Lucas 1988 JME

$$\max_{c(t)} \int_0^\infty \left(\frac{c^{1-\sigma} - 1}{1-\sigma} \right) N(t) e^{-\rho t} dt$$

subject to:

$$Nc + \dot{K} = AK^\beta N^{1-\beta}, \quad \frac{\dot{A}}{A} = \mu$$

$$H = N\left(\frac{c^{1-\sigma} - 1}{1-\sigma} \right) + \theta(AK^\beta N^{1-\beta} - Nc)$$

where c is consumption, N is labor force that grows at the rate λ, K is physical capital, ρ is a positive discount factor, A and δ are positive technology parameters, β is the share of capital,and σ is the inverse of the intertemporal elasticity of substitution.

$$H = N\left(\frac{c^{1-\sigma} - 1}{1-\sigma} \right) + \theta(AK^\beta N^{1-\beta} - Nc)$$

$$c^{-\sigma} = \theta$$

$$\dot{\theta} = \theta(\rho - \beta AN^{1-\beta}K^{\beta-1}) \qquad \lim_{t\to\infty} e^{-\rho t}\theta K = 0$$

Now, along a **balanced growth** path where $\frac{\dot{c}}{c} \equiv \kappa$:

$$\frac{\dot{\theta}}{\theta} = -\frac{\sigma c^{-\sigma-1}}{c^{-\sigma}}\dot{c} = -\sigma\frac{\dot{c}}{c} = -\sigma\kappa$$

But from FOC,

$$\frac{\dot{\theta}}{\theta} = (\rho - \beta A N^{1-\beta} K^{\beta-1})$$

$$-\frac{\dot{\theta}}{\theta} + \rho = \beta A N^{1-\beta} K^{\beta-1}; \quad MPK = \rho + \sigma\kappa$$

If production is Cobb-Douglas:

$$MPK = \beta\frac{F(K, N)}{K} = \beta\frac{AN^{1-\beta}K^{\beta}}{K}$$

$$\frac{Nc + \dot{K}}{K} = A\left(\frac{N}{K}\right)^{1-\beta} = \beta^{-1}MPK = \frac{\rho + \sigma\kappa}{\beta}$$

So, if $\frac{\dot{K}}{K}$ is constant along a balanced growth path, so is $\frac{Nc}{K}$. This implies that

$$\frac{\dot{N}}{N} + \frac{\dot{c}}{c} = \frac{\dot{K}}{K} = \lambda + \kappa$$

($\lambda + \kappa$ is capital growth, and also output growth defined as the growth of Nc.)

Now some Algebra: On a BGP

$$\frac{dA(\frac{N}{K})^{1-\beta}}{dt} = \dot{A}\left(\frac{N}{K}\right)^{1-\beta} + A(1-\beta)\left(\frac{N}{K}\right)^{-\beta}\left(\frac{N\dot{K}-K\dot{N}}{K^2}\right)$$

$$\frac{\frac{dA\left(\frac{N}{K}\right)^{1-\beta}}{dt}}{A(\frac{N}{K})^{1-\beta}} = \frac{\dot{A}}{A} + (1-\beta)\left(\frac{\dot{N}}{N} - \frac{\dot{K}}{K}\right) = \mu + (1-\beta)(\lambda - \kappa - \lambda)$$

$$= 0$$

$$\frac{\mu}{1-\beta} = \kappa$$

Thus, κ is independent of ρ or σ.

Let the savings rate

$$s = \frac{\dot{K}}{Nc + \dot{K}}$$

$$\frac{Nc + \dot{K}}{K} = \frac{\rho + \sigma\kappa}{\beta}$$

$$s^{-1} = \left(\frac{Nc + \dot{K}}{K}\right)\frac{K}{\dot{K}} = \frac{\rho + \sigma\kappa}{\beta(\kappa + \lambda)}$$

$$s = \frac{\beta(\kappa + \lambda)}{\rho + \sigma\kappa}$$

Saving rate does not affect κ.

Checking Transversality along a balanced growth path:

$$\frac{\dot{K}}{K} = \kappa + \lambda; \quad K(t) = K(0)e^{(\kappa+\lambda)t}$$

$$\frac{\dot{\theta}}{\theta} = -\sigma\kappa \qquad \theta(t) = \theta(0)e^{-\sigma\kappa t}$$

$$\lim_{t\to\infty} e^{-\rho t}K(0)\theta(0)e^{(\kappa+\lambda-\sigma\kappa)} = 0$$

$$if\ \sigma\kappa + \rho > \kappa + \lambda$$

$$\kappa(\sigma - 1) + \rho - \lambda > 0$$

CALIBRATION:

$$\lambda = 0.013, \quad (1-\beta) = .75 \quad s = 0.1$$

$$\kappa + \lambda = (0.024,\ 0.029),\ say\ 0.027 \qquad \kappa = 0.014$$

$$\mu = \kappa(1-\beta) = 0.0105$$

Then, from $s = \frac{\beta(\kappa+\lambda)}{\rho+\sigma\kappa}$,

$$\rho + (0.014)\sigma = 0.0675$$

$$if \quad \sigma = 1, \quad \rho = 0.0535$$

Problem that remains: How to account for persistent growth rate differentials between countries? Also, technology, or the production function implies that per capita output and capital growth are related by, from the production function,

$$\frac{Y}{N} = A\left(\frac{K}{N}\right)^{\beta}$$

$$g_{yt} - \beta g_{kt} = \mu$$

which does not seem uniform across countries. Introducing another factor, like knowledge or human capital may help, if combined with the removal of diminishing returns in its production.

Finally, a good share of persitent growth is accounted by μ. What is it?

Introduce Human Capital

$$N = \int_0^\infty N(h)dh$$

Effective labor force, with fraction $u(h)$ working in production:

$$N^c = \int_0^\infty N(h)u(h)hdh$$

Production and wage:

$$Y = F(N^c, K), \quad Wage = F_N h$$

$$Earnings = F_N h u(h)$$

Average skill:

$$h_a = \frac{\int_0^\infty N(h)hdh}{\int_0^\infty N(h)dh}$$

If all workers have same skill, and choose same u, $N^c = uhN$.

Production, with external effect:

$$N(t)c(t) + \dot{K}(t) = AK^\beta [uh(t)N(t)]^{1-\beta}(h_a(t))^\gamma$$

Human capital accumulation:

$$\dot{h} = \delta h(1-u)$$

PLANNER: Directly to Hamiltonian

$$Max_{c,u}\ \left(\frac{c^{1-\sigma}-1}{1-\sigma}\right)$$
$$+\theta_1\left(AK^{\beta}[uh(t)N(t)]^{1-\beta}(h_a(t))^{\gamma}-Nc\right)$$
$$+\theta_2(\delta h(1-u))$$

FOC

$$\theta_1 = c^{-\sigma}$$

$$\theta_2\delta h = \theta_1(1-\beta)AK^{\beta}[uh(t)N(t)]^{-\beta}(h_a(t))^{\gamma}(hN)$$

$$\dot{\theta}_1 = \rho\theta_1-\theta_1\beta AK^{\beta-1}[uh(t)N(t)]^{1-\beta}(h_a(t))^{\gamma}$$

$$\dot{\theta}_2 = \rho\theta_2-\theta_1 AK^{\beta}[uN(t)]^{1-\beta}h(t)^{-\beta+\gamma}(1-\beta+\gamma)$$
$$-\theta_2\delta(1-u)$$

For the equilibrium solution however:

$$\dot{\theta}_2 = \rho\theta_2-\theta_1\beta AK^{\beta}[uN(t)]^{1-\beta}h(t)^{-\beta+\gamma}(1-\beta)$$
$$-\theta_2\delta(1-u)$$

BGP: Using, as before

$$\frac{\dot{\theta}_1}{\theta_1} = -\kappa\sigma$$

$$\frac{Nc + \dot{K}}{K} = AK^{\beta-1}[uh(t)N(t)]^{1-\beta}(h_a(t))^{\gamma}$$

$$= \frac{MPK}{\beta} = \frac{\rho + \sigma\kappa}{\beta} \qquad (*)$$

Again, if $\frac{\dot{K}}{K}$ is constant, so is $\frac{Nc}{K}$, and s is also constant. Let

$$v = \frac{\dot{h}}{h} = \delta(1 - u)$$

Differentiating $(*)$ above with A fixed,

$$(\beta - 1)\frac{\dot{K}}{K} + (1 - \beta + \gamma)\frac{\dot{h}}{h} + (1 - \beta)\frac{\dot{N}}{N} = 0$$

$$(\beta - 1)(\kappa + \lambda) + (1 - \beta + \gamma)v - (\beta - 1)\lambda = 0$$

$$\left(\frac{(1 - \beta + \gamma)}{(1 - \beta)}\right)v = \kappa$$

If $\gamma = 0$, $\kappa = v$, $(1 - \beta + \gamma)v$ is like μ in previous model: $\kappa = \frac{\mu}{1-\beta}$

DERIVATION OF v (assuming at first we are working on the efficient path so the planner takes γ into account.)

Differentiating FOC wrt u

$$\theta_1(1-\beta)AK^{\beta}[uh(t)N(t)]^{-\beta}(h_a(t))^{\gamma}(hN) = \theta_2\delta h$$

$$\frac{\dot{\theta}_1}{\theta_1} + \beta\frac{\dot{K}}{K} + (1-\beta)\frac{\dot{N}}{N} + (1-\beta+\gamma)\frac{\dot{h}}{h} = \frac{\dot{\theta}_2}{\theta_2} + \frac{\dot{h}}{h}$$

$$-\sigma\kappa + \beta(\kappa+\lambda) + (1-\beta)\lambda + (\gamma-\beta)v = \frac{\dot{\theta}_2}{\theta_2}$$

$$(\beta-\sigma)\kappa + \lambda + (\gamma-\beta)v = \frac{\dot{\theta}_2}{\theta_2}$$

Now lets get $\left(\frac{\dot{\theta}_2}{\theta_2}\right)$ from FOC.

$$\dot{\theta}_2 = \rho\theta_2 - \theta_1 AK^{\beta}[uN(t)]^{1-\beta}h(t)^{\gamma-\beta}(1-\beta-\gamma)$$
$$- \theta_2\delta(1-u)$$

To get at $\left(\frac{\dot{\theta}_2}{\theta_2}\right)$, we must eliminate midle term with θ_1. But from FOC wrt u,

$$\theta_1(1-\beta)AK^{\beta}[uh(t)N(t)]^{-\beta}(h_a(t))^{\gamma}(hN) = \theta_2\delta h$$

Multiplying both sides by $\left(\frac{(1-\beta+\gamma)}{1-\beta}\right)\left(\frac{u}{h}\right)$:

$$\theta_1 AK^{\beta}[uN(t)]^{1-\beta}(h_a(t))^{(\gamma-\beta)}(1-\beta+\gamma)$$
$$= \frac{\theta_2\delta(1-\beta+\gamma)u}{1-\beta}$$

Substituting into FOC for the state equation in θ_2

$$
\begin{aligned}
\dot{\theta}_2 &= \rho\theta_2 - \frac{\theta_2\delta(1-\beta+\gamma)u}{1-\beta} - \theta_2\delta(1-u) \\
\frac{\dot{\theta}_2}{\theta_2} &= \rho - \delta\left(1 + \frac{\gamma}{1-\beta}\right)u - \delta(1-u) \\
&= \rho - \delta u - \frac{\gamma\delta u}{1-\beta} - \delta(1-u) \\
&= \rho - \frac{\gamma\delta u}{1-\beta} - \delta
\end{aligned}
$$

But since

$$
v = \delta(1-u) = \delta - \delta u
$$

$$
\frac{\dot{\theta}_2}{\theta_2} = \rho + \left(\frac{\gamma(v-\delta)}{1-\beta}\right) - \delta
$$

Now plugging this into the equation where we had differentiated the FOC wrt u, which gave us the other equation for $\frac{\dot{\theta}_2}{\theta_2}$:

$$
(\beta-\sigma)\kappa + \lambda + (\gamma-\beta)v = \frac{\dot{\theta}_2}{\theta_2}
$$

$$
(\beta-\sigma)\kappa + \lambda + (\gamma-\beta)v = \rho + \left(\frac{\gamma(v-\delta)}{1-\beta}\right) - \delta
$$

Solving:

$$
\begin{aligned}
v^* &= -\sigma^{-1}\left[\frac{(1-\beta)(\rho-\delta-\lambda)-\delta\gamma}{(1-\beta+\gamma)} \right] \\
&= -\sigma^{-1}\left[\frac{(1-\beta)(\rho-\lambda)}{(1-\beta+\gamma)} - \frac{(1-\beta)\delta+\gamma\delta}{(1-\beta+\gamma)} \right] \\
&= -\sigma^{-1}\left[\frac{(1-\beta)(\rho-\lambda)}{(1-\beta+\gamma)} - \delta \right]
\end{aligned}
$$

Note: if $\gamma = 0, v^* = \sigma^{-1}(\delta + \lambda - \rho)$.

However, on the competitive equilibrium path,agents take $(h_a)^\gamma$ as exogenous:

$$\dot{\theta}_2 = \rho\theta_2 - \frac{\theta_2\delta(1-\beta)u}{1-\beta} - \theta_2\delta(1-u)$$

$$\frac{\dot{\theta}_2}{\theta_2} = \rho - \delta$$

$$v = -\frac{[(1-\beta)(\rho-\delta-\lambda)-\delta\gamma]}{\sigma(1-\beta+\gamma)-\delta}$$

$$If \quad \gamma = 0, \; \kappa = v, \quad If \quad \sigma = 1,$$

$$v^* - v = \frac{\gamma}{(1-\beta+\gamma)}(\rho-\lambda)$$

Computing Balanced Growth Path:

$$\frac{Nc + \dot{K}}{K} = AK^{\beta - 1}[uh(t)N(t)]^{1-\beta}(h_a(t))^{\gamma} = \frac{\rho + \sigma\kappa}{\beta}$$

$$\rho + \sigma\kappa$$

$$= \beta A(K(0)e^{(\kappa+\lambda)t})^{\beta-1}[uh(0)e^{vt}N(0)e^{\lambda t}]^{1-\beta}(h(0)e^{vt})^{\gamma}$$

$$= \beta A(K(0)e^{\kappa t})^{\beta-1}[uN(0)]^{1-\beta}(h(0)e^{vt})^{(1-\beta+\gamma)}$$

$$= \beta A(K(0))^{\beta-1}[uN(0)]^{1-\beta}$$

$$\bullet\ (h(0))^{(1-\beta\gamma)}e^{\kappa(\beta-1)t}e^{(1-\beta+\gamma)vt}$$

But, on a BGP we have

$$(1-\beta+\gamma)v = (1-\beta)\kappa$$

so $e^{\kappa(\beta-1)t}e^{(1-\beta+\gamma)vt} = e^0 = 1$ and

$$\beta A(K(0))^{\beta-1}[uN(0)]^{1-\beta}(h(0))^{(1-\beta+\gamma)} = \rho + \sigma\kappa$$

which defines the BGP relation.

Let initial conditions correspond to BGP levels, $K(0) = z_1,\ h(0) = z_2$.

$$\beta A(z_1)^{\beta-1}[uN(0)]^{1-\beta}(z_2)^{(1-\beta+\gamma)} = \rho + \sigma\kappa$$

Then

$$z_1 = K(t)e^{-(\kappa+\lambda)t}$$

$$z_2 = h(t)e^{-\nu t}$$

along the BGP. Can draw graph of z_1 against z_2.(increasing). Position will depend on υ, κ, ρ. A higher ν, will imply a higher κ, and shift the curve to the right, so for given z_1, we would get a higher z_2.

So

$$v = -\frac{[(1-\beta)(\rho-\delta-\lambda)-\delta\gamma]}{\sigma(1-\beta+\gamma)-\delta}$$

$$\kappa = \left(\frac{(1-\beta+\gamma)}{(1-\beta)}\right)v$$

$$v = \delta(1-u);\ \ u = \frac{\delta - v}{\delta}$$

Calibration:

Denison who gives output or capital growth at roughly $0.027 = \kappa + \lambda$

$$\lambda = 0.013,\ \kappa = 0.014,$$

$$\beta = 0.25,\quad \delta = 0.05,$$

$$and\ v = 0.009 \text{ (from Denison)}$$

$$v = \delta(1-u)\ \ \rightarrow u = 0.82$$

From $\frac{(1-\beta+\gamma)}{(1-\beta)}v = \kappa,$ we get

$$\gamma = 0.417$$

which is enormous. As before $\rho + \sigma\kappa = .0675$

Also Lucas computes, from

$$v^* = -\sigma^{-1}\left[\frac{(1-\beta)(\rho-\lambda)}{(1-\beta+\gamma)} - \delta \right]$$

σ	v^*	u^*	κ^*
1	.024	.52	.037
2	.016	.68	.025
3	.014	.72	.022

Note that as a result, u is too high. Optimally, we need more time in education. This model explains the US data not much better than Solow's model: by choosing exogenous technical change μ, Solow's model does well, but the Lucas model can be consistent with permanent differentials in income and growth rates (note that in the model with human capital v depends on preference parameters ρ, σ) across countries.

Multiple equilibria and BGP: **Luck**?

$$Max_{n(i),L(i),\{a(i)\}} \int_0^\infty \Big(\ell n(A(n(i))^\beta) - L(i)\Big) e^{-\rho t} dt$$

$$\dot{A} = A^{1-\eta}(a(i))^\eta (L(i) - n(i))^\gamma (L-n)^\theta,$$

$$initial\ \ a(i)\ given$$

A is aggregate human capital, L is aggregate labor, n is aggregate labor allocated to human capital production. Agents take $A^{1-\eta}$ as given and optimize wrt private human capital $a(i)$, $\int_0^1 a(i)di,\ i \in [0,1]$, labor $L(i),\ L = \int_0^1 L(i)di,$ and the allocation of labor $n(i),\ n = \int_0^1 n(i)di.$ Then $A^{1-\eta}(L-n)^\theta$ represents the externalities/spillovers from total human capital and total labor allocated to human capital production.

Supressing index i,

$$H = (\ell n(An^{\beta}) - L) + \lambda(A^{1-\eta}a^{\eta}(L-n)^{\gamma}(L-n)^{\theta})$$

$$H_L = 0 \rightarrow 1 = \gamma A\lambda(L-n)^{\theta+\gamma-1}$$

$$H_n = 0 \rightarrow \frac{\beta}{n} = \gamma A\lambda(L-n)^{\theta+\gamma-1}$$

$$\beta = n; \; (L-n) = (\lambda A\gamma)^{\frac{1}{1-\gamma-\theta}}$$

$$(L-n)^{\gamma+\theta} = (\lambda A\gamma)^{\frac{\gamma+\theta}{1-\gamma-\theta}}$$

$$\dot{A} = A(L-n)^{\theta+\gamma}$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial a} = \lambda(\rho - \eta(L-n)^{\theta+\gamma}) - A^{-1}$$

Let $z = \lambda A$:

$$\dot{z} = z(\rho - \eta(L-n)^{\theta+\gamma}) + z(L-n)^{\theta+\gamma} - 1$$

$$\dot{z} = z(\rho + (1-\eta)(L-n)^{\theta+\gamma}) - 1$$

$$\dot{z} = z\left(\rho + (1-\eta)(\lambda A\gamma)^{\frac{\gamma+\theta}{1-\gamma-\theta}}\right) - 1$$

$$\dot{z} = z\rho + (1-\eta)(z\gamma)^{\frac{\gamma+\theta}{1-\gamma-\theta}+1} - 1$$

$$\dot{z} = \rho z + Mz^{\frac{1}{1-\theta-\gamma}} - 1$$

$$0 = \lim_{t\to\infty} e^{-\rho t} z(t)? \quad \textit{Transversality}\,?$$

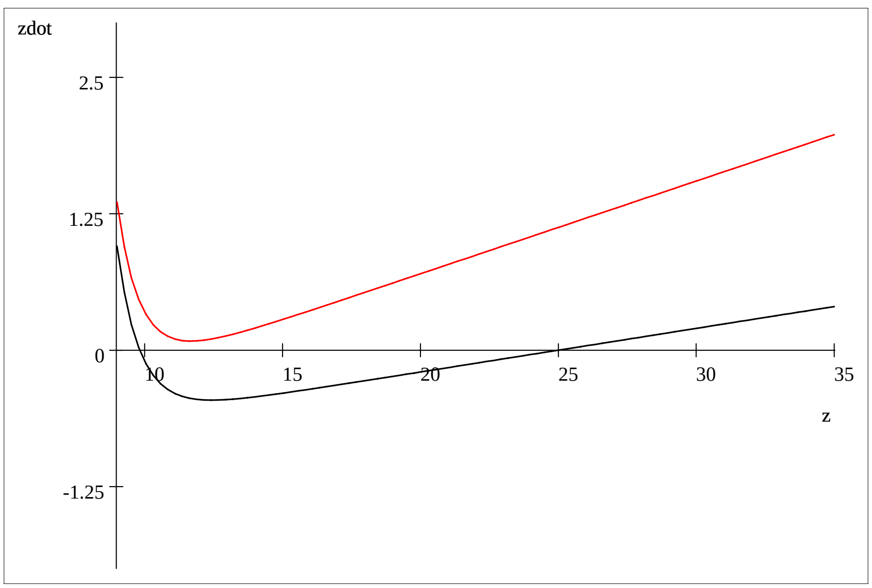
BGP for z

Let $\theta + \gamma > 1,$ a high labor externality.

$$\rho z + Mz^{\frac{1}{1-\theta-\gamma}} = 1$$

Note that $\rho z + Mz^{\frac{1}{1-\theta-\gamma}}$ is concave. Either two or no balanced growth solutions except on measure zero. Examples:

$$\rho z + (1-.5)\left((0.1)^{\frac{1.1}{-0.1}}\right)z^{\frac{1.1}{-0.1}} = 1$$



Black:$\rho =.04,$ Red: $\rho =.085$

But lower BGP is stable, $\lim_{t\to\infty} e^{-\rho t}z(t) = 0.$ How to determine initial $\lambda,$ which gives L? Any $\lambda(0)$ between 0 and the higher BGP converges to lower BGP. Indeterminacy.

A simple endogenous growth structure:

On a balanced growth path, along which the ratio of the asset to consumption will be constant:

$$c = sk$$

Utility of consumption is logarithmic, and that the production function is of the Cobb-Douglas form,

$$y = k^{\alpha}\bar{k}^{(1-\alpha)}L^{\beta}; \quad \frac{\partial y}{\partial k} = \alpha L^{\beta} \equiv w_1(L)$$

where $\bar{k}$ represents an external effect. Euler equation, where r is the discount rate and g is population growth plus depreciation, is:

$$\frac{\dot{c}}{c} = (w_1(L) - (r + g))$$

The goods market equilibrium as

$$\frac{\dot{k}}{k} = \frac{y}{k} - s - g = a(L) - s - g = L^{\beta} - s - g$$

Since s is a constant along balanced growth, the difference between right sides of the two equations above must be zero:

$$a(L) - s - w_1(L) + r = r - s + a(L)(1 - \alpha) = 0$$
$$s = r + a(L)(1 - \alpha)$$

The second equality follows because the marginal and average products of capital, $a(L)$ and $w_1(L)$, are proportional, and in our Cobb-Douglas example their difference is $a(L)(1 - \alpha)$. We can also express s as a function of L by using the labor market equilibrium condition given by equation where marginal utilty times marginal product of labor equals marginal utilty of leisure:

$$c^{-1} w_0(L) = V'(1 - L)$$
$$s = \frac{c}{k} = \frac{w_0(L)}{k} \frac{1}{V'(1 - L)} = \frac{m(L)}{V'(1 - L)}$$
$$= v(L)$$

where $m(L) = \frac{w_0(L)}{k}$ is the marginal product of labor divided by k. Substituting this expression into $s = r + a(L)(1 - \alpha)$ we have:

$$r = v(L) - a(L)(1 - \alpha)$$

This equation can have one, two or no solutions corresponding to the balanced growth paths, depending on the parameters of the model. The right-hand side of equation monotonic in L if $v(L)$ is decreasing, but if $v(L)$ is increasing, there may be two balanced growth paths. An increasing $v(L)$ however is only possible if the marginal product of labor is increasing, and this requires a significant labor externality.