The Perils of Taylor Rules

Note: Figures can be found in the paper
“To summarize, the research proposed here shows the surprising efficiency and robustness of simple policy rules in which the reaction of the interest rate is above a critical threshold. The analysis also shows that the estimated gains reported in some research from following alternative rules are not robust to a variety of models considered in this paper”

2 Introduction

\[ \frac{\dot{\lambda}}{\lambda} = r + \pi - R(\pi) \]

*Steady State*: \( r + \pi - R(\pi) = 0 \)

*Active policy* \( R'(\pi^*) > 1 \)

*Passive policy* \( R'(\pi^*) > 1 \)

\( R(\pi) \geq 0 \)
3 A Sticky-Price Model

Firm $j$:

$$Y^j = y(h^j) = Y^d d(P^j / P)$$

$$d(1) = 1 \quad d'(1) < -1.$$

$$U^j = \int_0^\infty e^{-rt} \left[ u(c^j, m^j) - z(h^j) - \frac{\gamma}{2} \left( \frac{\dot{P}^j}{P^j} - \pi^* \right)^2 \right] dt$$

$$\pi^* > -r.$$

$$\dot{a}^j = (R - \pi)a^j - Rm^j + \frac{P^j}{P} y(h^j) - c^j - \tau$$

and

$$\lim_{t \to \infty} e^{-\int_0^t[R(s) - \pi(s)] ds} a^j(t) \geq 0,$$

The household chooses sequences for $c^j, m^j, P^j \geq 0$, and $a^j$ so as to maximize $U^j$ taking as given $a^j(0), P^j(0)$, and the time paths of $\tau, R, Y^d$, and $P$. 
The first-order conditions associated with the household’s optimization problem are

\[ u_c(c^j, m^j) = \lambda^j \]

\[ u_m(c^j, m^j) = \lambda^j R \]

\[ z'(h^j) = \lambda^j \frac{P^j}{P} y'(h^j) - \mu^j y'(h^j) \]

\[ \dot{\lambda}^j = \lambda^j (r + \pi - R) \]

\[ \lambda^j \frac{P^j}{P} y(h^j) + \mu^j \frac{P^j}{P} Y^d d' \left( \frac{P^j}{P} \right) = \gamma r (\pi^j - \pi^*) - \gamma \dot{\pi}^j \]

\[ \lim_{t \to \infty} e^{-\int_{0}^{t}[R(s) - \pi(s)] ds} \alpha^j(t) = 0 \]

where \( \pi^j \equiv \frac{P^j}{P^j} \).
3.0 Monetary and Fiscal Policy

\[ R = R(\pi) \equiv R^* e^{\frac{A}{R^*}(\pi - \pi^*)} \]  

The instant budget constraint of the government is given by

\[ \dot{a} = (R - \pi)a - Rm - \tau, \]  

That is, the monetary-fiscal regime ensures that total government liabilities converge to zero in present discounted value for all (equilibrium or off-equilibrium) paths of the price level or other endogenous variables:

\[ \lim_{t \to \infty} e^{-\int_0^t [R(s) - \pi(s)] ds} a(t) = 0. \]

\[ ^1 \text{As discussed in Benhabib, Schmitt-Grohé and Uribe (1998), an example of a Ricardian monetary-fiscal regime is an interest-rate feedback rule like (\ref{eq:RicardianRule}) in combination with the fiscal rule } \tau + Rm = \alpha a; \alpha > 0. \text{ In the case in which } \alpha = R, \text{ this fiscal rule corresponds to a balanced-budget requirement.} \]
3.0 Equilibrium

\[ c = y(h). \] (1) \\

\[ m = m(c, R). \] (2)

Let \( \eta = d'(1) < -1 \) denote the equilibrium price elasticity of the demand function faced by the individual firm

\[ \dot{\lambda} = \lambda [r + \pi - R(\pi)] \]

\[ \dot{\pi} = r(\pi - \pi^*) - \frac{y(h(\lambda, \pi))\lambda}{\gamma} \left[ 1 + \eta - \frac{\eta z'(h(\lambda, \pi))}{\lambda y'(h(\lambda, \pi))} \right] \]

We define a perfect-foresight equilibrium as a pair of sequences \( \{\lambda, \pi\} \) satisfying (1) and (2). Given the equilibrium sequences \( \{\lambda, \pi\} \), the corresponding equilibrium sequences \( \{h, c, R, m\} \) are uniquely determined.

4 Steady-state equilibria

\[ 0 = r + \pi - R^* e^{\hat{A}(\pi - \pi^*)} \]

\[ 0 = r (\pi - \pi^*) - \frac{\lambda y(h(\lambda, \pi))}{\gamma} \left( 1 + \eta - \eta \frac{z'(h(\lambda, \pi))}{\lambda y'(h(\lambda, \pi))} \right) \]
5 Local equilibria

\[
\begin{pmatrix}
\dot{\lambda} \\
\dot{\pi}
\end{pmatrix}
= J
\begin{pmatrix}
\lambda - \lambda^* \\
\pi - \pi^*
\end{pmatrix}
\]

\[
J = \begin{bmatrix}
0 & u_c(1 - A) \\
J_{21} & J_{22}
\end{bmatrix}
\]

\[
J_{21} = \frac{y' \eta}{y' \gamma} \left[ \left( z'' - \frac{z'y''}{y'} \right) h_\lambda - \frac{z'}{\lambda} \right] > 0
\]

\[
J_{22} = r + \frac{y' \eta}{y' \gamma} \left( z'' - \frac{z'y''}{y'} \right) h_\pi
\]

The sign of \( J_{22} \) depends on the sign of \( h_\pi \), which in turn depends on whether consumption and real balances are Edgeworth substitutes or complements. Specifically, \( J_{22} \) is positive if \( u_{cm} \geq 0 \), and may be negative if \( u_{cm} < 0 \).

If monetary policy is active at \( \pi^* \) (\( A > 1 \)), the determinant of \( J \) is positive, and the real part of the roots of \( J \) have the same sign. Since both \( \lambda \) and \( \pi \) are jump variables, the equilibrium is locally determinate if and only if the trace of \( J \) is positive. It follows that if \( u_{cm} \geq 0 \), the equilibrium is locally determinate. If \( u_{cm} < 0 \), the equilibrium may be determinate or indeterminate. If monetary policy is passive at \( \pi^* \), (\( A < 1 \)), the determinant of \( J \) is negative, so that the real part of the roots of \( J \) are of opposite sign. In this
case, the equilibrium is locally indeterminate.
6 Global equilibria

\[ u(c, m) - z(h) = \left[ \left( xc^q + (1 - x)mq^q \right)^{\frac{1}{q}} \right]^w - \frac{h^{1+v}}{1 + v} \]

\[ q, w \leq 1, v > 0 \]

The restrictions imposed on \( q \) and \( w \) ensure that \( u(\cdot, \cdot) \) is concave, \( c \) and \( m \) are normal goods, and the interest elasticity of money demand is strictly negative. Note that the sign of \( u_{cm} \) equals the sign of \( w - q \). The production function takes the form

\[ y(h) = h^\alpha; \quad 0 < \alpha < 1 \]

In the recent related literature on determinacy of equilibrium under alternative specifications of Taylor rules, it is typically assumed that preferences are separable in consumption and real balances (e.g., Woodford, 1996; Bernanke and Woodford, 1997; Clarida, Galí, and Gertler, 1998). We therefore characterize the equilibrium under this preference specification first, before turning to the more general case.
6.1 Separable preferences $q = w$:

Throughout this subsection we will assume that
\[ R^* = r + \pi^*. \]

Let $p \equiv \pi - \pi^*$ and $n \equiv \ln(\lambda/\lambda^*)$.
\[
\begin{align*}
\dot{n} &= R^* + p - R^* e^{\left(\frac{A}{R^*}\right)p} \\
\dot{p} &= rp - \gamma^{-1} (1 + \eta) (\lambda^*)^{\omega} e^{\omega n} x^{\alpha \theta} \\
&\quad + \alpha^{-1} \gamma^{-1} \eta (\lambda^*)^{\beta} e^{\beta n} x^{\theta(1+\nu)}
\end{align*}
\]

The main result of this subsection is: if $r, A - 1 > 0$ and sufficiently small, then there exists an infinite number of trajectories originating in the neighborhood of the steady state $(\lambda^*, \pi^*)$ (where monetary policy is active) that are consistent with a perfect foresight equilibrium.

Proposition 1 (Global indeterminacy under active monetary policy and separable preferences) Suppose preferences are separable in consumption and real balances ($q = w$). Then, for $r$ and $A - 1$ positive and sufficiently small, the equilibrium exhibits indeterminacy as follows: trajectories originating in the neighborhood of the steady state $(\lambda, \pi) = (\lambda^*, \pi^*)$, where monetary policy is active, converge either to a limit cycle or to the other steady state, $(\bar{\lambda}, \bar{\pi})$, where monetary policy is passive. In the first case, the dimension of indeterminacy is two, while in the latter it is one.
6.2 Non-separable preferences ($q \neq w$)

\[ \dot{n} = r + \pi^* + p - R^* e^{(A/R^*)p} \]

\[ \dot{p} = r p - \frac{1 + \eta}{\gamma} x^{\alpha\theta} (\lambda^*)^\omega e^{\omega n} \left[ (R^*)^\chi \left( \frac{1 - x}{x} \right)^{1-\chi} e^{A\chi p} + 1 \right]^{\alpha \xi} \]

\[ + \frac{\eta}{\alpha \gamma} x^{(1+v)\theta} (\lambda^*)^\beta e^{\beta n} \left[ (R^*)^\chi \left( \frac{1 - x}{x} \right)^{1-\chi} e^{A\chi p} + 1 \right]^{(1+v)\xi}, \]

where $n$, $p$, $\beta$, and $\omega$ are defined as in the previous subsection and $\chi \equiv q/(q-1)$, $\xi \equiv (w - q)/[\alpha q(1 - w)] \neq 0$, and $\theta \equiv w/[\alpha q(1 - w)]$.

Proposition 2 Suppose $w < 0$. Then, the steady states of the system satisfy: (i) for each steady-state value of $p$ there exists a unique steady-state value of $n$; and (ii) the steady state at which monetary policy is active is either a sink or a source and the steady state at which monetary policy is passive is always a saddle.
The next proposition contains the main result of this subsection. Namely, that if the steady state at which monetary policy is active is locally the unique equilibrium (i.e., the steady state is a source), then the equilibrium is globally indeterminate. Specifically, there exist equilibrium trajectories originating arbitrarily close to the steady state at which monetary policy is active that converge either to a limit cycle or to another steady state at which monetary policy is passive.

Proposition 3  For parameter specifications \((r, A)\) sufficiently close to \((r^c, 1)\), the economy with non-separable preferences exhibits indeterminacy as follows: There always exist an infinite number of equilibrium trajectories originating arbitrarily close to the steady state at which monetary policy is active that converge either to: (i) that steady state; (ii) a limit cycle; or (iii) the other steady state at which monetary policy is passive. In cases (i) and (ii) the dimension of indeterminacy is two, whereas in case (iii) it is one.

Corollary 4  \textbf{(Periodic equilibria)} If \(-\frac{1-B}{B(1+v+\alpha)} < \xi < 0\), then there exists a region in the neighborhood of \((r, A) = (r^c, 1)\) for which the active steady state is a source surrounded by a stable limit cycle. On the other hand, if \(\xi > 0\) or \(\xi < -\frac{1-B}{B(1+v+\alpha)}\), then stable limit cycles do not exist.
7 Simulations

\[ u(c, m, h) = w^{-1}\left( (xc^q + (1 - x)m^q)^{\frac{1}{q}} \right)^w - (1 + v)^{-1}h^{(1 + v)} \]

\[ w \leq 1, \quad q \leq 1; \quad y(h) = h^\alpha; \quad R(\pi) = R^*e^{(\frac{A}{R^*})(\pi - \pi^*)} \]

where \( R^* = r + \pi^* \) so that we have a steady state \((y, n) = (0, 0)\) at the specified parameter values. We start with:

\[ R^* = 0.07; \quad r = 0.03; \quad \gamma = 5; \quad \alpha = 0.7; \quad A = 1.45; \]

\[ \eta = -50; \quad v = 1; \quad w = 0.15; \quad q = 0.15; \quad x = 0.975 \]

Note in particular, that money receives a very minor in the utility function, with \((1 - x) = 0.025\). The coefficient of adjustment costs \( \gamma = 5 \), and \( \eta = -50 \) reflecting a fairly competitive economy. \( A = 1.45 \) is roughly the value specified by Taylor. Setting \( \omega = q \) gives a utility function separable in money and consumption. The other parameters are standard. In general we should note that little changes in the simulations if parameters are changed, except of course where changes of stability of the steady state \((0, 0)\) occurs, corresponding to the curves H and P in Figure 5, but indeterminacy of some form always persists.
A simulation in Figure 6, illustrates an unstable focus at \((0, 0)\) with trajectories diverging away, with one of the divergent trajectories becoming the stable manifold of the saddle \((\bar{y}, \bar{n})\), indicating one-dimensional indeterminacy. Since \((0, 0)\) is a focus, for every \(y\) inside the broken homoclinic loop, there are two values of \(n\) which places the economy on the stable manifold of \((\bar{y}, \bar{n})\).\(^2\)

A third simulation (Figure 7) shows a very similar picture, this time allowing \(u_{cm} > 0\), by setting \(w - q > 0\), with \(w = 0.15\), \(q = 0.125\). As in the previous case the trace of the Jacobian at the steady state \((0, 0)\) remains positive so that \((0, 0)\) is an unstable focus. Another simulation (not shown in a graph) with the same parameters but under the assumption \(\tilde{\pi}(t) = \pi(t)\), makes difference to the qualitative aspects of the picture and looks identical to Figure 7 because the trace at \((0, 0)\) remains positive, and \((0, 0)\) is still an unstable focus.

\(^2\) Note that if we set \(r = 0\), we obtain an integrable Hamiltonian system with closed curves an a homoclinic orbit enveloping them, as in Figure 4. Such a system arises in the model of Calvo(1983) with staggered contracts under an active policy, where the \(\tilde{\pi}\) equation depends only on output and not on \(\pi\).
Finally to illustrate the full range of bifurcations we start by considering a case (Figure 8) where $u_{cm} < 0^3$ by setting $w = 0.15$, $q = 0.4$, $w - q < 0$ first the assumption $\tilde{\pi}(t) = \pi^*$. Now the term $u_{cm} < 0$ is large enough to offset the effect of $r = 0.03$ on the trace of the Jacobian at the steady state $(0, 0)$ so the trace becomes negative. The steady state at $(0, 0)$ becomes a stable focus, and attracting points enclosed in the broken homoclinic loop, including the an unstable manifold of the steady state $(\tilde{y}, \tilde{n})$. Indeterminacy in this case we have two dimensional indeterminacy.

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3 Remember that the case with $u_{cm} < 0$ is similar to the case with no money in the utility function but money entering the production function, as in Benhabib, Schmitt-Grohe and Uribe.
If we increase the trace we observe a Hopf Bifurcation as in the bifurcation diagram in Figure 5. As the steady state \((0, 0)\) looses stability a stable cycle emerges, attracting nearby trajectories, including an unstable manifold of \((\bar{y}, \bar{n})\). Inside the periodic orbit trajectories converge to \((0, 0)\), so indeed in the limit we have \(\lim_{t \to \infty} \pi(t) = \pi^*\). Here we note that the trace is not a monotonic function of the parameter \(q\). When \(u_{cm} < 0, (\omega < q)\) the second term of the trace, \(r + M \left(\frac{A}{R^*}\right) \omega (1 - x) v\), is negative and is a non-linear and non-monotonic function of in \(q\), and when \(r > 0\), the trace can (and in this case does) move from positive to negative to positive. Thus we increase the trace when we set \(q = 0.9\), keeping other parameters constant. Figure 9 illustrates the stable cycle and the trajectories converging to it. Again we have two dimensional indeterminacy.
Finally we can cross the curve P of homoclinic loops and destroy the cycles by either increasing the trace more (say at a value of \( q = 0.151 \)), or by changing the value of the other parameter, representing a northward move in Figure 5. We illustrate this by keeping \( q = 0.9 \), but by setting \( A = 1.2 \). Figure 10 illustrates this situation, where the steady state \((0,0)\) has become an unstable focus, with one trajectory emanating from it becoming the stable manifold of \((\bar{y}, \bar{n})\), indicating one dimensional indeterminacy.\(^4\) We note that in the last two cases simply doing local analysis around the steady state \((0,0)\) yields an unstable focus and may mislead one to conclude that equilibrium is unique: it is not even locally unique because trajectories originating in the neighborhood of \((0,0)\) converge to the cycle or to the other steady state.

\(^4\) One dimensional indeterminacy has the important feature that under Non-Ricardian policies and sticky prices, if the initial value of the price is fixed, the needed to choose the initial condition to assure government solvency results in a unique equilibrium, a situation that would lead to non-existence if the equilibrium were unique under Ricardian policies. For a discussion see Benhabib, Schmitt-Grohe and Uribe (1998).
7.1 A staggered contract model of Calvo(1983)

\[
R(\pi) = \frac{U_m}{U_c}
\]

\[
\dot{c} = \frac{c}{\varepsilon_c} (R(\pi) - r - \pi)
\]

\[
\dot{\pi} = b(q - c)
\]

Let \( n = \ln c \):

\[
\dot{n} = \frac{1}{\varepsilon_c} (R(\pi) - r - \pi)
\]

\[
\dot{\pi} = bq - ce^n
\]

If we define

\[
\frac{F(\pi)}{d\pi} = (r + \pi - R(\pi))
\]

\[
\frac{dG(n)}{dz} = bq - ce^n
\]

and the Hamiltonian Function:

\[
\mathcal{H} = F(\pi) - G(n)
\]

solutions are level sets of \( \mathcal{H} \), and

\[
\frac{d\mathcal{H}}{dt} = 0.
\]
Theorem (Kopell and Howard, 1975, Theorem 7.1): Let \( \dot{X} = F_{\mu, \nu}(X) \) be a two-parameter family of ordinary differential equations on \( \mathbb{R}^2 \), \( F \) smooth in all of its four arguments, such that \( F_{\mu, \nu}(0) = 0 \). Also assume:

1. \( dF_{0,0}(0) \equiv A \) has a double zero eigenvalue and a single eigenvector \( e \).

2. The mapping \((\mu, \nu) \rightarrow (\det dF_{\mu, \nu}(0), \text{tr} dF_{\mu, \nu}(0))\) has a nonzero Jacobian at \((\mu, \nu) = (0, 0)\).

3. Let \( Q(X, X) \) be the \( 2 \times 1 \) vector containing the terms quadratic in the \( x_i \) and independent of \((\mu, \nu)\) in a Taylor series expansion of \( F_{\mu, \nu}(X) \) around 0. Then \([dF_{(0,0)}(0), Q(e, e)]\) has rank 2.

Then: There is a curve \( f(\mu, \nu) = 0 \) such that if \( f(\mu_0, \nu_0) = 0 \), then \( \dot{X} = F_{\mu_0, \nu_0}(X) \) has a homoclinic orbit. This one-parameter family of homoclinic orbits (in \((X, \mu, \nu)\) space) is on the boundary of a two-parameter family of periodic solutions. For all \(|\mu|, |\nu|\) sufficiently small, if \( \dot{X} = F_{\mu, \nu}(X) \) has neither a homoclinic orbit nor a periodic solution, there is a unique trajectory joining the critical points.

The following proposition shows that the equilibrium conditions of the economy with separable preferences satisfy the hypotheses of this theorem.
Theorem (Normal form representation [Kuznetsov, 1995, Theorem 8.4]): Suppose that a planar system
\[ \dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}^2, \]
with smooth \( f \), has at \( \alpha = 0 \) the equilibrium \( x = 0 \) with a double zero eigenvalue. Via a Taylor series expansion around \( x = 0 \) and transformation of variables, this system can be expressed as:
\[
\dot{y}_1 = y_2 + a_{00}(\alpha) + a_{10}(\alpha)y_1 + a_{01}(\alpha)y_2 \\
+ \frac{1}{2} a_{20}(\alpha)y_1^2 + a_{11}(\alpha)y_1y_2 + \frac{1}{2} a_{02}(\alpha)y_2^2 + P_1(y, \alpha)
\]
\[
\dot{y}_2 = b_{00}(\alpha) + b_{10}(\alpha)y_1 + b_{01}(\alpha)y_2 \\
+ \frac{1}{2} b_{20}(\alpha)y_1^2 + b_{11}(\alpha)y_1y_2 + \frac{1}{2} b_{02}(\alpha)y_2^2 + P_2(y, \alpha),
\]
where \( a_{lk}(\alpha), b_{lk}(\alpha) \), and \( P_{1,2}(y, \alpha) = O(\|y\|^3) \) are smooth functions of their arguments. Assume that
\( a_{00}(0) = a_{10}(0) = a_{01}(0) = b_{00}(0) = b_{10}(0) = b_{01}(0) = 0 \)
and that the following nondegeneracy conditions are satisfied:
(BT.0) the Jacobian matrix \( A(0) = \frac{\partial f}{\partial x}(0, 0) \neq 0 \);
(BT.1) \( a_{20}(0) + b_{11}(0) \neq 0 \);
(BT.2) \( b_{20}(0) \neq 0 \);
(BT.3) the map
\[
(x, \alpha) \mapsto \left( f(x, \alpha), \text{tr} \left( \frac{\partial f(x, \alpha)}{\partial x} \right), \text{det} \left( \frac{\partial f(x, \alpha)}{\partial x} \right) \right)
\]
is regular at point \( (x, \alpha) = (0, 0) \).
Then there exist smooth invertible variable trans-