OLG

\[ \begin{align*}
& \text{Max } U (c_0 (t), c_1 (t + 1)) \\
& S.T. \ (e_0 - c_0 (t)) \rho_t = (c_1 (t + 1) - e_1) \\
& \text{FOC: } \\
& \frac{U_0 (c_0 (t), c_1 (t + 1))}{U_1 (c_0 (t), c_1 (t + 1))} = \rho_t \\
& (e_0 - c_0 (t)) \left( \frac{U_0 (c_0 (t), c_1 (t + 1))}{U_1 (c_0 (t), c_1 (t + 1))} \right) = (c_1 (t + 1) - e_1) \\
& \text{Market Clearing: } \\
& c_0 (t) + c_1 (t) = e_0 + e_1 \\
& e_0 - c_0 (t) = c_1 (t) - e_1 \\
& \text{Equilibrium: } \\
& (e_0 - c_0 (t)) \left( \frac{U_0 (c_0 (t), e_0 + (e_1 - c_0 (t + 1)))}{U_1 (c_0 (t), e_0 + (e_1 - c_0 (t + 1)))} \right) \\
& = (e_0 - c_0 (t + 1))
\end{align*} \]
Under some assumptions, this is a first order difference equation:

Note: if

\[ U(c_0(t), c_1(t + 1)) = V(c_0(t)) + W(c_1(t + 1)), \]

then

\[ (c_0(t) - c_0) V'(c_0(t)) = W'(c_1(t + 1)) (c_1 - c_1(t + 1)) \]

If \( (c_0(t) - c_0) > 0 \) (implying \( (c_1 - c_1(t + 1)) > 0 \)) RHS is large for small positive \( c_1(t + 1) \) and decreasing, so that by the implicit function theorem, \( c_1(t + 1) = F(c_0(t)) \). Similar argument if utility is homothetic, that is \( u_0/U_1 \) is a decreasing function of \( (c_0/c_1) \).
Let $\gamma = 1$, that is Population growth is zero.
Samuelson Case
Steady States:
1. Autarkic, \((e_0 - c_0) = 0\);
2. Trading \(\rho = \left( \frac{U_0(c_0, e_0 + e_1 - c_0)}{U_1(c_0, e_0 + e_1 - c_0)} \right) = 1\), golden rule.

TRY
1. \(U = ac_0 - b (c_0)^2 + c_1\)
\[a, b > 0, \ (e_0, e_1) = (0, w), \ w > \frac{a}{b}.\]

Try \(a = 4, 2, 1.5\)

2. \(U = \frac{\lambda(c_0 + b)^{1-a}}{1-a} + c_1,\)
\[a > 0, \ a \neq 1, \ \lambda, b > 0;\]

Let \(c_0 - e_0 > 0.\) Try \(\lambda = 51, \ a = 5.1\)

OLG WITH GOVERNMENT

\[\text{Max} \ V(c_0(t)) + c_1(t + 1)\]

S.T.
\[e_0 + e_1 - g = c_0(t) + c_1(t)\]
\[c_1(t) - e_1 = e_0 - c_0(t) - g\]
\[(e_0 - c_0(t + 1) - g) = V_0(c_0(t))(e_0 - c_0(t))\]
\[x(t + 1) = x(t) V_0(e_0 - x(t)) + g\]

Does an increase in government expenditures and taxes increase or decrease savings?
Try interpretation with money:

\[ m_t = p_t (e_0 - c_0(t)) \]

\[ r_t m_t + p_{t+1} e_1 = p_{t+1} (c_1(t + 1)) \]

\[ \frac{r_t p_t}{p_{t+1}} (e_0 - c_0(t)) = (c_1(t + 1) - e_1) \]

Max \( U \left( c_0(t), e_1 + \frac{r_t p_t}{p_{t+1}} (e_0 - c_0(t)) \right) \)

\[ \frac{r_t p_t}{p_{t+1}} = \frac{U_0 (c_0(t), c_1(t + 1))}{U_1 (c_0(t), c_1(t + 1))} \]

\[ e_0 - c_0(t + 1) = \frac{U_0 (c_0(t), c_1(t + 1))}{U_1 (c_0(t), c_1(t + 1))} (e_0 - c_0(t)) \]

\[ \frac{m_{t+1}}{p_{t+1}} = \frac{U_0 \left( e_0 - \frac{m_t}{p_t}, e_1 + \frac{m_{t+1}}{p_{t+1}} \right) }{U_1 \left( e_0 - \frac{m_t}{p_t}, e_1 + \frac{m_{t+1}}{p_{t+1}} \right) } \left( \frac{m_t}{p_t} \right) \]

<table>
<thead>
<tr>
<th></th>
<th>Trading</th>
<th>Autarkic</th>
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<tbody>
<tr>
<td>Samuelsonian</td>
<td>Unstable, PO</td>
<td>Stable, Not PO</td>
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<tr>
<td>Classical</td>
<td>Stable, Not PO</td>
<td>Unstable, Not PO</td>
</tr>
<tr>
<td>( \rho = 1, e_0 \neq c_0 )</td>
<td>( \rho \neq 1, e_0 = c_0 )</td>
<td></td>
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</tbody>
</table>
DIAMOND

Max \( (1 - \beta) \ln c_0 + \beta \ln (w - c_0) (1 + R) \)

FOC

\[
\frac{1 - \beta}{c_0} = \frac{\beta}{w - c_0}
\]

\((1 - \beta)(w - c_0) = \beta c_0; \quad (w - c_0) = \beta w\)

\[f(k) = k^\alpha + (1 - \delta) k\]

\[k_{t+1} = \frac{\beta w}{1 + n} = \frac{\beta}{1 + n} (f - f'k) = \frac{\beta}{1 + n} (1 - \alpha) k^\alpha\]

SS (Why only one?)

\[k = \left(\frac{\beta (1 - \alpha)}{1 + n}\right)^{\frac{1}{1 - \alpha}}\]

\[\bar{k} = \left(\frac{\beta (1 - \alpha)}{1 + n}\right)^{\frac{1}{1 - \alpha}}\]

Golden Rule:

\[(1 + n) k_{t+1} = (k_t)^\alpha + (1 - \delta) k_t - c_t\]

\[c_t = (k_t)^\alpha + (1 - \delta) k_t - (1 + n) k_{t+1}\]

\[c' = \alpha k^{\alpha - 1} - (\delta + n) = 0\]

\[k^* = \left(\alpha^{-1} (\delta + n)\right)^{\frac{1}{\alpha - 1}}\]
Note possible inefficiency \((\alpha k^{\alpha-1} \text{ does not contain } \delta)\):

\[
\alpha k^{\alpha-1} = \alpha \left( \frac{(1 + n)}{\beta (1 - \alpha)} \right) \geq (\delta + n)
\]

\[
\bar{k} = \left( \frac{\beta (1 - \alpha)}{1 + n} \right)^{\frac{1}{1-\alpha}} \geq (\alpha^{-1} (\delta + n))^{\frac{1}{\alpha-1}} = k^*
\]

**Stability of SS**

\[
\frac{dk_{t+1}}{dk_t} = \frac{\alpha \beta (1 - \alpha)}{1 + n} k^{\alpha-1}
\]

\[
= \alpha \left( \frac{\beta (1 - \alpha)}{1 + n} \right) \left( \frac{\beta (1 - \alpha)}{1 + n} \right)^{-1} = \alpha < 1
\]

**Simple OLG:**

\[
k_{t+1} = S (r (k_t), w (k_t))
\]

\[
\frac{dk_{t+1}}{dk_t} = \frac{1 - s_r r'}{w'} > 0 \quad s_r, w' > 0, \quad r' < 0
\]
OLG with endogenous labor, consumption in second period:

\[ \max U( wtLtR_{t+1}) - V(L_t) \]

FOC and Dynamics:

\[ U'(w(k_t, L_t), L_tR(k_{t+1}, L_{t+1})) w(k_t, L_t) R(k_{t+1}, L_{t+1}) = V'(L_t) \]

\[ k_{t+1} = w(k_t, L_t) L_t \]

These are two difference equations in \((k_t, L_t)\).
Simplify by assuming log utility:

\[ \max (1 - \beta) \ln(1 - L_t) + \beta \ln (L_tw_tR_t) \]

FOC

\[ \frac{1 - \beta}{1 - L_t} = \frac{\beta}{L_t}; \quad L_t = \beta \]

Production (note change in exponents of Prod. fn): Golden Rule

\[ \max_k c(k) = k^{1 - \alpha} \beta^\alpha + (1 - \delta) k - k \]
\[ \delta = (1 - \alpha) k^{-\alpha} \beta^\alpha \]
\[ k^* = \delta^{-\frac{1}{\alpha}} (1 - \alpha)^{\frac{1}{\alpha}} \beta \]
Dynamics

\[ k_{t+1} = w_t L_t = \alpha (k_t)^{1-\alpha} \beta^\alpha \]

Steady State

\[ \bar{k} = (\alpha \beta^\alpha)^{\frac{1}{\alpha}} = \alpha^{\frac{1}{\alpha}} \beta \]

Stability

\[ \frac{dk_{t+1}}{dk_t} = \alpha (1 - \alpha) (\bar{k})^{-\alpha} \beta^\alpha = 1 - \alpha < 1 \]

Note: With log utility, dimension is reduced by 1.
OLG with Money

\[ Max \ U (c_{t+1}) - V (L_t) \]

S.T.

\[ k_{t+1} + M_{t+1}Q_t = w_tL_t \]
\[ c_{t+1} = R_{t+1}k_{t+1} + M_{t+1}Q_{t+1} \]

FOC and dynamics

\[ R_{t+1}w_tU' (R_{t+1}w_tL_t) = V' (L_t) \]
\[ k_{t+1} + M_{t+1}Q_t = w_tL_t \]
\[ Q_{t+1} = R_{t+1}Q_t \]

These are three difference equations in \((L, k, Q)\)

Note that there are two steady states, with \(Q = 0\) and with \(Q \neq 0, \ R = 1\) (golden rule).
Simplify with Log Utility and \( F = k^{1-\alpha} \beta^\alpha + (1 - \delta) k \). Then \( L = \beta \).

Dynamics:

\[
k_{t+1} = \alpha (k_t)^{1-\alpha} \beta^\alpha - M_{t+1} Q_t
\]
\[
Q_{t+1} = R_{t+1} Q_t
\]

At golden rule steady state:

\[
R \left( \hat{k}, \beta \right) = (1 - \alpha) \left( \hat{k} \right)^{-\alpha} \beta^\alpha = 1
\]
\[
\hat{k} = (1 - \alpha)^{\frac{1}{\alpha}} \beta
\]

Note that \( MQ \geq 0 \) since:

\[
\hat{k} = \alpha \left( \hat{k} \right)^{1-\alpha} \beta^\alpha - MQ
\]
\[
\alpha \left( \hat{k} \right)^{1-\alpha} \beta^\alpha = \alpha \left( (1 - \alpha)^{\frac{1}{\alpha}} \beta \right)^{1-\alpha} \beta^\alpha
\]
\[
= \left( \frac{\alpha}{1 - \alpha} \right) (1 - \alpha)^{\frac{1}{\alpha}} \beta \geq (1 - \alpha)^{\frac{1}{\alpha}} \beta = \hat{k}
\]
0.1 Stationary Sunspot Eq. In a Finance Constrained Economy, JET 1986 (Hand to Mouth Workers?)

Workers:

\[ \max_{\{c_t\}} \sum_{t=1}^{\infty} \gamma^{t-1} u(c_t^w) - \gamma^t v(n_t) \]

\[ u' + cu''(c) > 0, v(0) = 0. \]

\[
    p_t \left( c_t^w + (k_t^w - dk_{t-1}^w) \right) \leq M_t^w + r_t k_{t-1}^w \\
    p_t \left( c_t^w + (k_t^w - dk_{t-1}^w) \right) + M_{t+1}^w \leq M_t^w + r_t k_{t-1}^w + w_t n_t \\
    u(c_t^w) = u_t^w \left( \frac{M_t^w + r_t k_{t-1}^w + w_t n_t - M_{t+1}^w - p_t \left( k_t^w - dk_{t-1}^w \right)}{p_t} \right) \\
    -u'(c_t^w) \frac{1}{p_t} + \gamma E u'(c_{t+1}^w) \frac{1}{p_{t+1}} \leq 0 
\]

First constraint is a finance constraint stating that a given period’s expenditures must be financed either out of money held at the beginning of the period, or by borrowing against the value of capital held then. (Both the rents \( r_t k_{t-1}^w \) and wages \( w_t n_t \), are received only at the end of period \( t \) but the former can be borrowed against by pledging the capital as collateral, while future wage payments cannot be borrowed against, because of moral hazard problems.) The depreciation factor is \( d \), where a fraction \( 1 - d \) of capital depreciates. Second constraint is the usual budget constraint.

FOC with respect to \( k_{t+1} \): Look at equilibria such that:

\[ u'(c_t^w) > \gamma_t E \left[ u'(c_{t+1}^w) \left( \frac{r_{t+1}}{p_{t+1}} + d \right) \right] \]

Also return to holding money \( \frac{p_t}{p_{t+1}} \) is less than return capital \( \frac{r_{t+1} + dp_{t+1}}{p_{t+1}} \):

\[ r_{t+1} + dp_{t+1} > p_t, \quad \frac{r_{t+1}}{p_{t+1}} + d > \frac{p_t}{p_{t+1}} \]

so workers set \( k_t^w = 0 \) since current marginal utility of forgoing a unit of current consumption exceeds discounted expected utility obtained by investing it (they can’t borrow to consume more today), even though the net return on investing capital is positive: \( \frac{r_{t+1} + dp_{t+1}}{p_{t+1}} > 1 \). Then they maximize a static labor market problem

\[
    \max_{n_t} E \left[ u(c_{t+1}) - v(n_t) \right] \quad \text{S.T.} \quad p_{t+1} c_{t+1} = w_t n_t 
\]

to get (since workers set \( k_t^w = 0 \), \( c_{t+1} = \frac{w_t n_t}{p_{t+1}} = \frac{M_t}{p_{t+1}} \), \( u'(c_{t+1}) \frac{p_t}{p_{t+1}} = v'(n_t) \)):

\[ n_t u'(n_t) = V(n_t) = E \left[ u'(c_{t+1}) \frac{w_t n_t}{p_{t+1}} \right] = E \left[ c_{t+1} u'(c_{t+1}) \right] = E \left[ U \left( \frac{w_t n_t}{p_{t+1}} \right) \right] \]
\( U(c) = cu'(c) \). (Like olg with a month; workers consume their wage bill \( \frac{w_t n_t}{p_t + 1} = \frac{M_{t+1}}{T} + 1 \))

Capitalists

\[
\sum_{t=1}^{\infty} \beta^{t-1} \ln c_t^e
\]

ST

\[ p_t (c_t^e + (k_t^e - dk_{t-1}^e)) + M_{t+1}^e \leq M_t^e + r_t k_t^e \]

Solution:

\[
c_t^e = (1 - \beta) \left( \frac{r_t}{p_t} + d \right) k_{t-1}
\]

\[
k_t = \beta \left( \frac{r_t}{p_t} + d \right) k_{t-1}
\]

with

\[
\frac{r_t}{p_t} + d > bd, \quad b = \beta^{-1},
\]

so

\[
k_t = \beta \left( \frac{r_t}{p_t} + d \right) k_{t-1} > \beta (\beta^{-1} d) k_{t-1} = dk_{t-1}
\]

so that \( k_t > dk_{t-1} \), all capital from last period is held and then added to–no decumulation. Thus Capitalists hold no money if \( \frac{r_t}{p_t} + d > bd \).

Fixed Coeff. production: Must use \( m \) units of labor with each unit of capital to produce \( a > b - d \) units of new output plus \( d \) units of old undepreciated capital. (Needed so steady state capital >0).

EQ:

\[
w_t n_t = M_{t+1}^w = M: \quad \text{Money Market}
\]

\[
n_t = mk_{t-1} \quad \text{Labor Market}
\]

Since \( pc_t^w = M_t^w = w_{t-1} n_{t-1} \) and \( c_t^e = (1-\beta) \left( \frac{r_t}{p_t} + d \right) k_{t-1} = (1-\beta) \left( \frac{r_t}{p_t} + d \right) \frac{b_k}{\beta (\frac{r_t}{p_t} + d)} = (b-1) k_t \),

\[
c_t^w + c_t^e + k_t = \frac{M}{p_t} + bk_t = (a + d) k_{t-1} = \text{output}
\]

\[
\frac{M}{p_t} = (a + d) k_{t-1} - bk_t = c_t^w
\]

Using these in \( V(n_t) = E \left[ U \left( \frac{w_t n_t}{p_t + 1} \right) \right] \) we obtain

\[
V(n_t) = E \left[ U \left( \frac{w_t n_t}{p_t + 1} \right) \right] = E \left[ U (c_{t+1}) \right]
\]

\[
V(mk_{t-1}) = E_t \left( (a + d) k_t - bk_{t+1} \right)
\]
\[ V(mk_{t-1}) = E_t(U((a + d)k_t - bk_{t+1})) \]

Difference eq. that can produce indeterminacy and sunspots. See Reichlin (JET 1986, set \( b = 1 \)), Benhabib and Laroque (JET 1988) for more general setups. Let the elasticity of labor supply \( \varepsilon = \frac{w}{n} \left( \frac{\partial n}{\partial w} \right) = \left[ \frac{(a)V'}{cU'} - 1 \right]^{-1} \). The unique steady state \( k^* \) is stable if:

\[ \varepsilon > \frac{a + d - b}{2b - a - d} \]

giving indeterminacy: for initial \( k_0 \) locally all trajectories of \( \{k_t\}_{t=1}^\infty \) converge to \( k^* \).

If we assign the pure rates the values \( \rho = 0.04/\text{year}, \ g = 0.33/\text{year} \) (reciprocal of the U.S. capital/output ratio), and \( \delta = 0.10/\text{year} \), then for \( \Delta = 3/\text{months} \), indeterminacy requires \( \varepsilon > 0.05 \). For smaller values of \( \Delta \), the required elasticity is even smaller.
YAARI
Utility

$$\max_{c(t)} \int_0^T \alpha(t) g(c(t)) \, dt$$

Wealth is accumulated savings, on which interest is earned:

$$S(t) = \int_0^t e^{\int_\tau^t j(x) \, dx} (m(\tau) - c(\tau)) \, d\tau$$

where $m(\tau)$ is earnings, $c(\tau)$ is consumption, and $j(\tau)$ is the interest rate and $S(T) \geq 0$.

$$\dot{S} = j(t) \int_0^t e^{\int_\tau^t j(x) \, dx} (m(\tau) - c(\tau)) \, d\tau + m(t) - c(t)$$

$$= j(t) S(t) + m(t) - c(t)$$

$$c(t) = j(t) S(t) + m(t) - \dot{S}(t)$$
So
\[ \max_{S(t)} \int_{0}^{T} \alpha(t) g \left[ j(t) S(t) + m(t) - \dot{S}(t) \right] dt \]

ST
\[ \int_{0}^{T} e^{\int_{t}^{\tau} j(x) dx} (m(\tau) - c(\tau)) d\tau \geq 0 \]

\[ \max_{S(t)} \int_{0}^{T} \alpha(t) g \left[ j(t) S(t) + m(t) - \dot{S}(t) \right] \]

Use calculus of variations:

\[ F(\dot{S}, S, t) = \alpha(t) g \left[ j(t) S(t) + m(t) - \dot{S}(t) \right] \]

\[ F_S = \alpha g' j = -\dot{\alpha} g' - \alpha g'' \dot{c} = \dot{F}_S \]

\[ \dot{c} = -\frac{\dot{\alpha} g'}{\alpha g''} - \frac{g'}{g''} j = \frac{g'}{g''} \left( -j - \frac{\dot{\alpha}}{\alpha} \right) \]

\[ \dot{\frac{\dot{c}}{c}} = \left( \frac{g'}{g'' c} \right) \left( -j - \frac{\dot{\alpha}}{\alpha} \right) \]

Note: if \( \alpha = e^{-\delta t} \), \( \frac{\dot{\alpha}}{\alpha} = -\delta \).
Random Death

\[ T \in [0, \bar{T}] ; \quad \text{Density} : \ \pi(t) \geq 0 \]
\[ \int_0^{\bar{T}} \pi(t) \, dt = 1 \]

Define

\[ \Omega(t) = \int_t^{\bar{T}} \pi(t) \, dt ; \quad \pi_t(\tau) = \frac{\pi(\tau)}{\Omega(t)} \]

\( \pi_t(\tau) \) is the conditional density of \( \pi(\tau) \) given that death must occur in \([t, \bar{T}]\). So \( \Omega(t) \) is the probability that death occurs after \( t \), or the probability of being alive at \( t \).
Utility:

$$EV = \int_0^T \pi(t) \int_0^t \alpha(t) g(c(t)) d\tau \, dt$$

Utility mass up to $t$ multiplied by probability of being alive until $t$ (dying at $t$). Let $\alpha(t) g(c(t)) = f(t)$. Let

$$\int_0^t f(s) = F(t) - F(0)$$

Then

$$EV = \int_0^T \pi(t) (F(t) - F(0)) \, dt$$

Now integrate by parts ($u = (F(t) - F(0))$, $dv = \pi(t)$)

$$(1 - \Omega(t))(F(t) - F(0)) \bigg|_0^T - \int_0^T (1 - \Omega(t)) f(t) \, dt$$

$$= F(T) - F(0) - \int_0^T f(t) \, dt + \int_0^T \Omega(t) f(t) \, dt$$

$$= F(T) - F(0) - F(T) + F(0) + \int_0^T \Omega(t) f(t) \, dt$$
So

\[ EV = \int_0^T \Omega (t) \alpha (t) g (c (t)) \, dt \]

So

\[ F (\dot{S}, S, t) = \Omega (t) \alpha (t) g (c (t)) \]

\[ F_S = \Omega (t) \alpha g' j = -\dot{\Omega} (t) \alpha (t) g' - \Omega (t) \dot{\alpha} g' - \Omega (t) \alpha g'' \]

\[ \dot{c} = \frac{-g'}{g''} \left( j + \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\Omega} (t)}{\Omega (t)} \right) \]

But

\[ \Omega (t) = \int_t^\bar{T} \pi (t) \, dt; \quad \dot{\Omega} (t) = -\pi (t) \]

\[ \frac{\dot{\Omega} (t)}{\Omega (t)} = \frac{-\pi (t)}{\Omega (t)} - \pi_t (t) \]

So

\[ \dot{c} = \frac{-g'}{g''} \left( j + \frac{\dot{\alpha}}{\alpha} - \pi_t (t) \right) \]

So the discount rate can be interpreted as \((\pi_t (t) - \frac{\dot{\alpha}}{\alpha})\) : the standard discount rate \((-\frac{\dot{\alpha}}{\alpha})\) plus an adjustment for the possibility of death (hazard rate) at \(t\).