v)}{\partial v} = \frac{1}{2}\kappa^2x^2\frac{\partial^2 w(x,v)}{\partial x^2}+(2\kappa^2-g)x\frac{\partial w(x,v)}{\partial x}+(\kappa^2-g)w(x,v)$$

Then

$$\frac{1}{2}\kappa^2x^2f''(x)+(2\kappa^2-g)xf'(x)+(\kappa^2-g)f(x) = \int_{0}^{+\infty} pe^{-pv}\frac{\partial w(x,v)}{\partial v}dv = pf(x)$$

This gives the characteristic equation

$$\frac{1}{2}\kappa^2 x^2 f''(x) + (2\kappa^2 - g)xf'(x) + (\kappa^2 - g - p)f(x) = 0$$

Then $f(x)$ has the functional form[47]

$$f(x) = \begin{cases} C_1 x^{-\beta_1} & \text{when } x \leq 1 \\ C_2 x^{-\beta_2} & \text{when } x \geq 1 \end{cases}$$

where β_1 and β_2 are the two roots of the characteristic equation

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - g)\beta + \kappa^2 - g - p = 0$$

Solving this equation, we have

$$\beta_1 = \frac{\frac{3}{2}\kappa^2 - g - \sqrt{(\frac{1}{2}\kappa^2 - g)^2 + 2\kappa^2 p}}{\kappa^2}$$

and

$$\beta_2 = \frac{\frac{3}{2}\kappa^2 - g + \sqrt{(\frac{1}{2}\kappa^2 - g)^2 + 2\kappa^2 p}}{\kappa^2}.$$

8.14. Simulation results-Gini coefficient

Sensitivity to bequest parameter χ

χ	*Gini*	Fraction of Bequest in the Aggregate Wealth
14	0.637806	0.00138273
15	0.636731	0.0014135
16	0.635706	0.00144288
17	0.634729	0.001471

Table 8.1: Gini and the Bequest Motive

[47] The two coefficients, C_1 and C_2, can be determined by $\int_0^{+\infty} f(x)dx = 1$ and the continuity condition at x^* as derived in section 8.9.

Sensitivity to q, with $p = 0.016$.

$\frac{q}{p}$	*Gini*	Fraction of Bequest in the Aggregate Wealth
0.7	0.640537	0.00131927
0.75	0.636731	0.0014135
0.8	0.632909	0.00150774
0.85	0.629073	0.00160197
0.9	0.625224	0.0016962
0.95	0.621363	0.00179044
1	0.617491	0.00188467

Table 8.2: Gini and q

Sensitivity to σ and χ

$\sigma \backslash \chi$	14	15	16	17
0.2	0.691555	0.690649	0.689788	0.688966
0.21	0.680519	0.67958	0.678686	0.677833
0.22	0.670422	0.669451	0.668527	0.667646
0.23	0.66118	0.66018	0.659228	0.658319
0.24	0.652716	0.651688	0.650711	0.649777
0.25	0.644917	0.643865	0.642864	0.641908
0.26	0.637806	0.636731	0.635706	0.634729

Table 8.3: Gini on the Volatility of the Risky Asset and the Bequest Motive