OLG

$$Max\ \ U(c_0(t), c_1(t+1))$$

$$S.T.\ \ (e_0 - c_0(t))\rho_t = (c_1(t+1) - e_1)$$

FOC:

$$\frac{U_0(c_0(t), c_1(t+1))}{U_1(c_0(t), c_1(t+1))} = \rho_t$$

$$(e_0 - c_0(t))\left(\frac{U_0(c_0(t), c_1(t+1))}{U_1(c_0(t), c_1(t+1))}\right) = (c_1(t+1) - e_1)$$

Market Clearing:

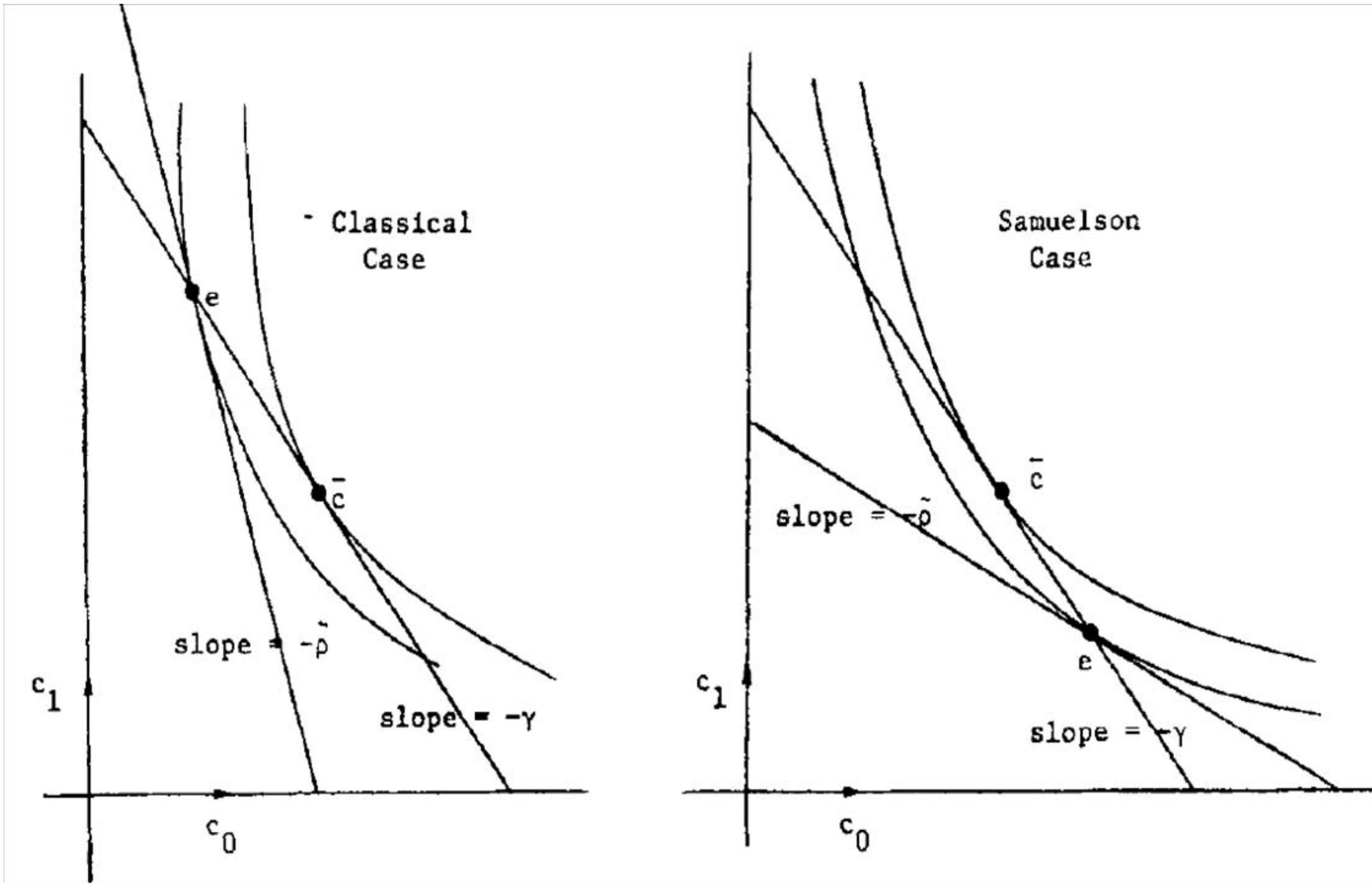
$$c_0(t) + c_1(t) = e_0 + e_1$$

$$e_0 - c_0(t) = c_1(t) - e_1$$

Equilibrium:

$$(e_0 - c_0(t))\left(\frac{U_0(c_0(t), e_0 + (e_1 - c_0(t+1)))}{U_1(c_0(t), e_0 + (e_1 - c_0(t+1)))}\right)$$

$$= (e_0 - c_0(t+1))$$


Classical
Case
e
$\bar{c}$
slope = $-\tilde{\rho}$
slope = $-\gamma$
c_1
c_0
Samuelson
Case
$\bar{c}$
slope = $-\tilde{\rho}$
e
slope = $-\gamma$
c_1
c_0

Under some assumptions, this is a first order difference equation:

Note: if

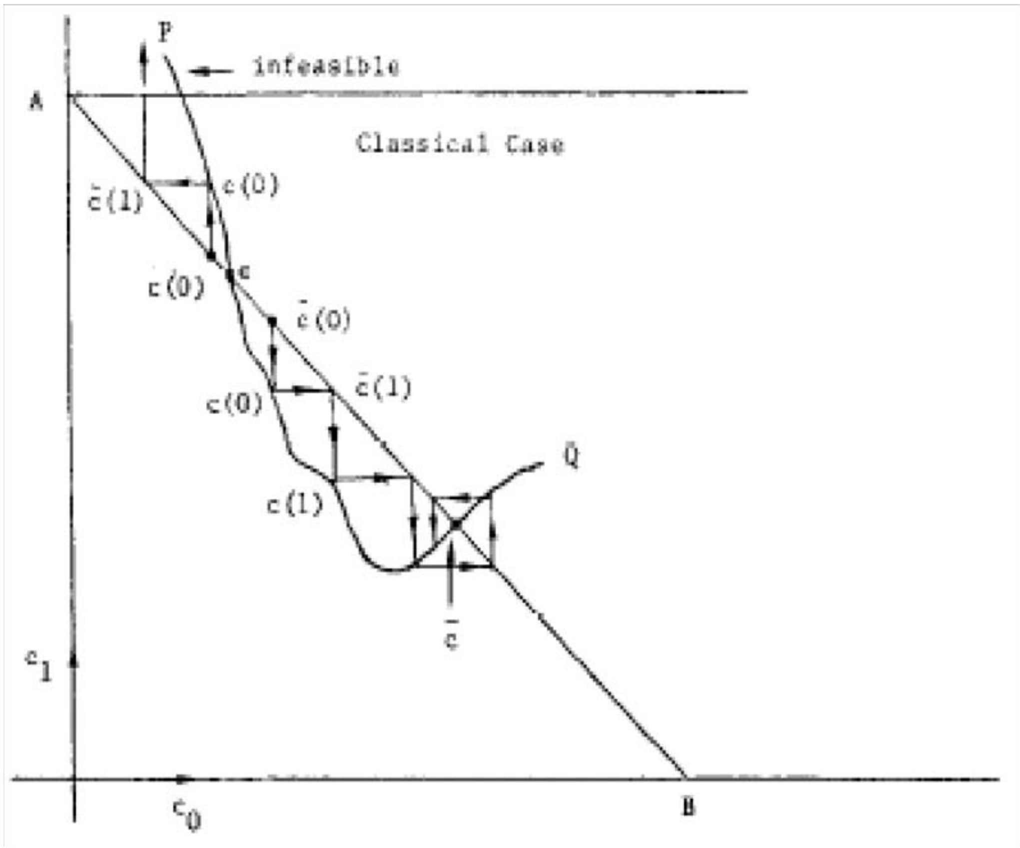
$$U\big(c_0(t), c_1(t+1)\big) = V(c_0(t)) + W\big(c_1(t+1)\big),$$

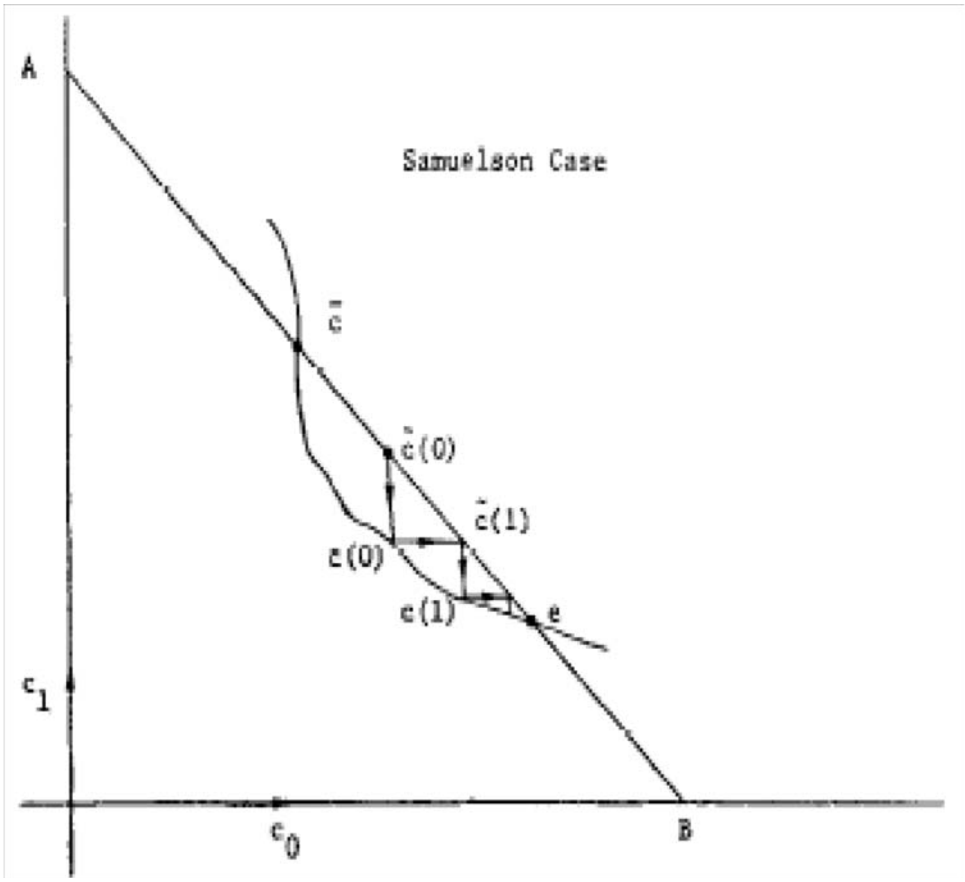
then

$$(c_0(t) - e_0)V'(c_0(t)) = W'\big(c_1(t+1)\big)\big(e_1 - c_1(t+1)\big)$$

If $(c_0(t) - e_0) > 0$ (implying $\big(e_1 - c_1(t+1)\big) > 0$) RHS is large for small positive $c_1(t+1)$ and decreasing, so that by the implicit function theorem, $c_1(t+1) = F(c_0(t))$. Similar argument if utility is homothetic, that is U_0/U_1 is a decreasing function of (c_0/c_1).

Let $\gamma = 1$, that is Population growth is zero.




A
Samuelson Case
$\bar{c}$
$\tilde{c}(0)$
$\tilde{c}(1)$
c(0)
c(1)
e
c_1
c_0
B

Steady States:

1. Autarkic, $(e_0 - c_0) = 0$;

2. Trading $\rho = \left(\frac{U_0(c_0, e_0 + e_1 - c_0)}{U_1(c_0, e_0 + e_1 - c_0)} \right) = 1,$ golden rule.

TRY

$$1. \quad U = ac_0 - b(c_0)^2 + c_1$$

$$a, b > 0, \quad (e_0, e_1) = (0, w), \; w > \frac{a}{b}.$$

$$Try \; a = 4, 2, 1.5$$

$$2. \quad U = \frac{\lambda(c_0 + b)^{1-a}}{1 - a} + c_1,$$

$$a > 0, \; a \neq 1, \; \lambda, b > 0;$$

$$Let \; c_0 - e_0 > 0. \quad Try \; \lambda = 51, \; a = 5.1$$

OLG WITH GOVERNMENT

$$Max \; V(c_0(t)) + c_1(t+1)$$

S.T.

$$e_0 + e_1 - g = c_0(t) + c_1(t)$$

$$c_1(t) - e_1 = e_0 - c_0(t) - g$$

$$(e_0 - c_0(t+1) - g) = V_0(c_0(t))(e_0 - c_0(t))$$

$$x(t+1) = x(t)V_0(e_0 - x(t)) + g$$

Does an increase in governement expenditures and taxes increase or decrease savings?

Try interpretation with money:

$$m_t = p_t(e_0 - c_0(t))$$

$$r_t m_t + p_{t+1} e_1 = p_{t+1}(c_1(t+1))$$

$$\frac{r_t p_t}{p_{t+1}}(e_0 - c_0(t)) = (c_1(t+1) - e_1)$$

$$Max\ U\left(c_0(t), e_1 + \frac{r_t p_t}{p_{t+1}}(e_0 - c_0(t))\right)$$

$$\frac{r_t p_t}{p_{t+1}} = \frac{U_0(c_0(t), c_1(t+1))}{U_1(c_0(t), c_1(t+1))}$$

$$e_0 - c_1(t+1) = \frac{U_0(c_0(t), c_1(t+1))}{U_1(c_0(t), c_1(t+1))}(e_0 - c_0(t))$$

$$\frac{m_{t+1}}{p_{t+1}} = \frac{U_0\left(e_0 - \frac{m_t}{p_t}, e_1 + \frac{m_{t+1}}{p_t+1}\right)}{U_1\left(e_0 - \frac{m_t}{p_t}, e_1 + \frac{m_{t+1}}{p_t+1}\right)}\left(\frac{m_t}{p_t}\right)$$

	Trading	Autarkic
Samuelsonian	Unstable,PO	Stable, Not PO
Classical	Stable,Not PO	Unstable, Not PO
	$\rho = 1,\ e_0 \neq c_0$	$\rho \neq 1,\ e_0 = c_0$

DIAMOND

$$Max\ (1-\beta)\ell n\ c_0 + \beta \ell n\ (w-c_0)(1+R)$$

FOC

$$\frac{1-\beta}{c_0} = \frac{\beta}{w-c_0}$$

$$(1-\beta)(w-c_0) = \beta c_0; \quad (w-c_0) = \beta w$$

$$f(k) = k^\alpha + (1-\delta)k$$

$$k_{t+1} = \frac{\beta w}{1+n} = \frac{\beta}{1+n}(f - f'k) = \frac{\beta}{1+n}(1-\alpha)k^\alpha$$

SS (Why only one?)

$$k = \left(\frac{\beta(1-\alpha)}{1+n}\right)k^\alpha$$

$$\bar{k} = \left(\frac{\beta(1-\alpha)}{1+n}\right)^{\frac{1}{1-\alpha}}$$

Golden Rule:

$$(1+n)k_{t+1} = (k_t)^\alpha + (1-\delta)k_t - c_t$$

$$c_t = (k_t)^\alpha + (1-\delta)k_t - (1+n)k_{t+1}$$

$$c' = \alpha k^{\alpha-1} - (\delta+n) = 0$$

$$k^* = (\alpha^{-1}(\delta+n))^{\frac{1}{\alpha-1}}$$

Note possible inefficiency
$\left(\alpha k^{\alpha-1}\right.$ does not contain $\left.\delta\right)$:

$$\alpha k^{\alpha-1} = \alpha\left(\frac{(1+n)}{\beta(1-\alpha)}\right) \gtrless (\delta+n)$$

$$\bar{k} = \left(\frac{\beta(1-\alpha)}{1+n}\right)^{\frac{1}{1-\alpha}} \gtrless (\alpha^{-1}(\delta+n))^{\frac{1}{\alpha-1}} = k^*$$

Stability of SS

$$\frac{dk_{t+1}}{dk_t} = \frac{\alpha\beta(1-\alpha)}{1+n}k^{\alpha-1}$$

$$= \alpha\left(\frac{\beta(1-\alpha)}{1+n}\right)\left(\frac{\beta(1-\alpha)}{1+n}\right)^{-1} = \alpha < 1$$

Simple OLG:

$$k_{t+1} = S(r(k_t), w(k_t))$$

$$\frac{dk_{t+1}}{dk_t} = \frac{1-s_r r'}{w'} > 0 \quad s_r, w' > 0,\ r' < 0$$

OLG with endogenous labor, consumption in second period:

$$Max\ \ U(w_t L_t R_{t+1}) - V(L_t)$$

FOC and Dynamics:

$$U'(w(k_t, L_t), L_t R(k_{t+1}, L_{t+1}))w(k_t, L_t)R(k_{t+1}, L_{t+1})$$

$$= V'(L_t)$$

$$k_{t+1} = w(k_t, L_t)L_t$$

These are two difference equations in (k_t, L_t).

Simplify by assuming log utility:

$$Max\ \ (1-\beta)\ell n(1-L_t) + \beta\ell n(L_t w_t R_t)$$

FOC

$$\frac{1-\beta}{1-L_t} = \frac{\beta}{L_t};\quad L_t = \beta$$

Production (note change in exponents of Prod. fn):Golden Rule

$$Max_k\quad c(k) = k^{1-\alpha}\beta^{\alpha} + (1-\delta)k - k$$

$$\delta = (1-\alpha)k^{-\alpha}\beta^{\alpha}$$

$$k^* = \delta^{\frac{-1}{\alpha}}(1-\alpha)^{\frac{1}{\alpha}}\beta$$

Dynamics

$$k_{t+1} = w_t L_t = \alpha (k_t)^{1-\alpha} \beta^{\alpha}$$

Steady State

$$\bar{k} = (\alpha \beta^{\alpha})^{\frac{1}{\alpha}} = \alpha^{\frac{1}{\alpha}} \beta$$

Stability

$$\frac{dk_{t+1}}{dk_t} = \alpha(1-\alpha)(\bar{k})^{-\alpha} \beta^{\alpha} = 1 - \alpha < 1$$

Note: With log utility, dimension is reduced by 1.

OLG with Money

$$Max\, U(c_{t+1}) - V(L_t)$$

S.T.

$$k_{t+1} + M_{t+1} Q_t = w_t L_t$$

$$c_{t+1} = R_{t+1} k_{t+1} + M_{t+1} Q_{t+1}$$

FOC and dynamics

$$R_{t+1} w_t U'(R_{t+1} w_t L_t) = V'(L_t)$$

$$k_{t+1} + M_{t+1} Q_t = w_t L_t$$

$$Q_{t+1} = R_{t+1} Q_t$$

These are three difference equations in (L, k, Q)
Note that there are two steady states, with $Q = 0$ and with $Q \neq 0,\ R = 1$ (golden rule).

Simplify with Log Utility and $F = k^{1-\alpha}\beta^{\alpha} + (1-\delta)k$. Then $L = \beta$.

Dynamics:

$$\begin{aligned} k_{t+1} &= \alpha(k_t)^{1-\alpha}\beta^{\alpha} - M_{t+1}Q_t \\ Q_{t+1} &= R_{t+1}Q_t \end{aligned}$$

At golden rule steady state:

$$R\left(\hat{k}, \beta\right) = (1-\alpha)\left(\hat{k}\right)^{-\alpha}\beta^{\alpha} = 1$$

$$\hat{k} = (1-\alpha)^{\frac{1}{\alpha}}\beta$$

Note that $MQ \gtrless 0$ since:

$$\hat{k} = \alpha\left(\hat{k}\right)^{1-\alpha}\beta^{\alpha} - MQ$$

$$\begin{aligned} \alpha\left(\hat{k}\right)^{1-\alpha}\beta^{\alpha} &= \alpha\left((1-\alpha)^{\frac{1}{\alpha}}\beta\right)^{1-\alpha}\beta^{\alpha} \\ &= \left(\frac{\alpha}{1-\alpha}\right)(1-\alpha)^{\frac{1}{\alpha}}\beta \gtrless (1-\alpha)^{\frac{1}{\alpha}}\beta = \hat{k} \end{aligned}$$

YAARI

Utility

$$Max_{c(\bullet)} \int_0^T \alpha(t) g(c(t)) dt$$

Wealth is accumulated savings, on which interest is earned:

$$S(t) = \int_0^t e^{\int_\tau^t j(x)dx} (m(\tau) - c(\tau)) d\tau$$

where $m(\tau)$ is earnings, $c(\tau)$ is consumption, and $j(\tau)$ is the interest rate and $S(T) \geq 0$.

$$\begin{aligned} \dot{S} &= j(t) \int_0^t e^{\int_\tau^t j(x)dx} (m(\tau) - c(\tau)) d\tau + m(t) - c(t) \\ &= j(t) S(t) + m(t) - c(t) \\ c(t) &= j(t) S(t) + m(t) - \dot{S}(t) \end{aligned}$$

So

$$Max_{S(\cdot)}\int_0^T \alpha(t)g[j(t)S(t)+m(t)-\dot{S}(t)]dt$$

ST

$$\int_0^T e^{\int_\tau^t j(x)dx}(m(\tau)-c(\tau))d\tau \geq 0$$

$$Max_{S(\cdot)}\int_0^T \alpha(t)g[j(t)S(t)+m(t)-\dot{S}(t)]$$

Use calculus of variations:

$$F(\dot{S},S,t) = \alpha(t)g[j(t)S(t)+m(t)-\dot{S}(t)]$$

$$F_S = \alpha g' j = -\dot{\alpha} g' - \alpha g'' \dot{c} = \dot{F}_{\dot{S}}$$

$$\dot{c} = -\frac{\dot{\alpha} g'}{\alpha g''} - \frac{g'}{g''} j = \frac{g'}{g''}\left(-j - \frac{\dot{\alpha}}{\alpha}\right)$$

$$\frac{\dot{c}}{c} = \left(\frac{g'}{g'' c}\right)\left(-j - \frac{\dot{\alpha}}{\alpha}\right)$$

Note: if $\alpha = e^{-\delta t}$, $\frac{\dot{\alpha}}{\alpha} = -\delta$.

Random Death

$$T \in [0, \bar{T}]; \quad Density: \quad \pi(t) \geq 0$$

$$\int_0^{\bar{T}} \pi(t)dt = 1$$

Define

$$\Omega(t) = \int_t^{\bar{T}} \pi(t)dt \quad ; \quad \pi_t(\tau) = \frac{\pi(\tau)}{\Omega(t)}$$

$\pi_t(\tau)$ is the conditional density of $\pi(\tau)$ given that death must occur in $[t, \bar{T}]$. So $\Omega(t)$ is the probability that death occurs after t, or the probaility of being alive at t.

Utility:

$$EV = \int_0^{\bar{T}} \pi(t) \int_0^t \alpha(t)g(c(t))d\tau dt$$

Utility mass up to t multiplied by probability of being alive until t (dying at t)). Let $\alpha(t)g(c(t)) = f(t)$. Let

$$\int_0^t f(s) = F(t) - F(0)$$

Then

$$EV = \int_0^T \pi(t)(F(t) - F(0))dt$$

Now integrate by parts

$$\left(u = (F(t) - F(0)),\ dv = \pi(t)\right)\ :$$

$$(1 - \Omega(t))(F(t) - F(0))\ \big|_0^{\bar{T}}\ - \int_0^T (1 - \Omega(t)) f(t)dt$$

$$= F(\bar{T}) - F(0) - \int_0^{\bar{T}} f(t)dt + \int_0^T \Omega(t) f(t)dt$$

$$= F(\bar{T}) - F(0) - F(\bar{T}) + F(0) + \int_0^T \Omega(t) f(t)dt$$

So

$$EV = \int_0^T \Omega(t)\alpha(t)g(c(t))dt$$

So

$$F(\dot{S}, S, t) = \Omega(t)\alpha(t)g(c(t))$$

$$F_S = \Omega(t)\alpha g' j = -\dot{\Omega}(t)\alpha(t)g' - \Omega(t)\dot{\alpha}g' - \Omega(t)\alpha g'' \dot{c} = \dot{F}_{\dot{S}}$$

$$\dot{c} = \frac{-g'}{g''}\left(j + \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\Omega}(t)}{\Omega(t)}\right)$$

But

$$\Omega(t) = \int_t^{\bar{T}} \pi(t)dt; \qquad \dot{\Omega}(t) = -\pi(t)$$

$$\frac{\dot{\Omega}(t)}{\Omega(t)} = \frac{-\pi(t)}{\Omega(t)} - \pi_t(t)$$

So

$$\dot{c} = \frac{-g'}{g''}\left(j + \frac{\dot{\alpha}}{\alpha} - \pi_t(t)\right)$$

So the discount rate can be interpreted as $\left(\pi_t(t) - \frac{\dot{\alpha}}{\alpha}\right)$: the standard discount rate $(-\frac{\dot{\alpha}}{\alpha})$ plus an adjusment for the possibility of death (hazard rate) at t.