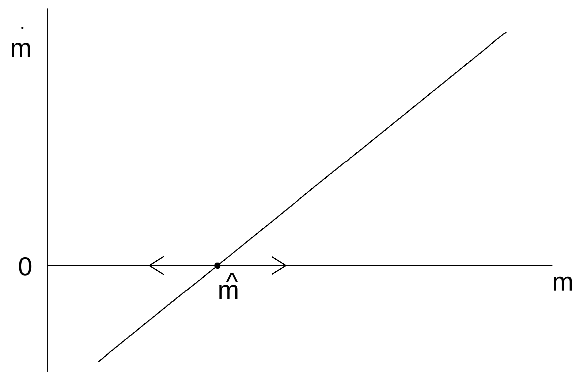
ṁ
0
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m

Models With Money:

$$\begin{aligned} y &= y(m), \quad \dot{c} = y'(m)\dot{m} \\ \pi &= \left(\sigma - \frac{\dot{m}}{m}\right) \\ p &= U'(c); \\ \frac{\dot{p}}{p} &= \frac{U''}{U'} y'(m)\dot{m} = r - (y'(m) - \pi) \\ &= r - \left(y'(m) - \left(\sigma - \frac{\dot{m}}{m}\right)\right) \end{aligned}$$

becomes:

$$\dot{m}\left(y'(m)\left(1 + \frac{U'(c)}{U''(c)\,m}\right)\right) = \frac{U'(c)}{U''(c)}(r + \sigma - y'(m)) \tag{1}$$

If we define the elasticities

$$\varepsilon_c = -\frac{U''(c)\,c}{U'(c)}, \qquad \varepsilon_m = \frac{y'(m)\,m}{y(m)}$$

$$\dot{m} = \frac{m(r + \sigma - y'(m))}{1 - \varepsilon_c \varepsilon_m} \tag{2}$$

“To summarize, the research proposed her shows the surprising efficiency and robustness of simple policy rules in which the reaction of the interest rate is above a critical treshold. The analysis also shows that the estimated gains reported in some research from following alternative rules are not robust to a variety of models considered in this paper”

John Taylor,“ The robustness and efficiency of monetary policy rules as guidelines for interest rate setting by the European Central Bank,” May 1998.

1 Multiple equilibria

Presence of multiple equilibria (which can be real or nominal) depends on:

I: The way in which money is modelled

- **a) Money in utility**
- **b) Money in prod. fn.**
- **c) Cash-in-advance,**
  cash and credit goods

II: Flexible vs. sticky prices

III: Type of monetary feedback rule

- **a) passive or active rule**
- **b) timing of the rule**

IV) Type of fiscal policy
(Ricardian-Non-Ricardian)

2 Introduction

$$\frac{\dot{\lambda}}{\lambda} = r + \pi - R(\pi)$$
$$Steady\ State: \quad r + \pi - R(\pi) = 0$$

$$\begin{aligned} Active\ policy\ R'(\pi^*) &> 1 \\ Passive\ policy\ R'(\pi^*) &> 1 \\ R(\pi) &\geq 0 \end{aligned}$$

3 A flexible–price model

$$Max\, U = \int_0^\infty e^{-rt} u\,(c, m^{np})\, dt \qquad (3)$$

Assumption 1 (1**)** $u(\cdot,\cdot)$ *is strictly increasing and strictly concave, and* c *and* m^{np} *are normal goods.*

Assumption 2 $y(m^p)$ *is positive, strictly increasing, strictly concave,* $\lim_{m^p\to 0} y'(m^p) = \infty$*, and* $\lim_{m^p\to\infty} y'(m^p) = 0$.

Assumption 2′ $y(m^p)$ is a positive constant.

$$a \equiv (M^{np} + M^p + B)/P \qquad (4)$$

$$\dot{a} = (R - \pi)a - R(m^{np} + m^p) + y(m^p) - c - \tau. \qquad (5)$$

$$\lim_{t \to \infty} e^{-\int_0^t [R(s) - \pi(s)]ds} a(t) \geq 0 \qquad (6)$$

taking as given $a(0)$ and the time paths of τ, R, and π. The optimality conditions associated with the household's problem are

$$u_c(c, m^{np}) = \lambda \qquad (7)$$

$$m^p \left[y'(m^p) - R \right] = 0 \qquad (8)$$

$$\frac{u_m(c, m^{np})}{u_c(c, m^{np})} = R \qquad (9)$$

$$\lambda \left(r + \pi - R \right) = \dot{\lambda} \qquad (10)$$

$$\lim_{t \to \infty} e^{-\int_0^t [R(s) - \pi(s)]ds} a(t) = 0 \qquad (11)$$

Assumption together with equation (8) and $R > 0$ implies

$$m^p = m^p(R), \tag{12}$$

with $m^{p\prime} \equiv dm^p/dR < 0$. Alternatively, equation (8), $R > 0$, and assumption $2'$ imply that $m^p = m^{p\prime} = 0$. Using equation (9) and assumption , m^{np} can be expressed as a function of consumption and the nominal interest rate,

$$m^{np} = m^{np}(c, R), \tag{13}$$

that is increasing in c and decreasing in R.

3.0 The government

$$R = \rho(\pi), \tag{14}$$

where $\rho(\cdot)$ is continuous, non–decreasing, and strictly positive and there exists at least one $\pi^* > -r$ such that $\rho(\pi^*) = r + \pi^*$. Following Leeper (1991), we will refer to the monetary policy as active if $\rho'(\pi^*) > 1$ and as passive if $\rho'(\pi^*) < 1$.

The sequential budget constraint of the government is given by

$$\dot{B} = RB - \dot{M}^{np} - \dot{M}^{p} - P\tau,$$

which can be written as

$$\dot{a} = (R - \pi)a - R(m^{np} + m^{p}) - \tau. \tag{15}$$

The nominal value of initial government liabilities, $A(0)$, is predetermined:

$$a(0) = \frac{A(0)}{P(0)}. \tag{16}$$

We classify fiscal policies into two categories: Ricardian fiscal policies and non-Ricardian. Ricardian fiscal policies are those that ensure that the present discounted value of total government liabilities converges to zero—that is, equation (68) is satisfied—under all possible, equilibrium or off–equilibrium, paths of endogenous variables such as the price level, the money supply, inflation, or the nominal interest rate

Throughout the paper we will restrict attention to one particular Ricardian fiscal policy that takes the form

$$\tau = R\,a - R(m^{np} + m^{p}) \tag{17}$$

We will also analyze a particular non–Ricardian policy consisting of an exogenous path for lump–sum taxes

$$\tau = \bar{\tau}. \tag{18}$$

Equilibrium

$$c = y(m^p). \tag{19}$$

Using equations (12)–(14) and (69) to replace m^p, m^{np}, R, and c in equation (7),

$$u_c(c, m^{np}) = \lambda = \lambda(\pi) \tag{20}$$

$$\lambda'(\pi) = \rho' \left[u_{cc} y' m^{p\prime} + u_{cm}(m_c^{np} y' m^{p\prime} + m_R^{np})\right] \tag{21}$$

where m_c^{np} and m_R^{np} denote the partial derivatives of m^{np} with respect to c and R, respectively. (12)–(14), and (69), equations (10), (68), (15), and (17) can be rewritten as

$$\dot{\pi} = \frac{\lambda(\pi)[r + \pi - \rho(\pi)]}{\lambda'(\pi)} \tag{22}$$

$$\begin{aligned} \dot{a} \;&=\; [\rho(\pi) - \pi]a \\ &\quad -\rho(\pi)[m^{np}(y(m^p(\rho(\pi)), \rho(\pi)) + m^p(\rho(\pi))] - \tau \end{aligned} \tag{23}$$

$$\lim_{t\to\infty} e^{-\int_0^t [\rho(\pi) - \pi(s)]ds} a(t) = 0 \tag{24}$$

$$\tau \;=\; \rho(\pi)\; [a - [m^{np}(y(m^p(\rho(\pi))), \rho(\pi)) + m^p(\rho(\pi))]]$$

$$OR\;\; \tau \;=\; \bar{\tau}$$

Definition 1 *(Perfect–foresight equilibrium in the flexible–price economy) In the flexible–price economy, a perfect–foresight equilibrium is a set of sequences* $\{\pi, a, \tau\}$ *and an initial price level* $P(0) > 0$ *satisfying (*16*), (*22*)–(*24*) and either if fiscal policy is non-Ricardian* $\tau = \bar{\tau}$ *or* $\tau = R\,a - R(m^{np} + m^{p})$ *if fiscal policy is Ricardian, given* $A(0) > 0$*.*

Given a sequence for π, equations (12)–(14), (69), and (20) uniquely determine the equilibrium sequences of $\{c, m^{np}, m^{p}, \lambda, R\}$ independently of whether the equilibrium price level is unique.

Definition 2 *(Real and Nominal Indeterminacy) The equilibrium displays real indeterminacy if there exists an infinite number of equilibrium sequences* $\{\pi\}$*. The equilibrium exhibits nominal indeterminacy if for any equilibrium sequence* $\{\pi\}$*, there exists an infinite number of initial price levels* $P(0) > 0$ *consistent with a perfect–foresight equilibrium.*

In what follows, we restrict the analysis to equilibria in which the inflation rate remains bounded in a neighborhood around a steady–state value, π^*, which is defined as a constant value of π that solves (22), that is, a solution to $r + \pi = \rho(\pi)$. By assumption π^* exists and is greater than $-r$. Note that π^* may not be unique. In particular, if there exists a steady state π^* with $\rho'(\pi^*) > 1$, then since $\rho(\cdot)$ is assumed to be continuous and strictly positive there must also exist a steady state with $\rho' < 1$.

It follows from equation (22) that if the sign of $\lambda'(\pi^*)$ is the opposite of the sign of $1 - \rho'(\pi^*)$, any initial inflation rate near the steady state π^* will give rise to an inflation trajectory that converges to π^*. If, on the other hand, $\lambda'(\pi^*)$ and $1 - \rho'(\pi^*)$ are of the same sign, the only sequence of inflation rates consistent with equation (22) that remains in the neighborhood of π^* is the steady state π^*.

Note that the case $\rho'(\pi) = 0$ for all π corresponds to a pure interest rate peg. In this case it follows from equation (21) that λ is constant, and therefore the inflation rate is also constant, which implies that under a pure interest rate peg the economy exhibits real determinacy.

Under a Ricardian fiscal policy the set of equilibrium conditions includes equation **(??)**. Given a sequence $\{\pi\}$ and an initial price level $P(0) > 0$, equations (23) and **(??)** can be used to construct a pair of sequences $\{a, \tau\}$. Because the fiscal policy is Ricardian, the transversality condition (24) is always satisfied. If instead the fiscal authority follows the non–Ricardian fiscal policy given in (18), combining (16), (23), and (24) yields

$$\frac{A(0)}{P(0)} = \int_0^\infty e^{-\int_0^t [\rho(\pi)-\pi]ds} \{\rho(\pi)\left[m^{np}(y(m^p(\rho(\pi))), \rho(\pi)) + m^p(\rho(\pi))\right] + \bar{\tau}\} ds \tag{25}$$

which given $A(0) > 0$ and a sequence for π uniquely determines the initial price level $P(0)$.

The above analysis demonstrates that for the class of monetary–fiscal regimes considered nominal determinacy depends only on fiscal policy and not on monetary policy — a result that has been emphasized in the recent literature on the fiscal determination of the price level and that we summarize in the following proposition:

Proposition 1 *If fiscal policy is Ricardian, the equilibrium exhibits nominal indeterminacy. Under the non-Ricardian fiscal policy given by (*18*), the equilibrium displays nominal determinacy.*

Proposition 2 *Suppose preferences are separable in consumption and money ($u_{cm} = 0$) and money is productive (assumption 2 holds), then if monetary policy is active ($\rho'(\pi^*) > 1$), the equilibrium displays real indeterminacy, whereas if monetary policy is passive ($\rho'(\pi^*) < 1$), there exists a unique perfect–foresight equilibrium in which the real allocation converges to the steady state.*

Proposition 3 *Suppose that money is not productive (assumption $2'$ holds) and consumption and money are substitutes ($u_{cm} < 0$). Then, if monetary policy is active ($\rho'(\pi^*) > 1$), the real allocation is indeterminate, and if monetary policy is passive ($\rho'(\pi^*) < 1$) there exists a unique perfect–foresight equilibrium in which the real allocation converges to the steady state.*

1.Real Indeterminacy in the Flexible-Price Model

Monetary Policy	Non-productive money ($y' = 0$)			Productive money ($y' > 0$)		
	$u_{cm} > 0$	$u_{cm} < 0$	$u_{cm} = 0$	$u_{cm} > 0$	$u_{cm} < 0$	$u_{cm} = 0$
Passive ($\rho'(\pi^*) < 1$)	I	D	D	A	D	D
Active ($\rho'(\pi^*) > 1$)	D	I	D	A	I	I

Note: The notation is: D, determinate; I, indeterminate; A, ambiguous. (Under A the real allocation may be determinate or indeterminate depending on specific parameter values.)

Proposition 4 *Suppose that money is not productive (assumption $2'$ holds) and consumption and money are complements ($u_{cm} > 0$). Then, if monetary policy is passive ($\rho'(\pi^*) < 1$), the real allocation is indeterminate, and if monetary policy is active ($\rho'(\pi^*) > 1$) there exists a unique perfect–foresight equilibrium in which the real allocation converges to the steady state.*

Combining the case of non-productive money (assumption $2'$) with preferences that are separable in consumption and real balances ($u_{cm} = 0$) results in the continuous time version of the economy analyzed in Leeper (1991). In this case equation (7) implies that λ is constant. It then follows that π, R and m^{np} are also constant, and the only equilibrium real allocation is the steady state. This result differs from that obtained by Leeper who finds that under passive monetary policy the inflation rate is indeterminate. The difference stems from the fact that in Leeper's discrete–time model the nominal interest rate in period t is assumed to be a function of the change in the price level between periods $t-1$ and t, whereas in the continuous time model analyzed here, the inflation rate is the right hand side derivative of the price level, so its discrete–time counterpart is better approximated by the change in the price level between periods t and $t+1$.

In fact, it is straightforward to show that if in Leeper's discrete–time model, with $u_{cm} = 0$ and an endowment economy, the feedback rule is assumed to be forward looking — that is, $R_t = \rho(P_{t+1}/P_t)$ — the equilibrium displays real determinacy.

Leeper

$$U'(\bar{c}) = \frac{\beta}{\pi_{t+1}} R_{t+1}(\pi_t) U'(\bar{c})$$

$$R_{t+1}(\pi_t) = R^* + a(\pi_t - \pi^*), \qquad R^* = r + \pi^*$$

$$\pi_{t+1} = a\beta\pi_t + (R^* - a\pi^*)$$

3.0.1 Discrete-time models

$$\begin{aligned} \rho_{cm}(\hat{m}_t - \hat{m}_{t+1}) &= \hat{R}_t - \hat{\pi}_{t+1} \\ \hat{m}_t &= -\epsilon_{mR}\hat{R}_t \\ \hat{R}_t &= \epsilon_\rho \hat{\pi}_{t+1} \end{aligned}$$

$$\hat{\pi}_{t+2} = \left[1 + \frac{\epsilon_\rho - 1}{\epsilon_\rho \rho_{cm} \epsilon_{mR}}\right] \hat{\pi}_{t+1}.$$

In an economy with with $u_{cm} = 0$ and productive money,

$$\begin{aligned} \rho_{cc}(\hat{y}_{t+1} - \hat{y}_t) &= \epsilon_\rho \hat{\pi}_{t+1} - \hat{\pi}_{t+1} \\ \hat{y}_t &= -\epsilon_{yR}\epsilon_\rho \hat{\pi}_{t+1} \end{aligned}$$

$$\hat{\pi}_{t+2} = \left[1 + \frac{1 - \epsilon_\rho}{\rho_{cc}\epsilon_{yR}\epsilon_\rho}\right] \hat{\pi}_{t+1}.$$

4 A sticky–price model

Specifically, we assume that there exists a continuum of household–firm units indexed by j, each of which produces a differentiated good Y^j and faces a demand function $Y^d d\left(\frac{P^j}{P}\right)$, where Y^d denotes the level of aggregate demand, P^j the price firm j charges for its output, and P the aggregate price level. Such a demand function can be derived by assuming that households have preferences over a composite good that is produced from differentiated intermediate goods via a Dixit-Stiglitz production function. The function $d(\cdot)$ is assumed to satisfy $d(1) = 1$ and $d'(1) < -1$. The restriction imposed on $d'(1)$ is necessary for the individual firm's problem to be well defined in a symmetric equilibrium. The production of good j is assumed to take real money balances, m^{pj}, as the only input

$$Y^j = y(m^{pj})$$

where $y(\cdot)$ satisfies assumption .

The household's lifetime utility function is assumed to be of the form

$$U^j = \int_0^\infty e^{-rt} \left[u(c^j, m^{npj}) - \frac{\gamma}{2} \left(\frac{\dot{P}^j}{P^j} - \pi^* \right)^2 \right] dt \tag{26}$$

$\pi^* > -r$ denotes the steady–state inflation rate. The household's instant budget constraint and no-Ponzi-game restriction are

$$\dot{a}^j = (R - \pi) a^j - R(m^{npj} + m^{pj}) + \frac{P^j}{P} y(m^{pj}) - c^j - \tau \tag{27}$$

$$\lim_{t \to \infty} e^{-\int_0^t [R(s) - \pi(s)] ds} a^j(t) \geq 0 \tag{28}$$

In addition, firms are subject to the constraint that given the price they charge, their sales are demand-determined

$$y(m^{pj}) = Y^d d\left(\frac{P^j}{P} \right) \tag{29}$$

The household chooses sequences for c^j, m^{npj}, m^{pj}, $P^j \geq 0$ and a^j so as to maximize (26) subject to (58)–(29) taking as given $a^j(0)$, $P^j(0)$, and the time paths of τ, R, Y^d, and P.

The demand for real balances for non-production purposes can be expressed as

$$m^{npj} = m^{np}(c^j, R) \tag{30}$$

which by assumption is increasing in c^j and decreasing in R.

Equilibrium

In a symmetric equilibrium all household–firm units choose identical sequences for consumption, asset holdings, and prices. Thus, $c^j = c$, $m^{pj} = m^p$, $m^{npj} = m^{np}$, $a^j = a$, $P^j = P$, $\lambda^j = \lambda$, $\mu^j = \mu$, and $\pi^j = \pi$. In addition,

$$u_c(y(m^p), m^{np}(y(m^p), \rho(\pi))) = \lambda. \tag{31}$$

$$m^p = m^p(\lambda, \pi); \qquad m^p_\lambda < 0,\ m^p_\pi u_{cm} < 0 \tag{32}$$

Let $\eta \equiv d'(1) < -1$ denote the equilibrium price elasticity of the demand function faced by the individual firm.

$$\begin{aligned}
\dot{\lambda} &= \lambda\left[r + \pi - \rho(\pi)\right] \\
\gamma\dot{\pi} &= \gamma r(\pi - \pi^*) - y(m^p)\lambda\left[1 + \eta\left(1 - \frac{\rho(\pi)}{y'(m^p)}\right)\right] \\
\dot{a} &= [\rho(\pi) - \pi]a - \rho(\pi)\left[m^{np}(y(m^p), \rho(\pi)) + m^p\right] - \tau \\
0 &= \lim_{t\to\infty} e^{-\int_0^t [\rho(\pi)-\pi]ds} a(t) \\
\tau &= -\rho(\pi)\left[m^{np}(y(m^p), \rho(\pi)) + m^p\right] + Ra \\
\tau &= \bar{\tau}
\end{aligned}$$

Definition 3 *(Perfect–foresight equilibrium in the sticky–price economy) In the sticky–price economy, a perfect–foresight equilibrium is a set of sequences $\{\lambda, \pi, \tau, a\}$ satisfying (**??**)–(**??**) and either $\tau = \bar{\tau}$ if the fiscal regime is non-Ricardian or $\tau = R\,a - R(m^{np} + m^{p})$. if the fiscal regime is Ricardian, given $a(0)$.*

Given the equilibrium sequences $\{\lambda, \pi, \tau, a\}$, the corresponding equilibrium sequences $\{c, m^{np}, m^{p}, R\}$ are uniquely determined by (14), (69), (30), and (32).

Ricardian fiscal policy

$$\begin{pmatrix} \dot{\lambda} \\ \dot{\pi} \end{pmatrix} = A \begin{pmatrix} \lambda - \lambda^* \\ \pi - \pi^* \end{pmatrix} \tag{33}$$

$$A = \begin{bmatrix} 0 & u_c(1-\rho') \\ A_{21} & A_{22} \end{bmatrix}$$

$$A_{21} \;=\; -\frac{u_c c^* \;\eta R^* y'' m_\lambda^p}{\gamma {y'}^2} > 0$$

$$A_{22} \;=\; r + \frac{u_c c^* \eta}{\gamma}\left[\frac{\rho'}{y'} - \frac{R^*}{{y'}^2} y'' m_\pi^p\right]$$

Proposition 5 *If fiscal policy is Ricardian and monetary policy is passive ($\rho'(\pi^*) < 1$), then there exists a continuum of perfect–foresight equilibria in which π and λ converge asymptotically to the steady state (π^*, λ^*).*

Proposition 6 *If fiscal policy is Ricardian and monetary policy is active ($\rho'(\pi^*) > 1$), then, if $A_{22} > 0 (< 0)$, there exists a unique (a continuum of) perfect–foresight equilibria in which π and λ converge to the steady state (π^*, λ^*).*

Consider a utility function that is separable logarithmic in consumption, so that $u_c c^* = 1$. In this case, the trace of A is given by

$$\text{trace}\,(A) = r + \frac{(1+\eta)\rho'}{\gamma R^*} \tag{34}$$

Let $\bar{\rho}' \equiv -\frac{r\,R^*\,\gamma}{1+\eta}$ denote the value of ρ' at which the trace vanishes. Clearly, $\bar{\rho}'$ may be greater or less than one. If $\bar{\rho}' \leq 1$, then the equilibrium is indeterminate for any active monetary policy. We highlight this result in the following corollary.

Corollary 7 *Suppose fiscal policy is Ricardian and preferences are log–linear in consumption and real balances. If* $\bar{\rho}' \equiv -\frac{r\,R^*\,\gamma}{1+\eta}$ *is less than or equal to one, then there exists a continuum of perfect–foresight equilibria in which* π *and* λ *converge to the steady state* (π^*, λ^*) *for any active monetary policy.*

Periodic perfect–foresight equilibria

Proposition 8 *Consider an economy with preferences given by* $u(c, m^{np}) = (1-s)^{-1}c^{1-s} + V(m^{np})$, *technology given by* $y(m^p) = (m^p)^{\alpha}$, *and monetary policy given by a smooth interest–rate feedback rule,* $\rho(\pi) > 0$, *which for* $\pi > \bar{\pi}$ *takes the form* $\rho(\pi) = R^* + a\,(\pi - R^* + r)$ *where* $s > 0$, $0 < \alpha < 1$, $R^* - r > \bar{\pi} > R^* - r - R^*/a$, $a > 0$, *and* $R^* > 0$. *Let fiscal policy be Ricardian and let the parameter configuration satisfy* $\bar{a} \equiv \frac{-r\alpha\gamma}{\eta}\left(\frac{\eta}{1+\eta}\frac{R^*}{\alpha}\right)^{\frac{1-\alpha s}{\alpha-1}} > 1$ *and* $1 < s < 1/\alpha$. *Then there exists an infinite number of active monetary policies satisfying* $a < \bar{a}$ *for each of which the perfect foresight equilibrium is indeterminate and* π *and* λ *converge asymptotically to a deterministic cycle.*

4.0.1 Non-Ricardian Fiscal Policy

Suppose now that the government follows the non-Ricardian fiscal policy described in equation (18), that is, a fiscal policy whereby the time path of real lump–sum taxes is exogenous. Using

$$\dot{a} = [\rho(\pi) - \pi]a \tag{35}$$
$$-\rho(\pi)\left[m^{np}(y(m^p(\lambda,\pi)),\rho(\pi)) + m^p(\lambda,\pi)\right] - \bar{\tau}.$$

(??), and (35), which can be written as

$$\begin{pmatrix} \dot{\lambda} \\ \dot{\pi} \\ \dot{a} \end{pmatrix} = \begin{bmatrix} A & 0 \\ \epsilon & r \end{bmatrix} \begin{pmatrix} \lambda - \lambda^* \\ \pi - \pi^* \\ a - a^* \end{pmatrix} \tag{36}$$

where A is defined in (33) and ϵ is a one by two vector whose elements are the steady–state derivatives of $R(m^{np} + m^p)$ with respect to λ and π.

Proposition 9 *If fiscal policy is non–Ricardian and monetary policy is passive (*$\rho'(\pi^*) < 1$*), then there exists a unique perfect–foresight equilibrium in which* $\{\lambda, \pi\}$ *converge asymptotically to the steady state* (π^*, λ^*)*.*

Proposition 10 *If fiscal policy is non–Ricardian and monetary policy is active (*$\rho'(\pi^*) > 1$*), then if* $A_{22} > 0 (< 0)$*, there exists no (a continuum of) perfect–foresight equilibria in which* $\{\lambda, \pi\}$ *converge asymptotically to the steady state* (π^*, λ^*).

In the case that monetary policy is active and both eigenvalues of A are positive, there may exist bounded equilibria that converge to a stable cycle around the steady state. Note that for the system (**??**), (**??**), and (35) the dynamics of $\{\lambda, \pi\}$ are independent of a, and thus the analysis of periodic equilibria of the previous section still applies. For example, in the special case introduced earlier if cycles exist, any initial condition for (λ, π) in the neighborhood of the steady state will converge to a cycle. To assure that a does not explode, however, we must restrict ourselves to a one dimensional manifold in $\{\lambda, \pi\}$. This follows because while cycles restricted to the $\{\lambda, \pi\}$ plane are attracting, in the three dimensional space the cycle in $\{\lambda, \pi, a\}$ will have only a two dimensional stable manifold: initial values of λ and π will have to be chosen to assure that the triple $\{\lambda, \pi, a\}$ converges to the cycle and a remains bounded.

5

2.Real indeterminacy in the Sticky-Price Model

Monetary Policy	Fiscal Policy: Ricardian	Fiscal Policy: Non-Ricardian
Passive ($\rho'(\pi^*) < 1$)	I	D
Active ($\rho'(\pi^*) > 1$)		
$A_{22} < 0$	I	I
$A_{22} > 0$	I or D	I or NE

Note: The notation is D, determinate; I, indeterminate; NE, no perfect-foresight equilibrium exists.

5.1 Backward- and forward-looking feedback rules

5.1.0 Flexible-price model

We now analyze a generalization of the interest-rate feedback rule in which the nominal interest rate depends not only on current but also on past or future rates of inflation. Consider first the following backward-looking feedback rule

$$R = \rho(q\pi + (1-q)\pi^p); \quad \rho' > 0; \; q \in [0,1] \tag{37}$$

where π^p is a weighted average of past rates of inflation and is defined as

$$\pi^p = b \int_{-\infty}^{t} \pi e^{b(s-t)} ds; \quad b > 0 \tag{38}$$

Differentiating this expression with respect to time yields

$$\dot{\pi}^p = b(\pi - \pi^p) \tag{39}$$

The rest of the equilibrium conditions are identical to those obtained earlier for a flexible price system. In particular, we have that

$$\lambda'(R)\dot{R} = \lambda(R)[r + \pi - R] \qquad (40)$$

where

$$\lambda'(R) = [u_{cc}y'm^{p\prime} + u_{cm}(m_c^{np}y'm^{p\prime} + m_R^{np})] \qquad (41)$$

Using equation (42) to eliminate π from (44) and (40) and linearizing around the steady state results in the following system of linear differential equations

$$\begin{bmatrix} \dot{R} \\ \dot{\pi}^p \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{\lambda'}\left(\frac{1}{\rho' q_1} - 1\right) & -\frac{\lambda}{\lambda'}\frac{(1-q_1)}{q_1} \\ b\frac{1}{\rho' q_1} & -\frac{b}{q_1} \end{bmatrix} \begin{bmatrix} R - R^* \\ \pi^p - \pi^* \end{bmatrix}$$

Let J denote the Jacobian matrix of this system. Because R is a jump variable and π^p is predetermined, the real allocation is locally unique if the real parts of the eigenvalues of J have opposite signs, or, equivalently, if the determinant of J is negative. On the other hand, the real allocation is locally indeterminate if both eigenvalues have negative real parts, that is, if the determinant of J is positive and its trace is negative. The determinant and trace of J are given by

$$det(J) = \frac{\lambda}{\lambda'} \frac{b}{\rho' q_1} (\rho' - 1)$$

$$trace(J) = \frac{\lambda}{\lambda'} \left(\frac{1}{\rho' q_1} - 1 \right) - \frac{b}{q_1}$$

. Now consider the two polar cases of money entering only through preferences ($y' = 0$) and money entering only through production ($u_{cm} = m_R^{np} = 0$). If money enters only through preferences and money and consumption are Edgeworth complements ($u_{cm} > 0$), then equation (41) implies that λ' is negative. It follows directly from the above two expressions that the conditions governing the local determinacy of R are identical to those obtained under a purely contemporaneous feedback rule. Namely, the equilibrium is unique under active monetary policy ($\rho' > 1$) and is indeterminate under passive monetary policy ($\rho' < 1$).

When money enters only through production or only through preferences with consumption and money being Edgeworth substitutes, λ' is positive. Thus, the equilibrium is always locally determinate under passive monetary policy, as was the case under purely contemporaneous feedback rules.

However, contrary to the case of purely contemporaneous feedback rules, if monetary policy is active, then equilibria in which R converges to its steady state may not exist. To see this, note that in this case the determinant of J is positive, so that the real parts of the roots of J have the same sign as the trace of J. However, the trace of J can have either sign. If the trace is positive, then no equilibrium exists. If it is negative, the equilibrium is indeterminate. For large enough values of ρ' the trace of J becomes negative. Thus, highly active monetary policy induces indeterminacy. Furthermore, the larger the emphasis the feedback rule places on contemporaneous inflation (q close to one) or the lower the weight it assigns to inflation rates observed in the distant past (b large), the smaller is the minimum value of ρ' beyond which the equilibrium becomes indeterminate. In the limit, as q approaches unity or b approaches infinity, the equilibrium becomes indeterminate under every active monetary policy, which is the result obtained under purely contemporaneous feedback rules. On the other hand, as the monetary policy becomes purely backward looking ($q \to 0$), no equilibrium in which R converges to its steady state exists under active monetary policy.

5.2 How backward looking should Taylor Rules be?

We now analyze a generalization of the interest-rate feedback rule in which the nominal interest rate depends not only on current but also on past or future rates of inflation. Consider first the following backward-looking feedback rule

$$R = \rho(\chi\pi + (1-\chi)\pi^p); \quad \rho' > 0; \;\; \chi \in [0, 1] \tag{42}$$

where π^p is a weighted average of past rates of inflation and is defined as

$$\pi^p = b\int_{-\infty}^{t} \pi e^{b(s-t)} ds; \quad b > 0 \tag{43}$$

Differentiating this expression with respect to time yields

$$\dot{\pi}^p = b(\pi - \pi^p) \tag{44}$$

5.2.0

Sticky-price model

The pattern that arises under sticky prices is that if monetary policy is active, the introduction of a backward-looking component in monetary policy makes determinacy more likely, whereas a forward-looking component makes indeterminacy more likely. To facilitate the analysis, we reproduce here the equilibrium conditions for the sticky-price model.

$$\begin{aligned}\dot{\lambda} &= \lambda\left[r+\pi-R\right]\\ \dot{\pi} &= r(\pi-\pi^*)-\gamma^{-1}y(m^p(\lambda,R))\lambda\left[1+\eta\left(1-\frac{R}{y'}\right)\right]\end{aligned}$$

where $m^p(\lambda, R)$ results from replacing $\rho(\pi)$ by R in equation. Also, from the first equations above:[1]

$$\dot{R}=q\rho'\dot{\pi}+b\rho'\left(\pi-\pi^*\right)-b(R-R^*).$$

[1] Take the case:

$$R=a\left[q\pi+(1-q)b\left(\int_{-\infty}^{t}\pi\left(s\right)e^{b(s-t)}ds\right)-\pi^*\right]+R^*$$

$$\begin{aligned}\dot{R} &= -b\left(R-R^*-aq\pi+a\pi^*\right)+ab(1-q)\pi+aq\dot{\pi}\\ &= -b\left(R-R^*\right)+ab\left(\pi-\pi^*\right)+aq\dot{\pi}\end{aligned}$$

Using this expression and linearizing equations (**??**) and (**??**), the evolution of λ, π, and R is described by the following system of differential equations:

$$\begin{pmatrix} \dot{\lambda} \\ \dot{\pi} \\ \dot{R} \end{pmatrix} = A \begin{pmatrix} \lambda - \lambda^* \\ \pi - \pi^* \\ R - R^* \end{pmatrix}$$

where

$$A = \begin{bmatrix} 0 & u_c & -u_c \\ A_{21} & r & A_{23} \\ \rho' q A_{21} & \rho'(b + qr) & -b + \rho' q A_{23} \end{bmatrix}$$

Note the sequence

$$-1 \quad Trace(A) \quad -B+\frac{Det(A)}{Trace(A)} \quad Det(A),$$

For local determinacy we need change in sign pattern from $(-,-,-,-)$ to $(-,-,+,-)$ as we vary b from ∞ to 0. This is because, DET remains negative, so no real roots change sign but we go from no variations in the sign pattern (no roots with positive real parts) to two variations (two roots with real parts). (See Gantmacher, vol 2, p.180: Routh' scheme, and also on page 196, Orlando's formula)

$$Det(A) = bu_c A_{21} (1-\rho')$$
$$Trace(A) = r - b + \rho' q A_{23}.$$

$$\begin{aligned} B &= \text{Sum of the principal minors of } A \\ &= -u_c A_{21}(1-q\rho') - b\,(r+\rho' A_{23})\,. \end{aligned}$$

$$
\begin{aligned}
& -B + \frac{Det(A)}{Trace(A)} \\
= \; & \left(u_c A_{21}(1 - q\rho') + b\left(r + \rho' A_{23}\right)\right) + \left(\frac{b u_c A_{21}\left(1 - \rho'\right)}{r - b + \rho' q A_{23}}\right) \\
A_{21} \; = \; & -\frac{u_c c^* \eta R^* y'' m_\lambda^p}{\gamma {y'}^2} > 0 \\
A_{23} \; = \; & \left(\frac{u_c c^* \eta}{\gamma y'}\right)\left(1 - \frac{R^*}{y'} y'' m_R^p\right).
\end{aligned}
$$

We'll assume

$$R = a\left[\chi\pi + (1-\chi)b\left(\int_{-\infty}^{t} \pi\left(s\right)e^{b(s-t)}ds\right) - \pi^*\right] + R^*$$

so $\rho' = a$.Also assume $\chi = 0$.

Transform variables:

$$q = \ell n\lambda, \quad x = \pi - \pi^* \; z = R - R^*$$

Then

$$\begin{aligned}
\dot{q} &= [r + x + \pi^* - z - R^*] = x - z \\
\dot{x}^{\cdot} &= rx - \gamma^{-1} y(m^p(e^q, z + R^*))e^q \\
&\quad \cdot \left[1 + \eta \left(1 - \frac{z + R^*}{y'(m^p(e^q, z + R^*))}\right)\right] \\
&= rx + f(q, z) \\
\dot{z} &= bax - bz = b(ax - z)
\end{aligned}$$

Note that $f\,(q, z)$ depends on the model: money in production, in utility etc. The rest is the same for different models. Also note steady state is independent of b.

Steady State:

$$(\bar{q}, \bar{x}, \bar{z}) = (\bar{q}, 0, 0) \quad where\ \bar{q}\ solves\ \frac{1+\eta}{\eta} = \frac{R^*}{y'(m^p(e^{\bar{q}}, R^*)}$$

So

$$A = \begin{bmatrix} 0 & 1 & -1 \\ A_{21} & r & A_{23} \\ 0 & ab & -b \end{bmatrix}$$

and

$$Det(A) = bA_{21}\,(1-a)$$
$$Trace(A) = r - b.$$

$B =$ Sum of the principal minors of $A = -A_{21} - b\,(r + aA_{23}$

$$-B + \frac{Det(A)}{Trace(A)} = A_{21} + b\,(r + aA_{23}) + \left(\frac{bA_{21}\,(1-a)}{r-b}\right)$$

Choose critical $\hat{b}$ to make the above quantity equal to zero. Below $\hat{b}$ do we have determinacy?

6

Conclusion

In this paper we have shown that the implications of particular interest rate feedback rules for the determinacy of equilibrium depend not only on the fiscal policy regime but also on the structure of preferences and technologies. An important consequence of this finding is that the design of monetary policy should be guided not just by the stance of fiscal policy but also by the knowledge of the deep structural parameters describing preferences and technologies, which significantly complicates the task of the monetary policy maker.

6.1 Derivation of the $\dot{\pi}$ equation

The household's lifetime utility function is assumed to be of the form

$$U^j = \int_0^\infty e^{-rt} \left[u(c^j, m^{npj}) - \frac{\gamma}{2} \left(\frac{\dot{P}^j}{P^j} - \pi^* \right)^2 \right] dt \tag{45}$$

where c^j denotes consumption of the composite good by household j, $m^{npj} \equiv M^{npj}/P$ denotes real money balances held by household j for non-productive purposes, M^{npj} denotes nominal money balances, and $\pi^* > -r$ denotes the steady–state inflation rate. The household's instant budget constraint and no-Ponzi-game restriction are

$$\dot{a}^j = (R - \pi)a^j - R(m^{npj} + m^{pj}) + \frac{P^j}{P} y(m^{pj}) - c^j - \tau \tag{46}$$

$$\lim_{t \to \infty} e^{-\int_0^t [R(s) - \pi(s)] ds} a^j(t) \geq 0 \tag{47}$$

In addition, given the price they charge, firm's sales are demand-determined

$$y(m^{pj}) = Y^d d\left(\frac{P^j}{P}\right) \tag{48}$$

The household chooses sequences for c^j, m^{npj}, m^{pj}, $P^j \geq 0$ and a^j so as to maximize utility subject to (58)–(29) taking as given $a^j(0)$, $P^j(0)$, and the time paths of τ, R, Y^d, and P. The Hamiltonian is

$$e^{-rt}\left\{\begin{array}{c} u(c^j, m^{npj}) - \frac{\gamma}{2}\left(\frac{\dot{P}^j}{P^j} - \pi^*\right)^2 \\ +\lambda^j\left[(R-\pi)a^j - R(m^{npj} + m^{pj})\right. \\ \left. +\frac{P^j}{P}y(m^{pj}) - c^j - \tau - \dot{a}^j\right] \\ +\mu^j\left[Y^d d\left(\frac{P^j}{P}\right) - y(m^{pj})\right] \end{array}\right\}$$

The first–order conditions associated with c^j, m^{npj}, m^{pj} and a^j, are, respectively,

$$u_c(c^j, m^{npj}) = \lambda^j \tag{49}$$

$$u_m(c^j, m^{npj}) = \lambda^j R \tag{50}$$

$$\lambda^j\left[\frac{P^j}{P}y'(m^{pj}) - R\right] = \mu^j y'(m^{pj}) \tag{51}$$

$$\dot{\lambda}^j = \lambda^j \left(r + \pi - R\right) \tag{52}$$

$$\lim_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]ds} a^j(t) = 0 \tag{53}$$

Combining equations (60) and (50), the demand for real balances for non-production purposes can be expressed as

$$m^{npj} = m^{np}(c^j, R) \tag{54}$$

which by assumption is increasing in c^j and decreasing in R. Consider now the first–order condition with respect to P^j. Define

$$K = e^{-rt}\left[-\frac{\gamma}{2}\left(\frac{\dot{P}^j}{P^j} - \pi^*\right)^2 + \lambda^j \frac{P^j}{P} y(m^{pj}) + \mu^j Y^d d\left(\frac{P^j}{P}\right)\right.$$

.Then the first–order condition with respect to P^j can be expressed as

$$K_{P^j} = \frac{dK_{\dot{P}^j}}{dt} \tag{55}$$

where K_{P^j} and $K_{\dot{P}^j}$ denote the partial derivatives of K with respect to P^j and $\dot{P}^j$, respectively. Letting $\pi^j \equiv \dot{P}^j / P^j$, yields

$$K_{P^j} = \frac{e^{-rt}}{P^j} \begin{bmatrix} \gamma \left(\pi^j - \pi^* \right) \pi^j + \frac{\lambda^j P^j}{P} y(m^{pj}) \\ +\mu^j \frac{P^j Y^d}{P} d' \left(\frac{P^j}{P} \right) \end{bmatrix}$$

$$K_{\dot{P}^j} = -\frac{\gamma e^{-rt}}{P^j} (\pi^j - \pi^*)$$

$$\frac{dK_{\dot{P}^j}}{dt} = \frac{\gamma e^{-rt}}{P^j} \left[(r + \pi^j)(\pi^j - \pi^*) - \dot{\pi}^j \right]$$

Equation (55) can then be written as

$$\lambda^j \frac{P^j}{P} y(m^{pj}) + \mu^j \frac{P^j}{P} Y^d d' \left(\frac{P^j}{P} \right) = \gamma r (\pi^j - \pi^*) - \gamma \dot{\pi}^j \tag{56}$$

Substituting $\lambda^j = u_c$, $\frac{\lambda^j\left[\frac{P^j}{P}y'(m^{pj})-R\right]}{y'(m^{pj})} = \mu^j$ from above, and using $P^j = P$ in equilibrium,

$$\begin{aligned} &\gamma r(\pi^j - \pi^*) - \gamma\dot{\pi}^j \\ =\ & u_c y(m^{pj}) + \left(\frac{u_c\left[y'(m^{pj}) - R\right]}{y'(m^{pj})}\right) Y^d d'\left(\frac{P^j}{P}\right) \\ =\ & u_c y(m^{pj})\left(1 + \eta\left(\frac{y' - R}{y'}\right)\right) \\ =\ & u_c y(m^{pj}))\eta\left(\frac{1+\eta}{\eta} - \frac{R}{y'}\right) \end{aligned}$$

LHS has interpretation of marginal revenue per unit $(\eta = d'\ in\, equilibrium\ are\ \ units, 1+ \eta\ is\ MR, \frac{R}{y'} = \frac{u_m}{y'u_c}$is MC per unit of foregone utility from money).

7 Optimal Monetary Policies

y is output, z is the natural rate, x is the output gap in logs: $x_t = y_t - z_t$

IS-Euler eq.:[2]

$x_t = -\phi\left[i_t - E\pi_{t+1}\right] + E_t x_{t+1} + g_t$

Phillips curve:

$$\pi_t = \lambda x_t + \beta E\pi_{t+1} + u_t$$

$$g_t \; = \; \mu g_{t-1} + \hat{g}_t$$
$$u_t \; = \; \rho u_{t-1} + \hat{u}_t$$

Iterating forward: (future beliefs affect current output)

$$x_t = E_t \sum_{i=0}^{\infty} -\phi\left[i_{t+i} - \pi_{t+1+i}\right] + g_{t+i}$$

$$\pi_t = E_t \sum_{i=0}^{\infty} \beta^i \left[\lambda x_{t+i} + u_{t+i}\right]$$

[2] If $Y_t = C_t + E_t$ where E is government consumption,

$$y_t - e_t = -\phi\left[i_t - E_t\pi_{t+1}\right] + E_t\left[y_{t+1} - e_{t+1}\right]$$

where $e_t = -\log\left(1 - \frac{E_t}{Y_t}\right).Then$

where u_{t+i} is cost push, whereas x_{t+i} is marginal costs associated with excess demand. u_{t+i} allows variations in inflation not due to excess demand.

Policy

$$\max -\frac{1}{2}E_t\sum_{i=0}^{\infty}\beta^i\left[\alpha x_{t+i}^2+\pi_{t+i}^2\right]$$

Discretion:

$$Max_{x_t,\pi_t}\ -\frac{1}{2}\left[\alpha x_t^2+\pi_t^2\right]+F_t$$

$$\begin{aligned}
ST\quad \pi_t &= \lambda x_t + f_t \\
F_t &= -\frac{1}{2}E_t\sum_{i=1}^{\infty}\beta^i\left[\alpha x_{t+i}^2+\pi_{t+i}^2\right] \\
f_t &= \beta E_t\pi_{t+1}+u_t
\end{aligned}$$

because under discretion, govt. reoptimizes each period. Solution:

$$x_t=-\frac{\lambda}{\alpha}\pi_t$$

Substituting into phillips curve:

$$
\begin{aligned}
\pi_t &= -\frac{\lambda^2}{\alpha}\pi_t + \beta E\pi_{t+1} + u_t \\
&= (1+\frac{\lambda^2}{\alpha})^{-1} E_t\beta \begin{bmatrix} (1+\frac{\lambda^2}{\alpha})^{-1}\beta E_t\pi_{t+2} \\ +(1+\frac{\lambda^2}{\alpha})^{-1}u_{t+1} \end{bmatrix} \\
&\quad +(1+\frac{\lambda^2}{\alpha})^{-1}u_t \\
&= \sum_{s=t}^{\infty}(1+\frac{\lambda^2}{\alpha})^{-(s-t+1)}u_s\beta^{s-t} \\
&= (1+\frac{\lambda^2}{\alpha})^{-1}\sum_{s=t}^{\infty}(1+\frac{\lambda^2}{\alpha})^{-(s-t)}u_t\left(\rho\beta\right)^{s-t} \\
&= \frac{u_t(1+\frac{\lambda^2}{\alpha})^{-1}}{1-(1+\frac{\lambda^2}{\alpha})^{-1}\rho\beta} = \frac{u_t}{(1+\frac{\lambda^2}{\alpha})-\rho\beta} \\
&= \frac{\alpha u_t}{\lambda^2+\alpha\left(1-\rho\beta\right)} \equiv \alpha q u_t
\end{aligned}
$$

Similarly:

$$x_t = -\lambda q u_t$$

Now find interest rule by substituting into IS curve:

$$\begin{aligned}
-\lambda q u_t &= -\phi\left[i_t - E\pi_{t+1}\right] - \lambda q\left(\rho u_t\right) + g_t \\
i_t &= E\pi_{t+1} + \phi^{-1} g_t - \phi^{-1}\lambda q\left(\rho u_t\right) \\
But\ \pi_t &= \alpha q u_t,\ E_t\pi_{t+1} = \alpha q \rho u_t \\
\phi^{-1}\lambda q\left(1-\rho\right) u_t &= \phi^{-1}\frac{\left(1-\rho\right)\lambda}{\alpha\rho} E_t\pi_{t+1} \\
i_t &= \left(1 + \frac{\left(1-\rho\right)\lambda}{\alpha\rho}\right) E\pi_{t+1} + \phi^{-1} g_t
\end{aligned}$$

Note $\left(1+\frac{(1-\rho)\lambda}{\alpha\rho}\right) > 1$, so active Taylor rule.

Under Commitment:

$$\max -\frac{1}{2} E_t \sum_{i=0}^{\infty} \beta^i \left[\begin{array}{c} \alpha x_{t+i}^2 + \pi_{t+i}^2 + \\ \phi_{t+i}\left(\pi_{t+i} - \lambda x_{t+i} - \beta\pi_{t+i+1} - u_{t+i}\right) \end{array}\right]$$

FOC

$$\begin{aligned}
-\phi_{t+i}\beta + 2\beta\pi_{t+i} + \beta\phi_{t+i+1} &= 0 \\
\phi_{t+i+1} &= \phi_{t+1} - 2\pi_{t+i+1} \\
\frac{2\alpha x_{t+i+1}}{\lambda} &= \frac{2\alpha x_{t+i}}{\lambda} - 2\pi_{t+i+1} \\
x_{t+i+1} - x_{t+i} &= -\frac{\lambda}{\alpha}\pi_{t+i+1} \\
x_t &= -\frac{\lambda}{\alpha}\pi_t
\end{aligned}$$

Now plug into IS

$$\begin{aligned}
x_t &= -\phi\left[i_t - E_t\pi_{t+1}\right] + E_t x_{t+1} + g_t \\
i_t &= E_t\pi_{t+1} + \phi^{-1}\left(E_t x_{t+1} - x_t + g_t\right) \\
i_t &= E_t\pi_{t+1} + \phi^{-1} g_t - \phi^{-1}\frac{\lambda}{\alpha}\pi_{t+1} \\
&= E_t\pi_{t+1}\left(1 - \phi^{-1}\frac{\lambda}{\alpha}\right) + \phi^{-1} g_t
\end{aligned}$$

Note the passive Taylor Rule. But note also

$$\begin{aligned}
i_t &= E_t\pi_{t+1} + \phi^{-1} g_t - \phi^{-1}\frac{\lambda}{\alpha}\pi_{t+1} \\
&= E_t\pi_{t+1}\left(1 - \phi^{-1}\frac{\lambda}{\alpha}\right) + \phi^{-1} g_t \\
&= aq\rho u_t\left(1 - \phi^{-1}\frac{\lambda}{\alpha}\right) + \phi^{-1} g_t
\end{aligned}$$

so i_t is a function of the two shocks and where $r_t = aq\rho u_t\left(1 - \phi^{-1}\frac{\lambda}{\alpha}\right) + \phi^{-1} g_t$ can be interpreted as a variable natural rate. To avoid indeterminacy add any active Taylor Rule-(off equilibrium threat) path to implement the precise optimal equilibrium :

$$i_t = aq\rho u_t\left(1 - \phi^{-1}\frac{\lambda}{\alpha}\right) + \phi^{-1} g_t + BE\pi_{t+1}, \quad B > 1$$

The Perils of Taylor Rules

Note: Figures can be found in the paper

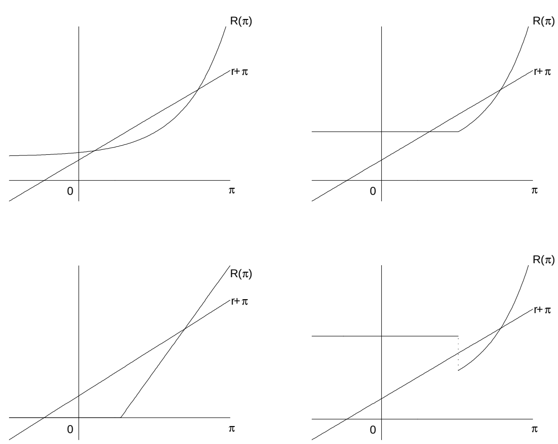
“To summarize, the research proposed her shows the surprising efficiency and robustness of simple policy rules in which the reaction of the interest rate is above a critical treshold. The analysis also shows that the estimated gains reported in some research from following alternative rules are not robust to a variety of models considered in this paper”

John Taylor,“ The robustness and efficiency of monetary policy rules as guidelines for interest rate setting by the European Central Bank,” May 1988.

8 Introduction

$$\frac{\dot{\lambda}}{\lambda} = r + \pi - R(\pi)$$
$$Steady\ State: \quad r + \pi - R(\pi) = 0$$

$$\begin{aligned} Active\ policy\ R'(\pi^*) &> 1 \\ Passive\ policy\ R'(\pi^*) &> 1 \\ R(\pi) &\geq 0 \end{aligned}$$



Taylor Rules, zero-bound on nominal rates, and multiple steady states

9 A Sticky-Price Model

Firm j :

$$\begin{aligned} Y^j &= y(h^j) = Y^d d(P^j/P) \\ d(1) &= 1 \quad d'(1) < -1. \end{aligned}$$

$$\begin{aligned} U^j &= \int_0^\infty e^{-rt} \left[u(c^j, m^j) - z(h^j) - \frac{\gamma}{2} \left(\frac{\dot{P}^j}{P^j} - \pi^* \right)^2 \right] dt \\ \pi^* &> -r. \end{aligned} \tag{57}$$

$$\dot{a}^j = (R - \pi)a^j - Rm^j + \frac{P^j}{P} y(h^j) - c^j - \tau \tag{58}$$

and

$$\lim_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]ds} a^j(t) \geq 0, \tag{59}$$

The household chooses sequences for c^j, m^j, $P^j \geq 0$, and a^j so as to maximize U^j taking as given $a^j(0)$, $P^j(0)$, and the time paths of τ, R, Y^d, and P.

The first-order conditions associated with the household's optimization problem are

$$u_c(c^j, m^j) = \lambda^j \tag{60}$$

$$u_m(c^j, m^j) = \lambda^j R \tag{61}$$

$$z'\left(h^j\right) = \lambda^j \frac{P^j}{P} y'(h^j) - \mu^j y'(h^j) \tag{62}$$

$$\dot{\lambda}^j = \lambda^j \left(r + \pi - R\right) \tag{63}$$

$$\lambda^j \frac{P^j}{P} y(h^j) + \mu^j \frac{P^j}{P} Y^d d'\left(\frac{P^j}{P}\right) = \gamma r(\pi^j - \pi^*) - \gamma \dot{\pi}^j \tag{64}$$

$$\lim_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]ds} a^j(t) = 0 \tag{65}$$

where $\pi^j \equiv \dot{P}^j / P^j$.

9.0 Monetary and Fiscal Policy

$$R = R(\pi) \equiv R^* e^{\frac{A}{R^*}(\pi - \pi^*)} \qquad (66)$$

The instant budget constraint of the government is given by

$$\dot{a} = (R - \pi)a - Rm - \tau, \qquad (67)$$

That is, the monetary-fiscal regime ensures that total government liabilities converge to zero in present discounted value for all (equilibrium or off-equilibrium) paths of the price level or other endogenous variables:[3]

$$\lim_{t \to \infty} e^{-\int_0^t [R(s) - \pi(s)]ds} a(t) = 0. \qquad (68)$$

[3] As discussed in Benhabib, Schmitt-Grohé and Uribe (1998), an example of a Ricardian monetary-fiscal regime is an interest-rate feedback rule like (73) in combination with the fiscal rule $\tau + Rm = \alpha a$; $\alpha > 0$. In the case in which $\alpha = R$, this fiscal rule corresponds to a balanced-budget requirement.

9.0 Equilibrium

$$c = y(h). \tag{69}$$

$$m = m(c, R). \tag{70}$$

Let $\eta \equiv d'(1) < -1$ denote the equilibrium price elasticity of the demand function faced by the individual firm

$$\dot{\lambda} \;=\; \lambda\left[r + \pi - R(\pi)\right]$$

$$\dot{\pi} \;=\; r(\pi - \pi^{*}) - \frac{y(h(\lambda,\pi))\lambda}{\gamma}\left[1 + \eta - \frac{\eta z'(h(\lambda,\pi))}{\lambda\, y'(h(\lambda,\pi))}\right]$$

We define a perfect-foresight equilibrium as a pair of sequences $\{\lambda, \pi\}$ satisfying **(??)** and **(??)**. Given the equilibrium sequences $\{\lambda, \pi\}$, the corresponding equilibrium sequences $\{h, c, R, m\}$ are uniquely determined
.

10 Steady-state equilibria

$$
\begin{aligned}
0 &= r + \pi - R^* e^{\frac{A}{R^*}(\pi - \pi^*)} \\
0 &= r\left(\pi - \pi^*\right) - \frac{\lambda y(h(\lambda, \pi))}{\gamma}\left(1 + \eta - \eta \frac{z'\left(h\left(\lambda, \pi\right)\right)}{\lambda y'\left(h\left(\lambda, \pi\right)\right)}\right)
\end{aligned}
$$

11 Local equilibria

$$\begin{pmatrix} \dot{\lambda} \\ \dot{\pi} \end{pmatrix} = J \begin{pmatrix} \lambda - \lambda^* \\ \pi - \pi^* \end{pmatrix}$$

$$J = \begin{bmatrix} 0 & u_c(1-A) \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{21} = \frac{y\eta}{y'\gamma}\left[\left(z'' - \frac{z'y''}{y'}\right)h_\lambda - \frac{z'}{\lambda}\right] > 0$$

$$J_{22} = r + \frac{y\eta}{y'\gamma}\left(z'' - \frac{z'y''}{y'}\right)h_\pi$$

The sign of J_{22} depends on the sign of h_π, which in turn depends on whether consumption and real balances are Edgeworth substitutes or complements. Specifically, J_{22} is positive if $u_{cm} \geq 0$, and may be negative if $u_{cm} < 0$.

If monetary policy is active at π^* ($A > 1$), the determinant of J is positive, and the real part of the roots of J have the same sign. Since both λ and π are jump variables, the equilibrium is locally determinate if and only if the trace of J is positive. It follows that if $u_{cm} \geq 0$, the equilibrium is locally determinate. If $u_{cm} < 0$, the equilibrium may be determinate or indeterminate. If monetary policy is passive at π^*, ($A < 1$), the determinant of J is negative, so that the real part of the roots of J are of opposite sign. In this case, the equilibrium is locally indeterminate.

12 Global equilibria

$$
\begin{aligned}
u(c,m) - z(h) &= \frac{\left[(xc^q + (1-x)m^q)^{\frac{1}{q}}\right]^w}{w} - \frac{h^{1+v}}{1+v} \\
q, w &\leq 1, v > 0
\end{aligned}
$$

The restrictions imposed on q and w ensure that $u(\cdot,\cdot)$ is concave, c and m are normal goods, and the interest elasticity of money demand is strictly negative.Note that the sign of u_{cm} equals the sign of $w-q$.The production function takes the form

$$y(h) = h^{\alpha}; \quad 0 < \alpha < 1$$

In the recent related literature on determinacy of equilibrium under alternative specifications of Taylor rules, it is assumed that preferences are separable in consumption and real balances (e.g., Woodford, 1996; Clarida, Galí, and Gertler, 1998). We therefore characterize the equilibrium under this preference specification first, before turning to the more general case.

12.1 Separable preferences

$q = w:$

Throughout this subsection we will assume that

$$R^* = r + \pi^*.$$

Let $p \equiv \pi - \pi^*$ and $n \equiv \ln(\lambda/\lambda^*)$.

$$\dot{n} = R^* + p - R^* e^{\left(\frac{A}{R^*}\right)p} \tag{71}$$

$$\begin{aligned} \dot{p} = & \; rp - \gamma^{-1}(1+\eta)(\lambda^*)^{\omega} e^{\omega n} x^{\alpha\theta} \\ & + \alpha^{-1}\gamma^{-1}\eta(\lambda^*)^{\beta} e^{\beta n} x^{\theta(1+v)} \end{aligned} \tag{72}$$

The main result of this subsection is: if $r, A - 1 > 0$ and sufficiently small, then there exists an infinite number of trajectories originating in the neighborhood of the steady state (λ^*, π^*) (where monetary policy is active) that are consistent with a perfect foresight equilibrium.

Proposition 11 ***(Global indeterminacy under active monetary policy and separable preferences)*** *Suppose preferences are separable in consumption and real balances ($q = w$). Then, for r and $A - 1$ positive and sufficiently small, the equilibrium exhibits indeterminacy as follows: trajectories originating in the neighborhood of the steady state $(\lambda, \pi) = (\lambda^*, \pi^*)$, where monetary policy is active, converge either to a limit cycle or to the other steady state, $(\bar{\lambda}, \bar{\pi})$, where monetary policy is passive. In the first case, the dimension of indeterminacy is two, while in the latter it is one.*

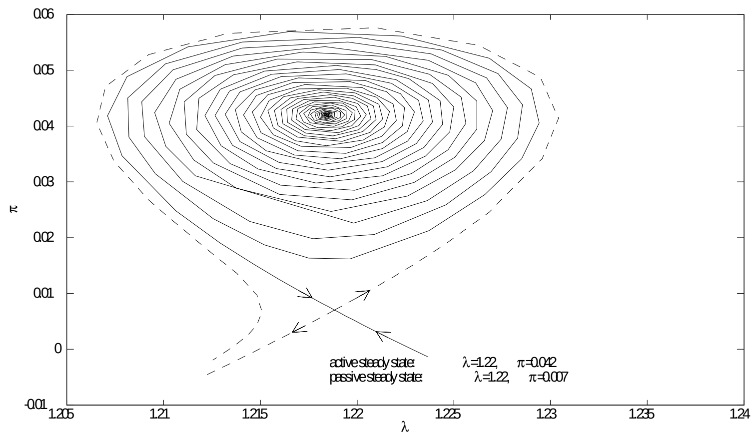
12.1.1 Simulations

$$u(c,m,h) = w^{-1}\left(\left(xc^{q}+(1-x)\,m^{q}\right)^{\frac{1}{q}}\right)^{w} - (1+v)^{-1}\,h^{(1+v)}$$
$$w \leq 1,\ q \leq 1;\ y(h) = h^{\alpha};\ R(\pi) = R^{*}e^{\left(\frac{A}{R^{*}}\right)(\pi-\pi^{*})}$$

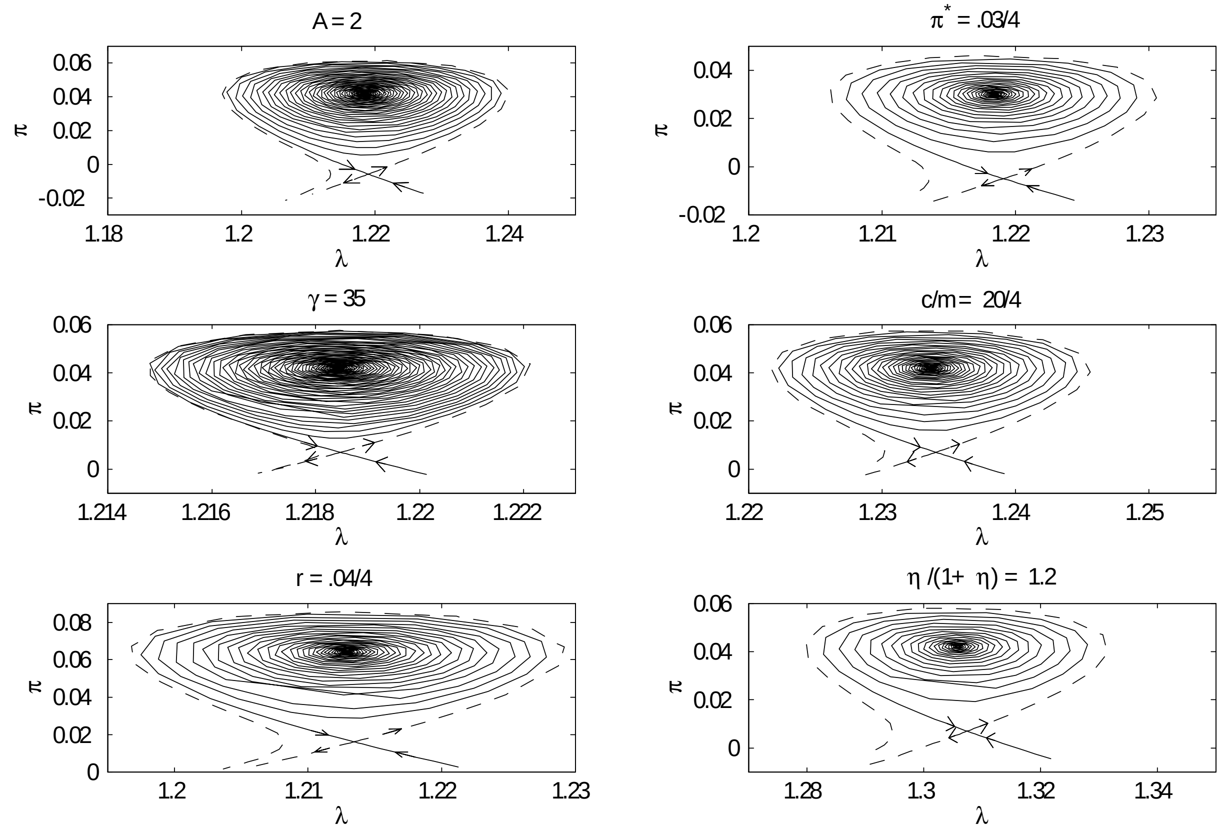
where $R^{*} = r + \pi^{*}$ so that we have a steady state $(y,n) = (0,0)$ at the specied parameter values. We start with:

$$R^{*} = 0.07; r = 0.03; \gamma = 5; \alpha = .7; A = 1.45;$$
$$\eta = -50; v = 1; w = .15; q = .15; x = .975$$

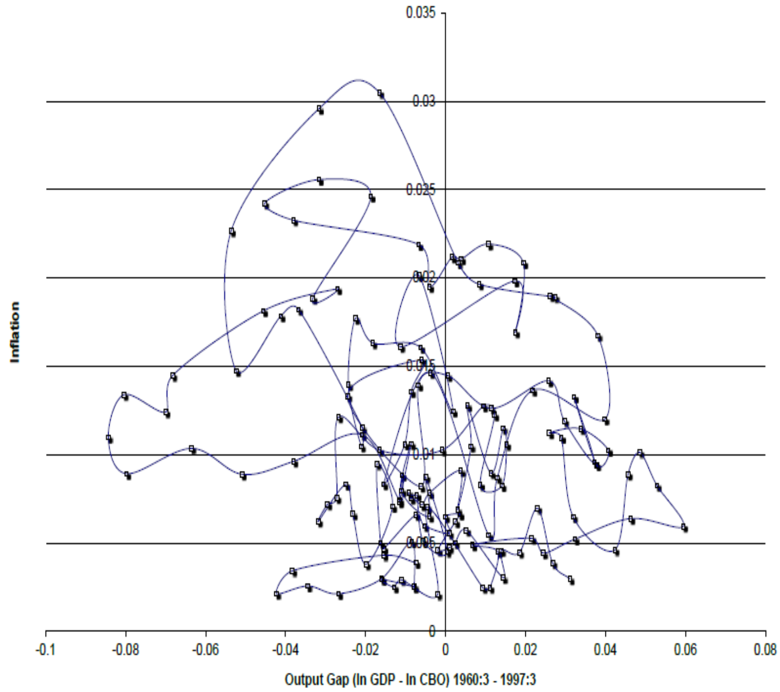
Note in particular, that money receives a very minor in the utility function, with $(1-x) = 0.025$. The coefficient of adjustment costs $\gamma = 5$, and $\eta = -50$ reflecting a fairly competitive economy. $A = 1.45$ is roughly the value specified by Taylor. Setting $\omega = q$ gives a utility function separable in money and consumption. The other parameters are standard. In general we should note that little changes in the simulations if parameters are changed, except of course where changes of stability of the steady state $(0, 0)$ occurs, corresponding to the curves H and P in Figure 5, but indeterminacy of some form always persists



Separable preferences: Saddle connection from the active to the passive steady state



Separable Preferences: Sensitivity analysis



Inflation
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
-0.1
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
Output Gap (ln GDP - ln CBO) 1960:3 - 1997:3

12.2 Non-separable pref.($q \neq w$)

$$\dot{n} = r + \pi^* + p - R^* e^{\left(\frac{A}{R^*}\right)p}$$

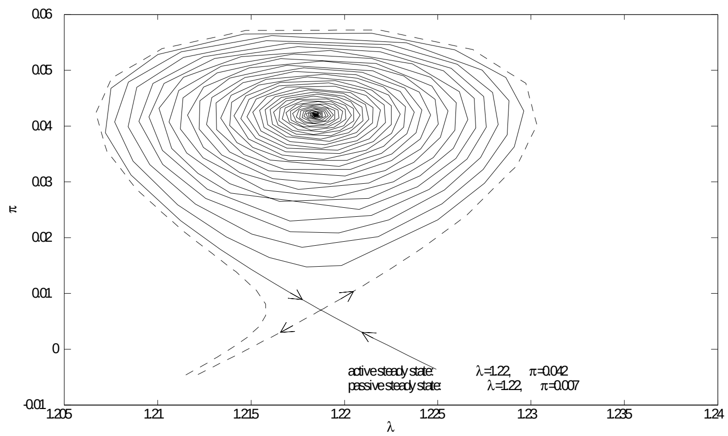
$$\begin{aligned}
\dot{p} \ = \ & rp - \frac{1+\eta}{\gamma} x^{\alpha\theta} (\lambda^*)^{\omega} e^{\omega n} \\
& \left[(R^*)^{\chi} \left(\frac{1-x}{x} \right)^{1-\chi} e^{\frac{A\chi p}{R^*}} + 1 \right]^{\alpha\xi} \\
& + \frac{\eta}{\alpha\gamma} x^{(1+v)\theta} (\lambda^*)^{\beta} e^{\beta n} \\
& \left[(R^*)^{\chi} \left(\frac{1-x}{x} \right)^{1-\chi} e^{\frac{A}{R^*}\chi p} + 1 \right]^{(1+v)\xi},
\end{aligned}$$

where n, p, β, ω are defined the previous section and $\chi \equiv q/(q-1)$, $\xi \equiv (w - q)/[\alpha q(1-w)] \neq 0$, $\theta \equiv w/[\alpha q(1-w)]$.

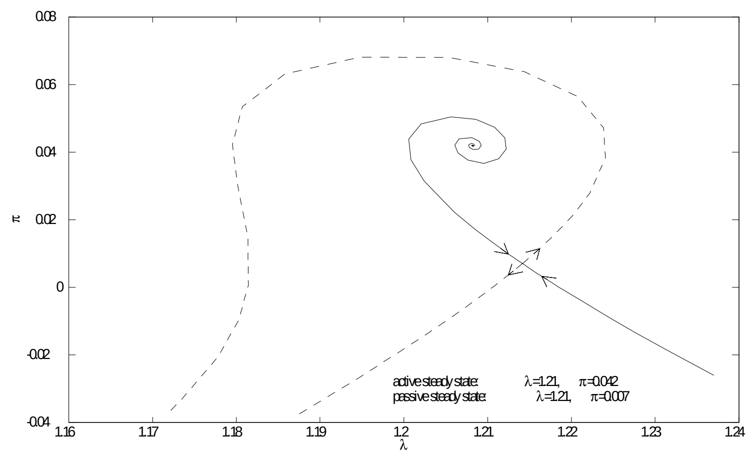
Proposition 12 *Suppose $w < 0$. Then, the steady states of the system satisfy: (i) for each steady-state value of p there exists a unique steady-state value of n; and (ii) the steady state at which monetary policy is active is either a sink or a source and the steady state at which monetary policy is passive is always a saddle.*

Proposition 13 *For parameter specifications* (r, A) *sufficiently close to* $(r^c, 1)$*, the economy with non-separable preferences exhibits indeterminacy as follows: There always exist an infinite number of equilibrium trajectories originating arbitrarily close to the steady state at which monetary policy is active that converge either to: (i) that steady state; (ii) a limit cycle; or (iii) the other steady state at which monetary policy is passive. In cases (i) and (ii) the dimension of indeterminacy is two, whereas in case (iii) it is one.*

Corollary 14 ***(Periodic equilibria)*** *If* $-\frac{1-B}{B(1+v+\alpha)} < \xi < 0$*, then there exists a region in the neighborhood of* $(r, A) = (r^c, 1)$ *for which the active steady state is a source surrounded by a stable limit cycle. On the other hand, if* $\xi > 0$ *or* $\xi < -\frac{1-B}{B(1+v+\alpha)}$*, then stable limit cycles do not exist.*



1.Non-separable preferences, w<q: saddle connection from the active to the passive steady state



Non-separable preferences, w>q: saddle connection from the active to the passive steady state

12.3 A staggered contract model of Calvo(1983)

$$R(\pi) = \frac{U_m}{U_c}$$

$$\dot{c} = \frac{c}{\varepsilon_c}(R(\pi) - r - \pi)$$

$$\dot{\pi} = b(q - c)$$

Let $n = \ell n\ c$:

$$\dot{n} = \frac{1}{\varepsilon_c}(R(\pi) - r - \pi)$$

$$\dot{\pi} = bq - ce^n$$

If we define

$$\frac{F(\pi)}{d\pi} = (r + \pi - R(\pi))$$

$$\frac{dG(n)}{dz} = bq - ce^n$$

and the Hamiltonian Function:

$$\mathcal{H} = F(\pi) - G(n)$$

solutions are level sets of $\mathcal{H}$, and

$$\frac{d\mathcal{H}}{dt} = 0.$$

Theorem (Kopell and Howard, 1975, Theorem 7.1): *Let $\dot{X} = F_{\mu,\nu}(X)$ be a two-parameter family of ordinary differential equations on R^2, F smooth in all of its four arguments, such that $F_{\mu,\nu}(0) = 0$. Also assume:*

1. $dF_{0,0}(0) \equiv A$ *has a double zero eigenvalue and a single eigenvector* e.
2. *The mapping* $(\mu, \nu) \to ($*det* $dF_{\mu,\nu}(0),$ *tr* $dF_{\mu,\nu}(0))$ *has a nonzero Jacobian at* $(\mu, \nu) = (0, 0)$.
3. *Let* $Q(X, X)$ *be the* 2×1 *vector containing the terms quadratic in the* x_i *and independent of* (μ, ν) *in a Taylor series expansion of* $F_{\mu,\nu}(X)$ *around* 0. *Then* $[dF_{(0,0)}(0), Q(e, e)]$ *has rank 2.*

Then: There is a curve $f(\mu, \nu) = 0$ such that if $f(\mu_0, \nu_0) = 0$, then $\dot{X} = F_{\mu_0,\nu_0}(X)$ has a homoclinic orbit. This one-parameter family of homoclinic orbits (in (X, μ, ν) space) is on the boundary of a two-parameter family of periodic solutions. For all $|\mu|$, $|\nu|$ sufficiently small, if $\dot{X} = F_{\mu,\nu}(X)$ has neither *a homoclinic orbit nor a periodic solution, there is a unique trajectory joining the critical points.*

The following proposition shows that the equilibrium conditions of the economy with separable preferences satisfy the hypotheses of this theorem.

322 8. Two-Parameter Bifurcations of Equilibria

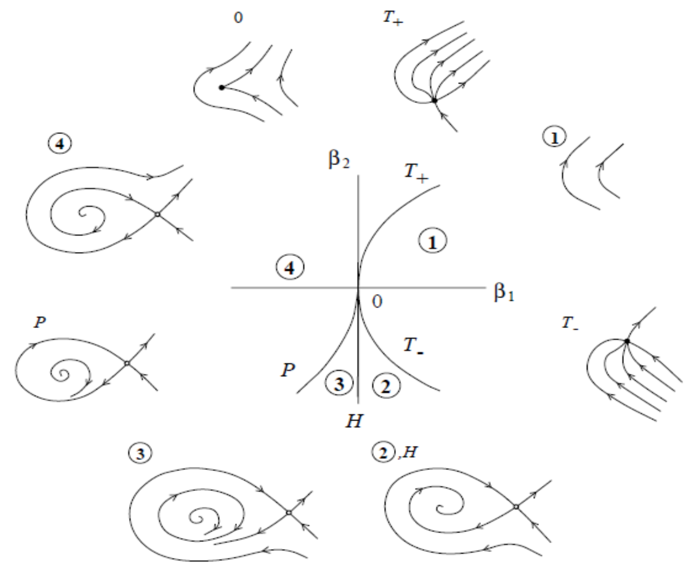


FIGURE 8.8. Bogdanov-Takens bifurcation.

Y. Kuznetsov, Elements of Applied Bifurcation Theory, Springer 1998.

Discrete time sticky price:

$$\max_{a,m,h,p^j} \sum_{t=0}^{\infty} \beta^t \left(\begin{array}{c} (1-\sigma)^{-1}(c_t)^{1-\sigma} - (1+\phi)^{-1}(h_t)^{1+\phi} \\ -\frac{\theta}{2}\left(\frac{p_t^j}{p_{t-1}^j} - \pi^*\right)^2 \end{array} \right)$$

ST

$$\begin{aligned}
\pi_t a_t + c_{t-1} &\leq R_{t-1}a_{t-1} - (1-R_{t-1})\, m_{t-1} \\
&\quad + \frac{p_{t-1}^j}{p_{t-1}} Y_t^d d\left(\frac{p_{t-1}^j}{p_{t-1}}\right) - \tau_{t-1} \\
f(m_t, h_t) &= Y_t^d d\left(\frac{p_t^j}{p_t}\right) \\
y &= f(m_t, h_t) = Y_t^d d\left(\frac{p_t^j}{p_t}\right) = c_t \\
y &= \left(a\,(m-\bar{m})^\rho + (1-a)\, h^\rho\right)^{\frac{1}{\rho}}
\end{aligned}$$

$$L = (1-\sigma)^{-1}\begin{pmatrix} R_{t-1}a_{t-1} - (1-R_{t-1})\,m_{t-1} \\ +\frac{p^j_{t-1}}{p_{t-1}}Y^d_t d\left(\frac{p^j_{t-1}}{p_{t-1}}\right) - \tau_{t-1} - \pi_t a_t \end{pmatrix}^{1-\sigma}$$

$$-(1+\phi)^{-1}(h_t)^{1+\phi} - \frac{\theta}{2}\left(\frac{p^j_t}{p^j_{t-1}} - \pi^*\right)^2$$

$$+\lambda_t\left(f(m_{t-1},h_{t-1}) - Y^d_t d\left(\frac{p^j_{t-1}}{p_{t-1}}\right)\right)$$

FOC:

$$(c_{t-1})^{-\sigma} = \beta(c_t)^{-\sigma}\frac{R_t}{\pi_t}$$

$$-\lambda_t f_m(m_{t-1},h_{t-1}) = \beta(c_t)^{-\sigma}\frac{R_t}{\pi_{t+1}} = (c_t)^{-\sigma}$$

$$\lambda_t = (c_t)^{-\sigma}\frac{R_t - 1}{R_t f_m(m_t,h_t)}$$

$$\frac{(h_t)^{-\phi}}{f_h(m_t,h_t)} = \lambda_t = (c_t)^{-\sigma}\frac{R_t-1}{R_t f_m(m_t,h_t)}$$

$$\frac{(h_t)^{-\phi}}{(c_t)^{-\sigma}} = f_h(m_t,h_t)\frac{R_t-1}{R_t f_m(m_t,h_t)} = w_t$$

$$\frac{R_t-1}{R_t w_t} = \frac{f_m(m_t,h_t)}{f_h(m_t,h_t)}$$

$$
\begin{aligned}
0 &= (c_t)^{-\sigma} \left(Y_t^d d \left(\frac{p_t^j}{p_t} \right) + \frac{p_t^j}{p_t} Y_t^d d' \left(\frac{p_t^j}{p_t} \right) \right) \left(\frac{1}{p_t} \right) \\
&\quad -\theta \left(\frac{p_t^j}{p_{t-1}^j} - \pi^* \right) \left(\frac{1}{p_{t-1}^j} \right) \\
&\quad -\lambda_t Y_t^d d' \left(\frac{p_t^j}{p_t} \right) \left(\frac{1}{p_t} \right) \\
&\quad +\beta\theta \left(\frac{p_{t+1}^j}{p_t^j} - \pi^* \right) \left(\frac{p_{t+1}^j}{p_t^j} \right) \left(\frac{1}{p_t^j} \right)
\end{aligned}
$$

$$
\begin{aligned}
0 &= (c_t)^{-\sigma} \left(Y_t^d d + Y_t^d d' \right) - \theta \left(\pi_t - \pi^* \right) \pi_t \\
&\quad -\lambda_t Y_t^d d' + \beta\theta \left(\pi_{t+1} - \pi^* \right) \pi_{t+1}
\end{aligned}
$$

Let $\eta = \frac{d'}{d}$.

$$
\begin{aligned}
0 &= Y_t^d d \left(c_t \right)^{-\sigma} (1 + \eta) - \theta \left(\pi_t - \pi^* \right) \pi_t \\
&\quad - \left(c_t \right)^{-\sigma} \frac{R_t - 1}{R_t f_m(m_t, h_t)} Y_t^d d' + \beta\theta \left(\pi_{t+1} - \pi^* \right) \pi_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
0 &= Y_t^d d \left(c_t \right)^{-\sigma} \left(1 + \eta - \frac{R_t - 1}{R_t f_m(m_t, h_t)} \eta \right) - \theta \left(\pi_t - \pi^* \right) \pi_t \\
&\quad +\beta\theta \left(\pi_{t+1} - \pi^* \right) \pi_{t+1}
\end{aligned}
$$

$$0 = Y_t^d d\left(c_t\right)^{-\sigma}\left(1+\eta-\frac{w_t}{f_h(m_t,h_t}\eta\right)-\theta\left(\pi_t-\pi^*\right)\pi_t$$
$$+\beta\theta\left(\pi_{t+1}-\pi^*\right)\pi_{t+1}$$

$$\left(\pi_t-\pi^*\right)\pi_t = \beta\left(\pi_{t+1}-\pi^*\right)\pi_{t+1}+$$
$$\theta^{-1}Y_t^d d\left(c_t\right)^{-\sigma}\eta\left(\frac{1+\eta}{\eta}-\frac{w_t}{f_h(m_t,h_t)}\right)$$

Since $Y_t^d d\left(\frac{p_t^j}{p_t}\right)=c_t$.

$$\left(\pi_t-\pi^*\right)\pi_t = \beta\left(\pi_{t+1}-\pi^*\right)\pi_{t+1}+$$
$$\theta^{-1}\left(c_t\right)^{1-\sigma}\eta\left(\frac{1+\eta}{\eta}-\frac{w_t}{f_h(m_t,h_t)}\right)$$

Eqs to solve for system in (c,π) :

$$\frac{\left(h_t\right)^{-\phi}}{\left(c_t\right)^{-\sigma}} = f_h(m_t,h_t)\frac{R_t-1}{R_t f_m(m_t,h_t)}$$
$$f(m_t,h_t)=c_t$$

The two equations above give (m_t,h_t) as functions of c_t and R_t.The Taylor rule below gives R_t as a function of $\pi_{t.}$

$$R_t = g(\pi_t)$$

Then we can substitute into the two difference equations (correspondences?) below to get a two dimensional system.

$$(c_t)^{-\sigma} = \beta (c_{t+1})^{-\sigma} \frac{R_t}{\pi_{t+1}}$$

$$\begin{aligned}(\pi_t - \pi^*)\pi_t \; = \; & \beta (\pi_{t+1} - \pi^*)\pi_{t+1} + \\ & \theta^{-1} (c_t)^{1-\sigma} \eta \left(\frac{1+\eta}{\eta} - \frac{w_t}{f_h(m_t, h_t)} \right)\end{aligned}$$

13

Avoiding Liquidity Traps

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Policies to escape or avoid liquidity traps

1.Expansionary Fical policy: Commitment to deficit spending

2.Expansionary Monetary Policy: Commitment to high monetary growth

Early Models of Indeterminacy in Monetary Models

Brock (IER, 1975): Hyperinflation and hyperdeflation

Calvo (IER, 1979) Local Indeterminacies

F. Black (JET, 1978) Indeterminacies from monetary feedback rules

Model

$$\int_0^\infty e^{-rt}u(c, M/P)dt,$$

$$Pc + P\tau + \dot{M} + \dot{B} = RB + Py.$$

Let $m \equiv M/P$ denote real balances and $a \equiv (M + B)/P$ real financial wealth. Let $u_{cm} > 0$.

$$c + \tau + \dot{a} = (R - \pi)a - Rm + y,$$

where $\pi \equiv \dot{P}/P$ is the instant rate of inflation.

$$\lim_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]ds}a(t) \geq 0$$

$$u_c(c, m) = \lambda$$

$$u_m(c, m) = \lambda R$$

$$\begin{aligned}\dot{\lambda} &= \lambda\,(r + \pi - R)\\ 0 &= \lim_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]ds}a(t)\end{aligned}$$

Monetary Policy

$$R = R(\pi). \tag{73}$$

We refer to monetary policy as active at an inflation rate π if $R'(\pi) > 1$ and as passive if $R'(\pi) < 1$.

Assumptions

$$R'(\pi) \geq 0 \quad \forall \pi.$$
$$R(\pi) \geq 0 \quad \forall \pi.$$
$$\exists\, \pi^* > -r : R(\pi^*) = r + \pi^* \text{ and } R'(\pi^*) > 1$$

Examples

$$R = R(\pi) = R^* \left(\frac{\pi}{\pi^*}\right)^{\frac{A}{R^*}}$$
$$R = R(\pi) \equiv R^* e^{\frac{A}{R^*}(\pi - \pi^*)}$$

FISCAL POLICY

$$\dot{a} = (R - \pi)a - Rm - \tau \qquad a(0) = \frac{A(0)}{P(0)}$$

RICARDIAN

$$\lim_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]ds} a(t) = 0$$

$$\tau + Rm = \alpha\, a$$

α is chosen arbitrarily by the government subject to the constraint that $\alpha \geq \underline{\alpha} > 0$. This policy states that consolidated government revenues are always higher than a certain fraction $\underline{\alpha}$ of total government liabilities. A special case is a balanced-budget rule: tax revenues are equal to interest payments on the debt: $\alpha = R$, provided R is bounded away from zero.

$$\lim_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]ds} a(t) = a(0) \lim_{t\to\infty} e^{-\int_0^t \alpha(\pi(s))ds}$$

NON-RICARDIAN

$$\frac{A(t)}{P(t)} = \int_t^{\infty} e^{-\int_t^v [R(\pi)-\pi]ds} \left\{ R(\pi\,(v))m\,(\pi\,(v)) + \bar{\tau} \right\} dv$$

LOCALLY RICARDIAN

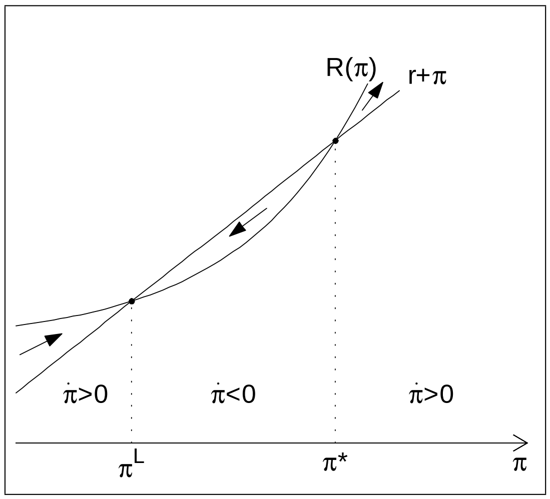
$$\tau + Rm = \alpha(\pi)a; \quad \alpha' > 0. \qquad (74)$$

Assume

$$\alpha(\pi^*) > 0, \qquad \alpha(\pi^L) \leq 0$$

FISHER EQ

$$R(\pi) = r + \pi$$



EQUILIBRIUM

$$c = y$$

Assuming that consumption and real balances are Edgeworth complements ($u_{cm} > 0$) and that the instant utility function is concave in real balances ($u_{mm} < 0$), equations

$$\lambda = L(R); \quad L' < 0.$$

$$\dot{\pi} = \frac{-L(R(\pi))}{L'(R(\pi))R'(\pi))}[R(\pi) - \pi - r]$$

$$\dot{a} = (R(\pi) - \pi - \alpha)a$$

Example 1: Targeting the growth rate of nominal government liabilities

$$\frac{\dot{A}}{A} = k,$$

where k is assumed to satisfy

$$R(\pi^L) \leq k < R(\pi^*)$$

Expressing $\dot{A}/A$ as $\dot{a}/a + \pi$ and combining the above fiscal policy rule with the instant government budget constraint yields

$$\tau + Rm = (R(\pi) - k)a$$

This fiscal policy rule is a special case of the one given when $\alpha(\pi)$ takes the form $R(\pi) - k$.

.

Under this policy, the government manages to fend off a low inflation equilibrium by threatening to implement a fiscal stimulus package consisting in a severe increase in the consolidated deficit should the inflation rate become sufficiently low. Interestingly, this type of policy prescription is what the U.S. Treasury as well as a large number of academic and professional economists are advocating as a way for Japan to lift itself out of its current deflationary trap.

Example 2: A balance-budget requirement

Consider now a fiscal policy rule consisting of a zero secondary deficit, that is:

$$P\tau = RB,$$

Recalling that $a = B/P + m$, we can rewrite the balanced budget rule as

$$\tau + Rm = R(\pi)\, a$$

It follows that a balanced budget requirement is a special case of the fiscal policy rule in which $\alpha(\pi) = R(\pi)$. Clearly, in this case $\alpha(\pi)$ is increasing because so is $R(\pi)$. Condition

$$\alpha(\pi^*) > 0$$

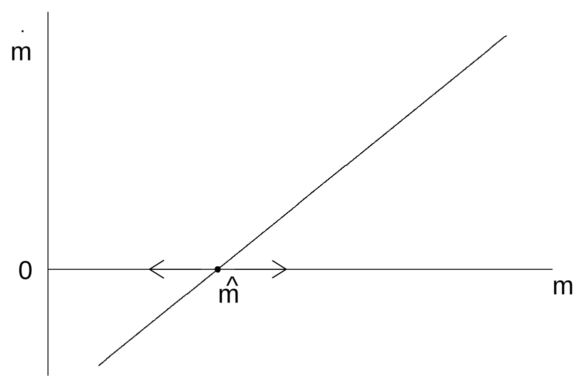
is satisfied because $R(\pi^*)$ is by assumption greater than zero. However, condition

$$\alpha(\pi^L) \leq 0$$

is only satisfied if $R(\pi^L) = 0$. This means that a balanced budget rule is effective in avoiding a liquidity trap only in the case where the Taylor rule leads the monetary authority to cut nominal rates all the way to zero at sufficiently low rates of inflation, so that the low inflation steady state occurs at a zero nominal interest rate.

Example 3. **Monetary Policy Regime Switch**

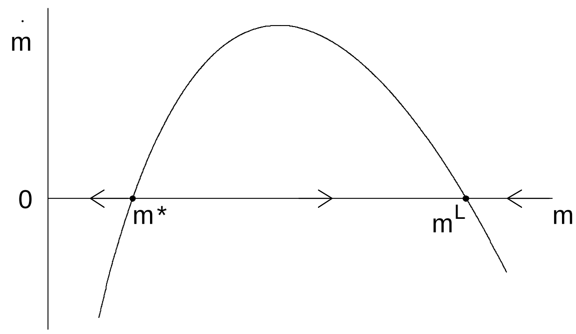
$$\frac{\dot{M}}{M} = \mu > -r \qquad\qquad \frac{\dot{m}}{m} = \mu - \pi$$



$$\dot{m} = \frac{r + \mu - \frac{u_m(y,m)}{u_c(y,m)}}{\frac{1}{m} + \frac{u_{cm}(y,m)}{u_c(y,m)}}$$

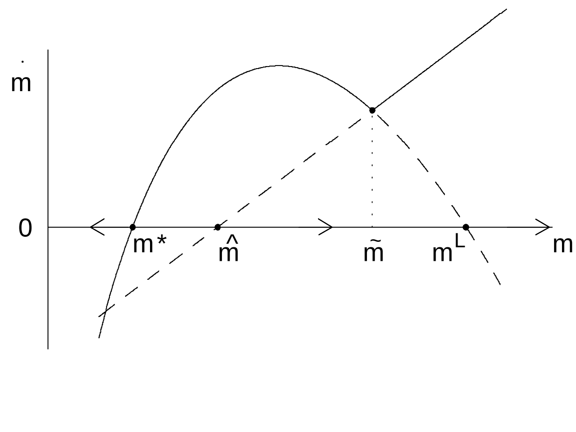
Transversality:

$$\lim_{t\to\infty} e^{-\int_0^t \left[\frac{u_m(y,m)}{u_c(y,m)} - \mu\right] ds} M(0) + e^{-\int_0^t \frac{u_m(y,m)}{u_c(y,m)} ds} B(t) = 0,$$



$$\dot{m} = \frac{u_c(y,m)}{u_{cm}(y,m)}\left[r + \pi(m) - R(\pi(m))\right],$$

$$0 \le B(t) \le \bar{B}e^{gt}; \quad \bar{B} \ge 0.$$



$$\dot{m} = \begin{cases} \frac{u_c(y,m)}{u_{cm}(y,m)} \left[r + \pi(m) - R(\pi(m))\right] & \text{for } m \leq \tilde{m} \\ \left[\frac{1}{m} + \frac{u_{cm}(y,m)}{u_c(y,m)}\right]^{-1} \left[r + \mu - \frac{u_m(y,m)}{u_c(y,m)}\right] & \text{for } m > \tilde{m} \end{cases}.$$

$$\begin{aligned} m^* &< \widehat{m} < \tilde{m} < m^L \\ \pi^L &< \mu < \pi^*. \end{aligned}$$

$$\begin{aligned} u_m(y, m^*)/u_c(y, m^*) &= r + \pi^* \\ u_m(y, \widehat{m})/u_c(y, \widehat{m}) &= r + \mu \\ u_m(y, m^L)/u_c(y, m^L) &= r + \pi^L. \end{aligned}$$

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Chaotic Interest Rate Rules

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Policies to escape or avoid liquidity traps

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Model

$$\sum_{t=0}^{\infty} \beta^t \frac{{c_t}^{1-\sigma}}{1-\sigma}$$

$$M_{t+1}+B_{t+1} = M_t+B_t(1+R_t)+p_t F\left(\frac{M_{t+1}}{p_t}\right)-p_t c_t-p_t \tau_t$$

$$c_t = \frac{M_t}{p_t}+\frac{B_t}{p_t}(1+R_t)+F\left(\frac{M_{t+1}}{p_t}\right)-\tau_t-\frac{M_{t+1}}{p_t}-\frac{B_{t+1}}{p_t} \tag{75}$$

Let $a_t \equiv (M_t + B_t)/p_t = m_t + b_t$ and let the inflation rate be defined by $\pi_t = \frac{p_{t+1}}{p_t}$.To prevent Ponzi games, households are subject to a borrowing constraint, where

$$\lim_{t\to\infty} \frac{a_t}{\prod_{j=0}^{t-1}((1+R_j)/\pi_{j+1})} \geq 0 \tag{76}$$

The marginal productivity of money at the optimum is equal to the opportunity cost of holding money:

$$F'\left(\frac{M_{t+1}}{p_t}\right) = \frac{R_{t+1}}{1+R_{t+1}}$$

$${c_t}^{-\sigma} = \frac{\beta\left(1+R_{t+1}\right)}{\pi_t}{c_{t+1}}^{-\sigma} \tag{77}$$

$$ {c_t}^{-\sigma} \left(1 - F' \left(\frac{M_{t+1}}{p_t} \right) \right) = \frac{\beta}{\pi_t} {c_{t+1}}^{-\sigma} \quad (78) $$

$$y_{t+1} = F\left(\frac{M_{t+1}}{p_t}\right) = \left(a\left(\frac{M_{t+1}}{p_t}\right)^{\rho} + (1-a)\,\bar{y}^{\rho}\right)^{\frac{1}{\rho}} \tag{79}$$

The implied money demand equation then is given by

$$\begin{aligned}\left(1 - F'\left(\frac{M_{t+1}}{p_t}\right)\right) &= \left(1 - a\left(\left(\frac{M_{t+1}}{p_t}\right)^{-1} y_{t+1}\right)^{1-\rho}\right) \\ &= (1 + R_{t+1})^{-1}\end{aligned}$$

for which real balances are a decreasing function of the nominal rate of interest R.

Note that the interest elasticity of real balances in this formulation is $1-\rho$, which is the reason that we prefer it to a Cobb-Douglas specification with unitary interest elasticity. Finally the transversality condition for the agent, the flip side of the no-Ponzi condition is given by:

$$\lim_{t\to\infty} \frac{a_t}{\prod_{j=0}^{t-1}((1+R_j)/\pi_{j+1})} \leq 0 \tag{80}$$

$$R_t = R^* \left(\frac{\pi_{t-1}}{\pi^*}\right)^{\frac{A}{R^*}}; \tag{81}$$

where $1 + R^* \equiv \pi^*/\beta > 1,\ \pi^* > 1,\ A > R^*$. Under this rule the nominal rate R_t on bonds B_t, set by the central bank in period $t-1$, depends (in the terminology of Carlstrom and Fuerst (2000)) on the forward-looking inflation rate between $t-1$ and t. Inverting this equation, we obtain:

$$\pi_t = \pi^* \left(\frac{R_{t+1}}{R^*}\right)^{\frac{R^*}{A}} \tag{82}$$

where $1 + R \equiv \pi/\beta > 1$.

FISCAL POLICY

Each period the government faces

$$M_t + B_t = M_{t-1} + (1 + R_{t-1}) B_{t-1} - P_t \tau_t$$

or

$$a_t = \frac{1 + R_{t-1}}{\pi_t} a_{t-1} - \frac{R_{t-1}}{\pi_t} m_{t-1} - \tau_t \quad (83)$$

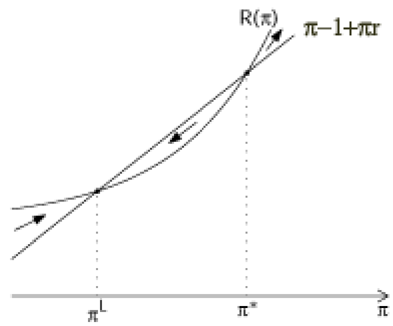
Total government liabilities in period t are given by liabilities carried over from the previous period, including interest, minus total consolidated revenues. Consolidated government revenues have two components, regular taxes and seignorage revenue. We assume that the fiscal regime consists of setting consolidated government revenues as a fraction of total government liabilities.

$$\frac{R_{t-1}}{\pi_t} m_{t-1} + \tau_t = \omega(\pi_t) a_{t-1}; \quad (84)$$

If $\omega(\cdot) > 0$ for all π, this expression implies

$$\lim_{t \to \infty} \frac{a_t}{\prod_{j=0}^{t-1} (1 + R_j) / \pi_{j+1}} = 0. \quad (85)$$

Therefore, government solvency is assured. On the other hand, if $lim_{t\to\infty} \pi_t = \pi^p$ on a candidate equilibrium path and $\omega(\pi^p) < 0$, transversality conditions will fail because the limit of discounted government liabilities held does not converge to zero. Such a fiscal policy, with $\omega(\pi^p) < 0$, will be locally non-Ricardian.



FISHER EQ

Let $\beta = (1+r)^{-1}$. At the steady state,$1 + R \equiv \pi/\beta = \pi\,(1+r) > 1$

$$R\,(\pi) = (\pi - 1) + \pi r$$

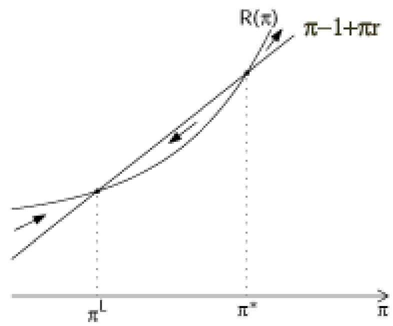
$$y_t = c_t = \left(a\left(\frac{M_{t+1}}{p_t}\right)^{\rho} + (1-a)\,\bar{y}^{\rho}\right)^{\frac{-\sigma}{\rho}}$$

$$F'\left(\frac{M_{t+1}}{p_t}\right) = \frac{R_{t+1}}{1+R_{t+1}}$$

$$\left(a\left(\frac{M_{t+1}}{p_t}\right)^{\rho} + (1-a)\,\bar{y}^{\rho}\right)^{\frac{-\sigma}{\rho}}$$

$$= \frac{\beta\,(1+R_{t+1})}{\pi^{*}\left(\frac{R_{t+1}}{R^{*}}\right)^{\frac{R^{*}}{A}}}\left(a\left(\frac{M_{t+2}}{p_{t+1}}\right)^{\rho} + (1-a)\,\bar{y}^{\rho}\right)^{\frac{-\sigma}{\rho}}$$

An equilibrium allocation is a sequence $\{\left(\frac{M_{t+1}}{p_t}\right), \pi_t, a_t\}_{t=1}^{\infty}$, given a_0, satisfying $R_t > 0$, the Euler equation, the equation for the dynamics of a,the Taylor Rule, and the transversality and no Ponzi conditions. Can write it as a difference equation in R_{t+1}.



Given a_0 and a pair of sequences $\{\left(\frac{M_{t+1}}{p_t}\right)_t, \pi_t\}_{t=1}^{\infty}$ satisfying the Euler equation, if $\omega(\cdot) > 0$, the transversality condition is satisfied. Thus, in such a case we can limit attention to the existence of sequences $\{\left(\frac{M_{t+1}}{p_t}\right), \pi_t\}_{t=1}^{\infty}$ satisfying the Euler equation and the Taylor rule. On the other hand, if $\omega(\pi) < 0$ for some π, some of the candidate equilibrium trajectories will fail to satisfy transversality, and may be ruled out.

Steady States

Consider constant solutions $x_t \equiv \left(\frac{M_t}{p_t}\right)^{\rho} = x^{ss} > 0$ to the Euler equation. Because x_t is not predetermined in period t (i.e., x_t is a "jump" variable), such solutions represent equilibrium real allocations. The steady state solutions can be more easily constructed first in terms of the nominal rate $R_t = R^{ss}$. The steady state solutions π^{ss} and $R^{ss} = R^* \left(\frac{\pi}{\pi^*}\right)^{\frac{A}{R^*}}$ must satisfy

$$1 + R^* \left(\frac{\pi}{\pi^*}\right)^{\frac{A}{R^*}} = \beta^{-1}\pi$$

By construction one such pair is (π^*, R^*) .There exists however another solution, (π^p, R^p) ,where π^p solves the above equation with $\pi^p < \pi^*$ and $0 < R^p < R^*$ and $\pi^p \equiv \beta(1 + R^p)$. It is straightforward to show that at the steady state equilibrium R^* monetary policy is active ($R'(\pi^*) > 1$), whereas at the steady-state equilibrium R^p monetary policy is passive ($\rho'(\pi^p) < 1$).

$$\left(a\left(\frac{M_{t+1}}{p_t}\right)^{\rho}+(1-a)\,\bar{y}^{\rho}\right)^{\frac{-\sigma}{\rho}}$$

$$=\frac{\beta\,(1+R_{t+1})}{\pi^*\left(\frac{R_{t+1}}{R^*}\right)^{\frac{R^*}{A}}}\left(a\left(\frac{M_{t+2}}{p_{t+1}}\right)^{\rho}+(1-a)\,\bar{y}^{\rho}\right)^{\frac{-\sigma}{\rho}}$$

Let $x_{t+1}=\left(\frac{M_{t+1}}{p_t}\right)^{\rho}$, $z_t=ax_t+(1-a)\,\bar{y}^{\rho}$, $q_t=\ell n\ z_t$, so that $z_t=e^{q_t}$. Then the Euler equation can be written as an explicit difference equation in z_t:

$$z_{t+1}=\left(\frac{\beta R^{*\frac{R^*}{A}}}{\pi^*}\frac{\left(1-a\left(\frac{az_t}{z_t-(1-a)\bar{y}^{\rho}}\right)^{\frac{1-\rho}{\rho}}\right)^{-1}}{\left(\left(1-a\left(\frac{az_t}{z_t-(1-a)\bar{y}^{\rho}}\right)^{\frac{1-\rho}{\rho}}\right)^{-1}-1\right)^{\frac{R^*}{A}}}\right)^{\frac{\rho}{\sigma}}z_t \quad (86)$$

$$q_{t+1} \;=\; q_t + \frac{\rho}{\sigma}\ell n \left(\frac{\beta R^{*\frac{R^*}{A}} \left(1 - a \left(\frac{ae^{q_t}}{e^{q_t} - (1-a)\bar{y}^{\rho}} \right)^{\frac{1-\rho}{\rho}} \right)^{-1}}{\pi^* \left(\left(1 - a \left(\frac{ae^{q_t}}{e^{q_t} - (1-a)\bar{y}^{\rho}} \right)^{\frac{1-\rho}{\rho}} \right)^{-1} - 1 \right)^{\frac{R^*}{A}}} \right.$$

$$q_{t+1} \;=\; q_t + \frac{\rho}{\sigma} h\left(e^{q_t}\right) \equiv q_t + \left(\frac{-\rho}{\sigma}\right) f\left(q_t\right) \qquad (8$$

Let $q_{t+1} = F_{\Delta}(q_t) = q_t + \Delta f(q_t)\,,\ \Delta = \frac{-\rho}{\sigma}$, where f is continuous in R^1.Assume the differential equation $\dot{y} = f(y)$ has two stationary points, one of which is asymptotically stable.

As pointed out in Yamaguti and Matano (1979), this assumption implies that after a linear transformation of variables, we have:

A) $f(0) = f(\bar{u}) = 0$ for some $\bar{u} > 0.$

B) $f(u) > 0\ for\ 0 < u < \bar{u}.$

C) $f(u) < 0\ for\ \bar{u} < u < \kappa$ where the constant κ is possibly $+\infty$.

Note that the two steady states x^* and x^p in the transformed variables q_p and q^* will correspond to the zeros of $f(q_t)$. Then defining $u_t = q_t - q_p$ provides the appropriate linear transformation required above. Furthermore, the assumption used by Yamaguti and Matano (1979) is weaker than the one above, requiring $F_{\Delta}(q_t)$ to have *at least* two steady states, or $f(u)$ to have *at least* two zeros. Requiring exactly two steady states, as above, assures that $\kappa = +\infty$, since otherwise there would be some $\tilde{u} > \bar{u}$ such that $f\left(\tilde{u}\right) = 0$, implying the existence of a third steady state. We use the fact that our stronger assumption of two steady states implies $\kappa = +\infty$ in Theorem 2 below.

THEOREM 1(Yamaguti and Matano(1979)) : There exits a positive constant c_1 such that for any $\frac{-\rho}{\sigma} > c_1$ the difference equation is chaotic in the sense of Li-Yorke (1975).

THEOREM 2(Yamaguti and Matano(1979)): Suppose the assumptions above hold and $\kappa = +\infty$. Then there exists another constant c_2 , $0 < c_1 < c_2$, such that for any $0 \leq \frac{\rho}{\sigma} \leq c_2$ (where $\Delta \equiv \left(\frac{-\rho}{\sigma}\right)$), the map $F_\Delta : q_t \rightarrow q_{t+1}$ given by $q_{t+1} = q_t + \Delta f\left(q_t\right),\ \Delta = \left(\frac{-\rho}{\sigma}\right)$, has a finite interval $[0, \alpha_\Delta]$ such that F_Δ maps $[0, \alpha_\Delta]$ into itself, with $\alpha_\Delta > \bar{u}$. Moreover, if $c_1 < \frac{-\rho}{\sigma} \leq c_2$,the chaotic phenomenon in the sense of Li-Yorke (1975) occurs in this confinement interval.

The theorems immediately apply to

our transformed equation since it has two steady states. However for an interior solution we must check that $\left(\frac{M_{t+1}}{p_t}\right)$ and $R_{t+1} = \left(1 - F'\left(\frac{M_{t+1}}{p_t}\right)\right)^{-1} - 1$, remain non-negative. This implies that there is a lower bound to $\left(\frac{M_{t+1}}{p_t}\right)$, and we must check whether the corresponding restrictions on the transforms of $\left(\frac{M_{t+1}}{p_t}\right)$, that is on $x_t,\ z_{t,}$ or q_t hold along these trajectories. In the simulations below real balances as well as the nominal interest rate remain positive and interior along the computed equilibrium trajectories.

CALIBRATION: The target nominal rate is 6%, which for a quarterly calibration implies $R^* = 0.0015$. The discount factor $\beta = (1.01/1.015) = 0.9951$. Since the target stationary inflation is given by $\pi^* = \beta(1 + R^*)$, we have $\pi^* = 1.01$, or an annual inflation target of 4%.The target nominal annual rate of 6% corresponds to the average yield on 3-month T-bills over the period 1960:Q1 to 1998:Q3. The annual inflation rate of 4% matches the average growth rate of the U.S. GDP deflator during 1960:Q1-1998:Q3, which is 4.2%.

The Taylor coefficient on the interest rate at the active steady state is given by $A/\pi^* = 1.60/1.01 = 1.5842$. The interest elasticity of money is given by $(\rho - 1)^{-1}$ .We set it to -0.1, implying a value of $\rho = -9$. The coefficient a is a measure of the productive effect of money, and can be calibrated from money demand, eq. (**??**). We set $a = 0.000359$, and the constant endowment income $\bar{y} = 1$.

The equilibrium condition (**??**) implies

$$F'\left(\frac{M_{t+1}}{p_t}\right) = \frac{R_{t+1}}{1 + R_{t+1}} = a\left(\left(\frac{M_{t+1}}{p_t}\right)^{-1} y_{t+1}\right)^{1-\rho}$$

where $\frac{R}{1+R} \approx R$. Taking logs it is easy to show that

$$\frac{d\ \ell n\left(\frac{M_{t+1}}{p_t}\right)}{d\ \ell n\left(\frac{R}{1+R}\right)} = \frac{1}{\rho - 1}$$

so that $\rho = -9$ seems reasonable.

Given an annual velocity of around 5.8, or a quarterly velocity of 1.45, yields a value for $a = 0.0003597$.For the intertemporal consumption elasticity $\sigma = 1.5$, and we explore other values as well.

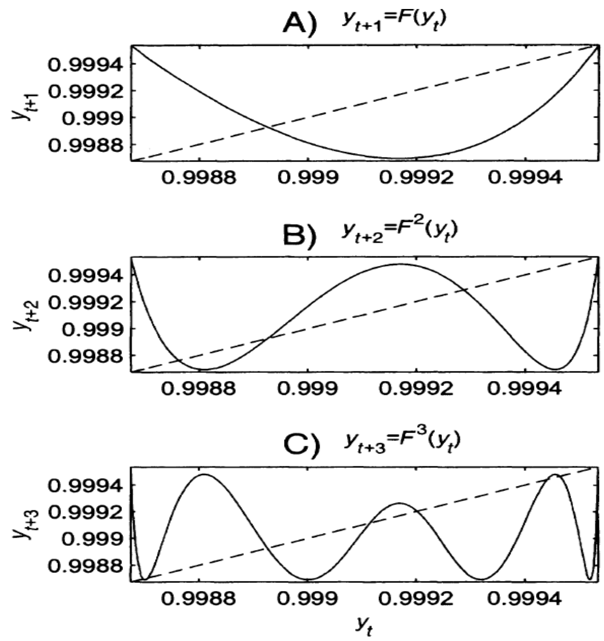


FIGURE 1. FORWARD-LOOKING TAYLOR RULES: THREE-PERIOD CYCLES

In certain cases, for which conditions are difficult to verify, the aperiodic chaotic equilibrium trajectories may exhibit ergodic chaos so that the $\lim_{t\to\infty} n^{-1}\sum_0^n x_t$ converges to a limit independently of the initial condition $x(0)$. This raises the possibility of eliminating chaotic trajectories as equilibria by designing the fiscal policy underlying the function $\omega(\cdot)$ appropriately, in analogy to the case where trajectories converging to the passive steady could be ruled out since they do not not respect government solvency or transversality conditions. In general however cyclic and aperiodic trajectories around the active steady state take values both above and below the steady state value, and it may not be possible to rule out such cyclic or chaotic trajectories as equilibria by the use of locally non-Ricardian fiscal policies.

14.0.1

Alternative Taylor Rules and Timing Conventions

14.0.2 A Linear Rule

It is important to realize however that the local properties of the ”active” steady state do not depend on the particular Taylor rule that we adopted. Consider for example the linear Taylor rule:

$$R_t = R^* + A(\frac{\pi_{t-1} - \pi^*}{\pi^*})$$

which, at the active steady state, has the same slope $\left(\frac{A}{\pi^*}\right)$ as our previous non-linear Taylor rule that respected the zero lower bound on the nominal rate. For $\sigma = 2.75$, for intial conditions in the neighborhood of the active steady state, this linear rule generates the same time-series picture as in the upper panel of Figure 1: the active steady state is indeterminate. For $\sigma = 2.65$ however, the active steady state is determinate, but now trajectories starting in a small neighborhood of it converge to a period 2 cycle, as in the upper panel of Figure 2. The amplitude of the cycle is zero at the bifurcation point $\hat{\sigma} \approx 2.709,$ and grows as σ is decreased. Thus local indeterminacy, with convergence either to the active steady state or the cycle surrounding it, does not depend on the non-linear Taylor rule.

14.0.3

Backward Looking Rules

We have also explored alternative formulations with forward looking Taylor rules, and with money entering the utility function in various non-separable ways. In such cases another potentially serious problem emerges: the instantaneous or temporary equilibrium is often non-unique. Given the Taylor rule, money balances at time t do not uniquely determine money balances at time t+1, so that the dynamics of the model, calibrated by respecting the standard parameter values, are ambigously defined even before the usual indeterminacy considerations come into play. To see thgis consider a backward looking rule given by

$$R_t = R^* \left(\frac{\pi_{t-2}}{\pi^*} \right)^{\frac{A}{R^*}}; \quad a\beta > 1, \tag{88}$$

which implies that

$$\pi_t = \pi^* \left(\frac{R_{t+2}}{R^*} \right)^{\frac{R^*}{A}} \tag{89}$$

Substituting into the Euler equation we get

$$
\begin{array}{l}
\left(a\left(\frac{M_{t+1}}{p_t}\right)^{\rho}+(1-a)\,\bar{y}^{\rho}\right)^{\frac{-\sigma}{\rho}} \\
= \left(\dfrac{\beta\left(1-a\left(a\left(\frac{M_{t+1}}{p_t}\right)^{\rho}+(1-a)\,\bar{y}^{\rho}\right)^{\frac{1-\rho}{\rho}}\left(\frac{M_{t+1}}{p_t}\right)^{\rho-1}\right)^{-1}}{\pi^{*}\left(\dfrac{\left(1-a\left(a\left(\frac{M_{t+2}}{p_{t+1}}\right)^{\rho}+(1-a)\bar{y}^{\rho}\right)^{\frac{1-\rho}{\rho}}\left(\frac{M_{t++2}}{p_{t+1}}\right)^{\rho-1}\right)^{-1}-1}{R^{*}}\right)^{\frac{R^{*}}{A}}} \right) \\
\left(a\left(\frac{M_{t+2}}{p_{t+1}}\right)^{\rho}+(1-a)\,\bar{y}^{\rho}\right)^{\frac{-\sigma}{\rho}}
\end{array}
$$

which is a difference correspondence. There often is not a unique $\left(\frac{M_{t+2}}{p_{t+1}}\right)$ defined for a given $\left(\frac{M_{t+1}}{p_t}\right)$: instantaneous equilibrium is not well-defined. The equilibrium trajectory may jump from one branch of solutions to the next, creating a more severe problem than the standard difficulties associated with multiple equilibria and indeterminacy.

14.0.4

Timing Conventions

A similar problem emerges if the production function depends on the initial rather than end of period balances: $c_t = y_t = F\left(\frac{M_t}{p_t}\right) = F\left(m_t\right)$.[4] In such a case the money and bond markets are in equilibrium if $F_m\left(m_t\right) = R_t$.The Euler equation becomes:

$$\left(F\left(m_t\right)\right)^{-\sigma}\pi_t = \beta\left(F\left(m_{t+1}\right)\right)^{-\sigma}\left(1+R_{t+1}\right)$$

With a forward looking Taylor rule this yields

$$\left(F\left(m_t\right)\right)^{-\sigma}\pi^*\left(\frac{F_m(m_{t+1})}{R^*}\right)^{\frac{R^*}{A}} = \beta\left(F\left(m_{t+1}\right)\right)^{-\sigma}\left(1+F_m\left(m_{t+1}\right)\right)$$

Again, whether we have a Cobb-Douglas or CES production function, m_t may not generate a unique m_{t+1}. If the Taylor Rule is backward looking,

$$\left(F\left(m_t\right)\right)^{-\sigma}\pi_t = \beta\left(F\left(m_{t+1}\right)\right)^{-\sigma}\left(1+R_{t+1}\right)$$

[4] For a general discussion of various timing assumptions and their impact on local determinacy properties, see Carlstrom and Fuerst (2001)

$$
\begin{aligned}
&\pi^* \left(\frac{F_m(m_{t+2})}{R^*}\right)^{\frac{R^*}{A}} \\
&= \beta \left(F\left(m_{t+1}\right)\right)^{-\sigma} \left(1 + F_m\left(m_{t+1}\right)\right) \left(F\left(m_t\right)\right)^{\sigma}
\end{aligned}
$$

$$
F_m\left(m_{t+2}\right) = \left(\begin{array}{c} \left(R^*\right)^{\frac{R^*}{A}} \left(\pi^*\right)^{-1} \beta \left(1 + F_m\left(m_t\right)\right) \\ \left(F\left(m_t\right)\right)^{\sigma} \left(F\left(m_{t+1}\right)\right)^{-\sigma} \end{array} \right)^{\frac{A}{R^*}}
$$

Equilibrium dynamics are defined by a second order difference equation and the solution can be complicated.