OLG

$$Max\, U\left(c_0\left(t\right), c_1\left(t+1\right)\right)$$

$$S.T.\;\left(e_0 - c_0\left(t\right)\right)\rho_t = \left(c_1\left(t+1\right) - e_1\right)$$

FOC:

$$\frac{U_0\left(c_0\left(t\right), c_1\left(t+1\right)\right)}{U_1\left(c_0\left(t\right), c_1\left(t+1\right)\right)} = \rho_t$$

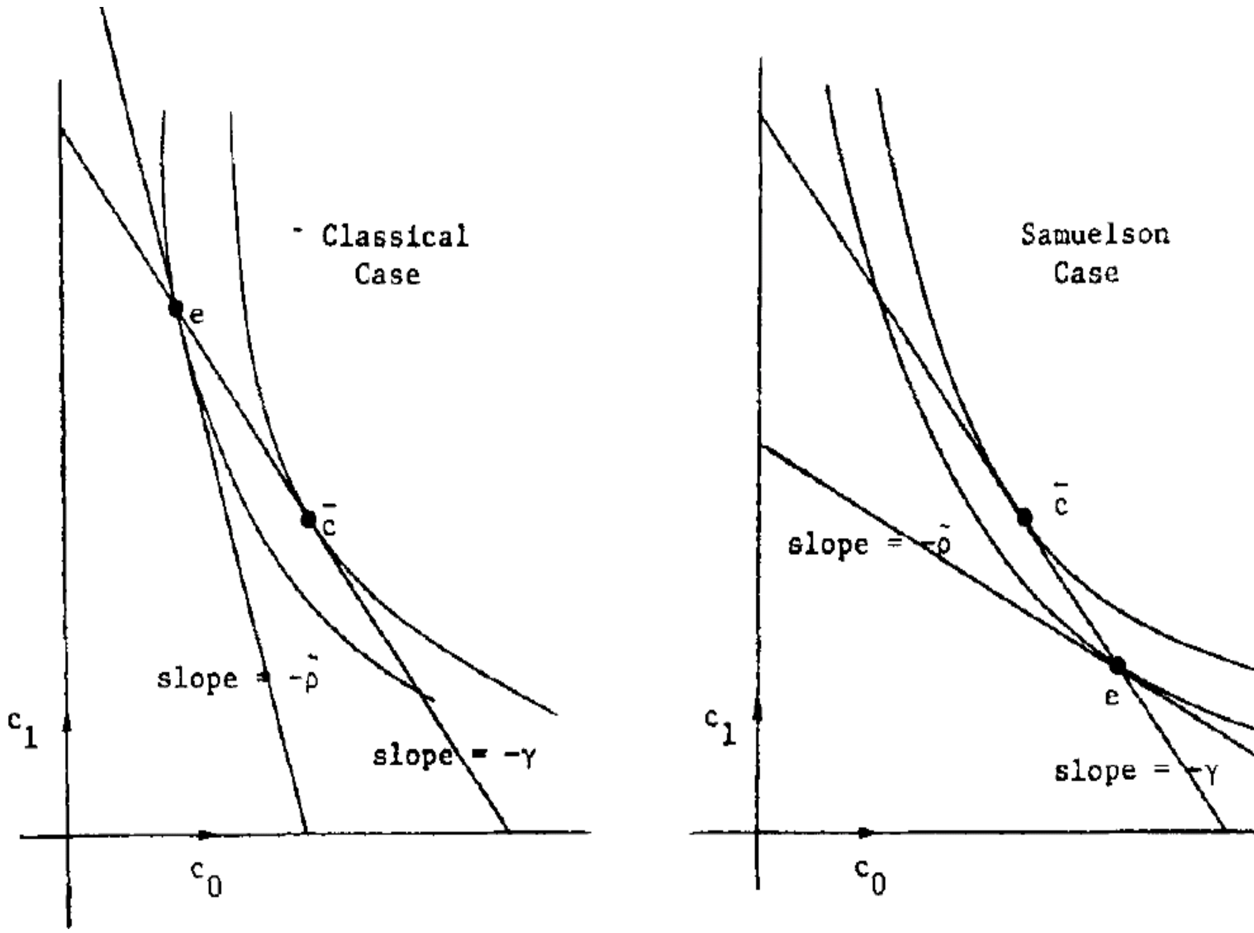
$$\left(e_0 - c_0\left(t\right)\right)\left(\frac{U_0\left(c_0\left(t\right), c_1\left(t+1\right)\right)}{U_1\left(c_0\left(t\right), c_1\left(t+1\right)\right)}\right) = \left(c_1\left(t+1\right) - e_1\right)$$

Market Clearing:

$$\begin{aligned} c_0\left(t\right) + c_1\left(t\right) &= e_0 + e_1 \\ e_0 - c_0\left(t\right) &= c_1\left(t\right) - e_1 \end{aligned}$$

Equilibrium:

$$\begin{aligned} &\left(e_0 - c_0\left(t\right)\right)\left(\frac{U_0\left(c_0\left(t\right), e_0 + \left(e_1 - c_0\left(t+1\right)\right)\right)}{U_1\left(c_0\left(t\right), e_0 + \left(e_1 - c_0\left(t+1\right)\right)\right)}\right) \\ =\; &\left(e_0 - c_0\left(t+1\right)\right) \end{aligned}$$



Under some assumptions, this is a first order difference equation:

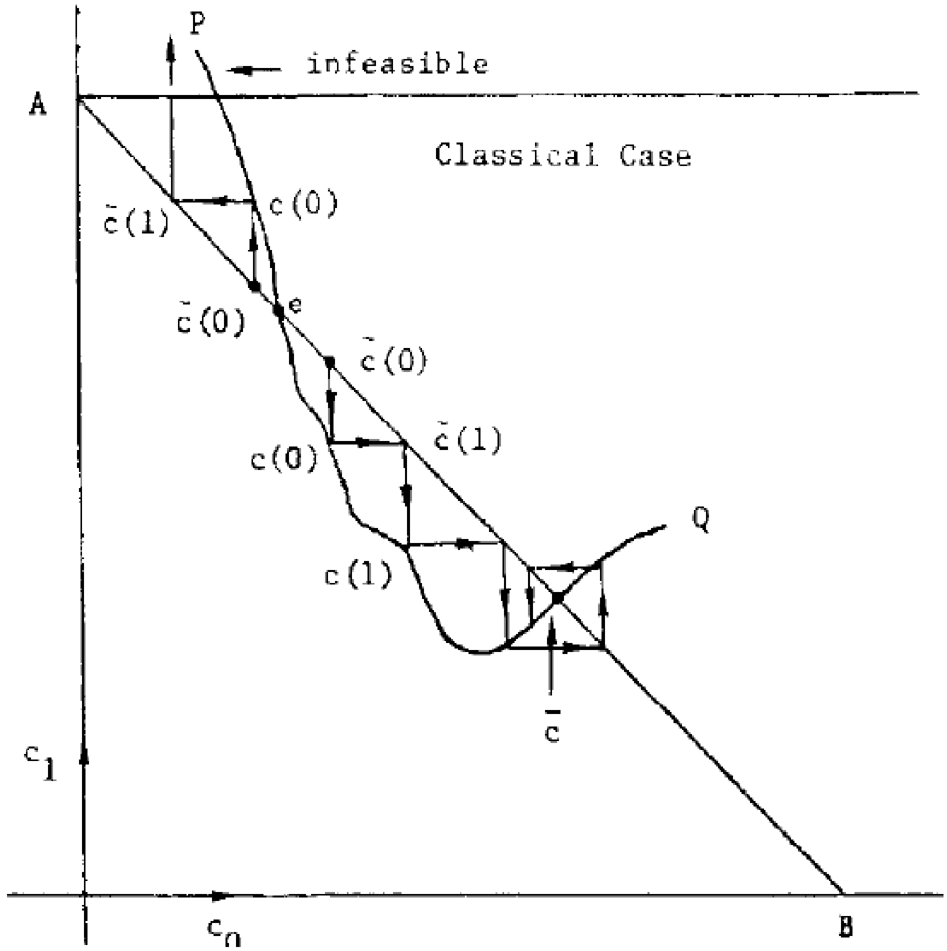
Note: if

$$U\left(c_0\left(t\right), c_1\left(t+1\right)\right) = V\left(c_0\left(t\right)\right) + W\left(c_1\left(t+1\right)\right),$$

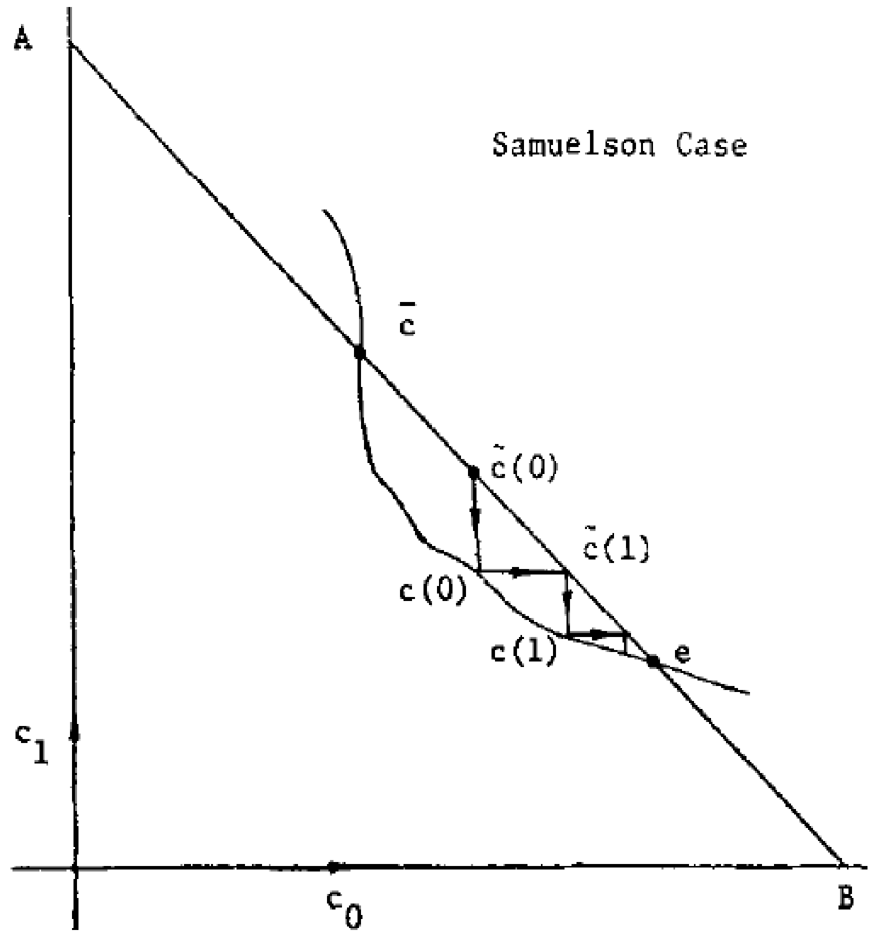
then

$$\left(c_0\left(t\right) - e_0\right) V'\left(c_0\left(t\right)\right) = W'\left(c_1\left(t+1\right)\right)\left(e_1 - c_1\left(t+1\right)\right)$$

If $\left(c_0\left(t\right) - e_0\right) > 0$ (implying $\left(e_1 - c_1\left(t+1\right)\right) > 0$) RHS is large for small positive $c_1\left(t+1\right)$ and decreasing, so that by the implicit function theorem, $c_1\left(t+1\right) = F\left(c_0\left(t\right)\right)$. Similar argument if utility is homothetic, that is U_0/U_1 is a decreasing function of $\left(c_0/c_1\right)$.



Let $\gamma = 1$, that is Population growth is zero.


A
Samuelson Case
$\bar{c}$
$\tilde{c}(0)$
$\tilde{c}(1)$
c(0)
c(1)
e
c_1
c_0
B

Steady States:

1. Autarkic, $(e_0 - c_0) = 0$;

2. Trading $\rho = \left(\frac{U_0(c_0,e_0+e_1-c_0)}{U_1(c_0,e_0+e_1-c_0)}\right) = 1$, golden rule.

TRY

$$\begin{aligned}
1.\ U &= ac_0 - b\left(c_0\right)^2 + c_1 \\
a, b &> 0,\ \ (e_0, e_1) = (0, w)\,,\ w > \frac{a}{b}. \\
Try\ a &= 4, 2, 1.5 \\
2.\ U &= \frac{\lambda(c_0 + b)^{1-a}}{1-a} + c_1, \\
a &> 0,\ a \neq 1,\ \lambda, b > 0; \\
Let\ c_0 - e_0 &> 0.\ \ Try\ \lambda = 51,\ a = 5.1
\end{aligned}$$

OLG WITH GOVERNMENT

$$Max\ V\left(c_0\left(t\right)\right) + c_1\left(t+1\right)$$

S.T.

$$\begin{aligned}
e_0 + e_1 - g &= c_0\left(t\right) + c_1\left(t\right) \\
c_1\left(t\right) - e_1 &= e_0 - c_0\left(t\right) - g
\end{aligned}$$

$$\begin{aligned}(e_0 - c_0(t+1) - g) &= V_0(c_0(t))(e_0 - c_0(t)) \\ x(t+1) &= x(t)\, V_0(e_0 - x(t)) + g\end{aligned}$$

Does an increase in governement expenditures and taxes increase or decrease savings?

Try interpretation with money:

$$m_t = p_t \left(e_0 - c_0 \left(t\right)\right)$$

$$\begin{aligned} r_t m_t + p_{t+1} e_1 &= p_{t+1} \left(c_1 \left(t+1\right)\right) \\ \frac{r_t p_t}{p_{t+1}} \left(e_0 - c_0 \left(t\right)\right) &= \left(c_1 \left(t+1\right) - e_1\right) \end{aligned}$$

$$Max\ U \left(c_0 \left(t\right), e_1 + \frac{r_t p_t}{p_{t+1}} \left(e_0 - c_0 \left(t\right)\right)\right)$$

$$\begin{aligned} \frac{r_t p_t}{p_{t+1}} &= \frac{U_0 \left(c_0 \left(t\right), c_1 \left(t+1\right)\right)}{U_1 \left(c_0 \left(t\right), c_1 \left(t+1\right)\right)} \\ e_0 - c_0 \left(t+1\right) &= \frac{U_0 \left(c_0 \left(t\right), c_1 \left(t+1\right)\right)}{U_1 \left(c_0 \left(t\right), c_1 \left(t+1\right)\right)} \left(e_0 - c_0 \left(t\right)\right) \\ \frac{m_{t+1}}{p_{t+1}} &= \frac{U_0 \left(e_0 - \frac{m_t}{p_t}, e_1 + \frac{m_{t+1}}{p_t+1}\right)}{U_1 \left(e_0 - \frac{m_t}{p_t}, e_1 + \frac{m_{t+1}}{p_t+1}\right)} \left(\frac{m_t}{p_t}\right) \end{aligned}$$

	Trading	Autarkic
Samuelsonian	Unstable,PO	Stable, Not PO
Classical	Stable,Not PO	Unstable, Not PO
	$\rho = 1,\ e_0 \neq c_0$	$\rho \neq 1,\ e_0 = c_0$

DIAMOND

$$Max \ \ (1-\beta)\,\ell n\ c_0 + \beta \ell n\ \ (w-c_0)\,(1+R)$$

FOC

$$\frac{1-\beta}{c_0} \ = \ \frac{\beta}{w-c_0}$$

$$(1-\beta)\,(w-c_0) \ = \ \beta c_0; \ \ (w-c_0) = \beta w$$

$$f\,(k) = k^{\alpha} + (1-\delta)\,k$$

$$k_{t+1} = \frac{\beta w}{1+n} = \frac{\beta}{1+n}\,(f - f'k) = \frac{\beta}{1+n}\,(1-\alpha)\,k^{\alpha}$$

SS (Why only one?)

$$k \ = \ \left(\frac{\beta\,(1-\alpha)}{1+n}\right) k^{\alpha}$$

$$\bar{k} \ = \ \left(\frac{\beta\,(1-\alpha)}{1+n}\right)^{\frac{1}{1-\alpha}}$$

Golden Rule:

$$(1+n)\,k_{t+1} \ = \ (k_t)^{\alpha} + (1-\delta)\,k_t - c_t$$

$$c_t \ = \ (k_t)^{\alpha} + (1-\delta)\,k_t - (1+n)\,k_{t+1}$$

$$c' \ = \ \alpha k^{\alpha-1} - (\delta+n) = 0$$

$$k^* \ = \ \left(\alpha^{-1}\,(\delta+n)\right)^{\frac{1}{\alpha-1}}$$

Note possible inefficiency $\left(\alpha k^{\alpha-1}\text{ does not contain }\delta\right)$:

$$\alpha k^{\alpha-1} = \alpha\left(\frac{(1+n)}{\beta(1-\alpha)}\right) \gtrless (\delta+n)$$

$$\bar{k} = \left(\frac{\beta(1-\alpha)}{1+n}\right)^{\frac{1}{1-\alpha}} \gtrless \left(\alpha^{-1}(\delta+n)\right)^{\frac{1}{\alpha-1}} = k^*$$

Stability of SS

$$\frac{dk_{t+1}}{dk_t} = \frac{\alpha\beta(1-\alpha)}{1+n}k^{\alpha-1}$$

$$= \alpha\left(\frac{\beta(1-\alpha)}{1+n}\right)\left(\frac{\beta(1-\alpha)}{1+n}\right)^{-1} = \alpha < 1$$

Simple OLG:

$$k_{t+1} = S\left(r(k_t), w(k_t)\right)$$

$$\frac{dk_{t+1}}{dk_t} = \frac{1 - s_r r'}{w'} > 0 \quad s_r, w' > 0,\ r' < 0$$

OLG with endogenous labor, consumption in second period:

$$Max\ U\left(w_t L_t R_{t+1}\right) - V\left(L_t\right)$$

FOC and Dynamics:

$$U'\left(w\left(k_t, L_t\right), L_t R\left(k_{t+1}, L_{t+1}\right)\right) w\left(k_t, L_t\right) R\left(k_{t+1}, L_{t+1}\right) = V'\left(L_t\right)$$

$$k_{t+1} = w\left(k_t, L_t\right) L_t$$

These are two difference equations in $\left(k_t, L_t\right)$.

Simplify by assuming log utility:

$$Max\ (1-\beta)\ell n(1-L_t)+\beta\ell n\left(L_t w_t R_t\right)$$

FOC

$$\frac{1-\beta}{1-L_t}=\frac{\beta}{L_t};\quad L_t=\beta$$

Production (note change in exponents of Prod. fn):Golden Rule

$$\begin{aligned} Max_k \quad c\left(k\right) &= k^{1-\alpha}\beta^{\alpha}+\left(1-\delta\right)k-k \\ \delta &= \left(1-\alpha\right)k^{-\alpha}\beta^{\alpha} \\ k^{*} &= \delta^{\frac{-1}{\alpha}}\left(1-\alpha\right)^{\frac{1}{\alpha}}\beta \end{aligned}$$

Dynamics

$$k_{t+1} = w_t L_t = \alpha \left(k_t\right)^{1-\alpha} \beta^{\alpha}$$

Steady State

$$\bar{k} = \left(\alpha \beta^{\alpha}\right)^{\frac{1}{\alpha}} = \alpha^{\frac{1}{\alpha}} \beta$$

Stability

$$\frac{dk_{t+1}}{dk_t} = \alpha \left(1-\alpha\right) \left(\bar{k}\right)^{-\alpha} \beta^{\alpha} = 1 - \alpha < 1$$

Note: With log utility, dimension is reduced by 1.

OLG with Money

$$Max\, U\left(c_{t+1}\right) - V\left(L_t\right)$$

S.T.

$$k_{t+1} + M_{t+1}Q_t = w_t L_t$$

$$c_{t+1} = R_{t+1}k_{t+1} + M_{t+1}Q_{t+1}$$

FOC and dynamics

$$\begin{aligned} R_{t+1}w_t U'\left(R_{t+1}w_t L_t\right) &= V'\left(L_t\right) \\ k_{t+1} + M_{t+1}Q_t &= w_t L_t \\ Q_{t+1} &= R_{t+1}Q_t \end{aligned}$$

These are three difference equations in (L, k, Q)

Note that there are two steady states, with $Q = 0$ and with $Q \neq 0$, $R = 1$ (golden rule).

Simplify with Log Utility and $F = k^{1-\alpha}\beta^{\alpha} + (1-\delta)\,k$. Then $L = \beta$.

Dynamics:

$$\begin{aligned} k_{t+1} &= \alpha\,(k_t)^{1-\alpha}\,\beta^{\alpha} - M_{t+1}Q_t \\ Q_{t+1} &= R_{t+1}Q_t \end{aligned}$$

At golden rule steady state:

$$\begin{aligned} R\left(\hat{k},\beta\right) &= (1-\alpha)\left(\hat{k}\right)^{-\alpha}\beta^{\alpha} = 1 \\ \hat{k} &= (1-\alpha)^{\frac{1}{\alpha}}\,\beta \end{aligned}$$

Note that $MQ \gtrless 0$ since:

$$\begin{aligned} \hat{k} &= \alpha\left(\hat{k}\right)^{1-\alpha}\beta^{\alpha} - MQ \\ \alpha\left(\hat{k}\right)^{1-\alpha}\beta^{\alpha} &= \alpha\left((1-\alpha)^{\frac{1}{\alpha}}\,\beta\right)^{1-\alpha}\beta^{\alpha} \\ &= \left(\frac{\alpha}{1-\alpha}\right)(1-\alpha)^{\frac{1}{\alpha}}\,\beta \gtrless (1-\alpha)^{\frac{1}{\alpha}}\,\beta = \hat{k} \end{aligned}$$

0.1 Stationary Sunspot Eq. In a Finance Constrained Economy, JET 1986 (Hand to Mouth Workers?)

(Hand to Mouth?)Workers:

$$Max_{\{c_t\}} \sum_{1}^{\infty} \gamma^{t-1} u\left(c_t^w\right) - \gamma^t v\left(n_t\right)$$

$u' + cu"(c) > 0, v(0) = 0.$

$$\begin{aligned} p_t\left(c_t^w + \left(k_t^w - dk_{t-1}^w\right)\right) &\leq M_t^w + r_t k_{t-1}^w \\ p_t\left(c_t^w + \left(k_t^w - dk_{t-1}^w\right)\right) + M_{t+1}^w &\leq M_t^w + r_t k_{t-1}^w + w_t n_t \end{aligned}$$

$$u\left(c_t^w\right) = u_t^w\left(\frac{M_t^w + r_t k_{t-1}^w + w_t n_t - M_{t+1}^w - p_t\left(k_t^w - dk_{t-1}^w\right)}{p_t}\right)$$

$$-u'\left(c_t^w\right)\frac{1}{p_t} + \gamma E u'\left(c_{t+1}^w\right)\frac{1}{p_{t+1}} \leq 0$$

First constraint is a finance constraint stating that a given period's expenditures must be financed either out of money held at the **beginning** of the period, or by borrowing against the value of capital held then. (Both the rents $r_t k_{t-1}^w$ and wages $w_t n_t$, are received only at the **end of period** t but the former can be borrowed against by pledging the capital as collateral, while future wage payments cannot be borrowed against, because of moral hazard problems.) The depreciation factor is d, where a fraction $1-d$ of capital depreciates. Second constraint is the usual budget constraint.

FOC with respect to k_{t+1}.Look at equilibria such that:

$$u'\left(c_t^w\right) > \gamma_t E\left[u'\left(c_{t+1}^w\right)\left(\frac{r_{t+1}}{p_{t+1}} + d\right)\right]$$

Also return to holding money $\frac{p_t}{p_{t+1}}$ is less than return capital $\frac{r_{t+1}}{p_{t+1}} + d$

$$r_{t+1} + dp_{t+1} > p_t, \quad \frac{r_{t+1}}{p_{t+1}} + d > \frac{p_t}{p_{t+1}}$$

so workers set $k_t^w = 0$ since current marginal utility of forgoing a unit of current consumption exceeds discounted expected utility obtained by investing it (they can't borrow to consume more today), even though the net return on investing capital is positive: $\frac{r_{t+1}+dp_{t+1}}{p_t} > 1$. Then they maximize a static labor market problem

$$Max_{n_t} E\left[u\left(c_{t+1}\right) - v\left(n_t\right)\right] \quad S.T. \;\; p_{t+1}c_{t+1} = w_t n_t$$

to get (since workers set $k_t^w = 0$, $c_{t+1} = \frac{w_t n_t}{p_{t+1}} = \frac{M_{t+1}}{p_{t+1}}$, $u'(c_{t+1})\frac{w_t}{p_{t+1}} = v'\left(n_t\right)$):

$$n_t v'\left(n_t\right) = V\left(n_t\right) = E\left(u'(c_{t+1})\frac{w_t n_t}{p_{t+1}}\right) = E\left[c_{t+1}u'\left(c_{t+1}\right)\right] = E\left[U\left(\frac{w_t n_t}{p_{t+1}}\right)\right]$$

where (bad notation!) $U(c) = cu'(c)$. (Like olg with a month; workers consume their wage bill $\frac{w_t n_t}{p_{t+1}} = \frac{M_{t+1}}{P_{t+1}}$!)

Capitalists

$$\sum_1^\infty \beta^{t-1} \ln c_t^c$$

ST

$$p_t \left(c_t^c + \left(k_t^c - dk_{t-1}^c\right)\right) + M_{t+1}^c \leq M_t^c + r_t k_{t-1}^c$$

Solution:

$$\begin{aligned} c_t^c &= (1-\beta)\left(\frac{r_t}{p_t} + d\right) k_{t-1} \\ k_t &= \beta\left(\frac{r_t}{p_t} + d\right) k_{t-1} \end{aligned}$$

with

$$\begin{aligned} \frac{r_t}{p_t} + d &> bd, \quad b = \beta^{-1}, \\ so\ k_t &= \beta\left(\frac{r_t}{p_t} + d\right) k_{t-1} > \beta\left(\beta^{-1} d\right) k_{t-1} = dk_{t-1} \end{aligned}$$

so that $k_t > dk_{t-1}$, all capital from last period is held and then added to–no decumulation. Thus Capitalists hold no money if $\frac{r_t}{p_t} + d > bd$.

Fixed Coeff. production: Must use m units of labor with each unit of capital to produce $a > b - d$ units of new output plus d units of old undepreciated capital. (Needed so steady state capital >0).

EQ:

$$\begin{aligned} w_t n_t &= M_{t+1}^w = M: \quad Money\ Market \\ n_t &= mk_{t-1} \quad Labor\ Market \end{aligned}$$

Since $p_t c_t^w = M_t^w = w_{t-1} n_{t-1}$ and $c_t^c = (1-\beta)\left(\frac{r_t}{p_t} + d\right) k_{t-1} = (1-\beta)\left(\frac{r_t}{p_t} + d\right) \frac{k_t}{\beta\left(\frac{r_t}{p_t} + d\right)} = (b-1) k_t$,

$$\begin{aligned} c_t^w + c_t^c + k_t &= \frac{M}{p_t} + bk_t = (a+d) k_{t-1} = output \\ \frac{M}{p_t} &= (a+d) k_{t-1} - bk_t = c_t^w \end{aligned}$$

Using these in $V(n_t) = E\left[U\left(\frac{w_t n_t}{p_{t+1}}\right)\right]$ we obtain

$$\begin{aligned} V(n_t) &= E\left[U\left(\frac{w_t n_t}{p_{t+1}}\right)\right] = E\left[U\left(c_{t+1}\right)\right] \\ V(mk_{t-1}) &= E_t\left(U\left((a+d) k_t - bk_{t+1}\right)\right) \end{aligned}$$

$$V\left(mk_{t-1}\right) = E_t\left(U\left(\left(a+d\right)k_t - bk_{t+1}\right)\right)$$

Difference eq. that can produce indeterminacy and sunspots. See Reichlin (JET 1986, set $b = 1$)) , Benhabib and Laroque (JET 1988) for more general setups. Let the elasticity of labor supply $\varepsilon = \left(\frac{w}{n}\right)\left(\frac{\partial n}{\partial w}\right) = \left[\frac{(n)V'}{cU'} - 1\right]^{-1}$. The unique steady state k^* is stable if:

$$\varepsilon > \frac{a+d-b}{2b-a-d}$$

giving indeterminacy: for initial k_0 locally all trajectories of $\{k\}_{t=1}^{\infty}$ converge to k^*.

If we assign the pure rates the values $\rho = 0.04/year$, $g = 0.33/year$ (reciprocal of the U.S. capital/output ratio), and $\delta = 0.10/year$, then for $\Delta = 3months$, indeterminacy requires $\varepsilon > 0.05$. For smaller values of Δ, the required elasticity is even smaller.

YAARI

Utility

$$Max_{c(\cdot)} \int_0^T \alpha(t)\, g(c(t))\, dt$$

Wealth is accumulated savings, on which interest is earned:

$$S(t) = \int_0^t e^{\int_\tau^t j(x)dx} (m(\tau) - c(\tau))\, d\tau$$

where $m(\tau)$ is earnings, $c(\tau)$ is consumption, and $j(\tau)$ is the interest rate and $S(T) \geq 0$.

$$\begin{aligned}
\dot{S} &= j(t) \int_0^t e^{\int_\tau^t j(x)dx} (m(\tau) - c(\tau))\, d\tau + m(t) - c(t) \\
&= j(t) S(t) + m(t) - c(t) \\
c(t) &= j(t) S(t) + m(t) - \dot{S}(t)
\end{aligned}$$

So

$$Max_{S(\cdot)} \int_0^T \alpha(t)\, g\left[j(t)\, S(t) + m(t) - \dot{S}(t)\right] dt$$

ST

$$\int_0^T e^{\int_\tau^t j(x)dx} \left(m(\tau) - c(\tau)\right) d\tau \geq 0$$

$$Max_{S(\cdot)} \int_0^T \alpha(t)\, g\left[j(t)\, S(t) + m(t) - \dot{S}(t)\right]$$

Use calculus of variations:

$$\begin{aligned} F\left(\dot{S}, S, t\right) &= \alpha(t)\, g\left[j(t)\, S(t) + m(t) - \dot{S}(t)\right] \\ F_S &= \alpha g' j = -\dot{\alpha} g' - \alpha g'' \dot{c} = \dot{F}_{\dot{S}} \\ \dot{c} &= -\frac{\dot{\alpha} g'}{\alpha g''} - \frac{g'}{g''} j = \frac{g'}{g''}\left(-j - \frac{\dot{\alpha}}{\alpha}\right) \\ \frac{\dot{c}}{c} &= \left(\frac{g'}{g'' c}\right)\left(-j - \frac{\dot{\alpha}}{\alpha}\right) \end{aligned}$$

Note: if $\alpha = e^{-\delta t}$, $\frac{\dot{\alpha}}{\alpha} = -\delta$.

Random Death

$$T \ \in \ \left[0, \bar{T}\right]; \ \ Density: \ \pi(t) \geq 0$$

$$\int_0^{\bar{T}} \pi(t)\, dt \ = \ 1$$

Define

$$\Omega(t) = \int_t^{\bar{T}} \pi(t)\, dt \ \ ; \quad \pi_t(\tau) = \frac{\pi(\tau)}{\Omega(t)}$$

$\pi_t(\tau)$ is the conditional density of $\pi(\tau)$ given that death must occur in $\left[t, \bar{T}\right]$. So $\Omega(t)$ is the probability that death occurs after t,or the probaility of being alive at t.

Utility:

$$EV = \int_0^{\bar{T}} \pi(t) \int_0^t \alpha(t) g(c(t)) \, d\tau dt$$

Utility mass up to t multiplied by probability of being alive until t (dying at t)). Let $\alpha(t) g(c(t)) = f(t)$. Let

$$\int_0^t f(s) = F(t) - F(0)$$

Then

$$EV = \int_0^T \pi(t) (F(t) - F(0)) \, dt$$

Now integrate by parts $(u = (F(t) - F(0)), \; dv = \pi(t))$

$$(1 - \Omega(t)) (F(t) - F(0)) \mid_0^{\bar{T}} - \int_0^T (1 - \Omega(t)) f(t) \, dt$$

$$= F(\bar{T}) - F(0) - \int_0^{\bar{T}} f(t) \, dt + \int_0^T \Omega(t) f(t) \, dt$$

$$= F(\bar{T}) - F(0) - F(\bar{T}) + F(0) + \int_0^T \Omega(t) f(t) \, dt$$

So

$$EV = \int_0^T \Omega(t)\,\alpha(t)\,g(c(t))\,dt$$

So

$$\begin{aligned} F\left(\dot{S}, S, t\right) &= \Omega(t)\,\alpha(t)\,g(c(t)) \\ F_S &= \Omega(t)\,\alpha g' j = -\dot{\Omega}(t)\,\alpha(t)\,g' - \Omega(t)\,\dot{\alpha} g' - \Omega \\ \dot{c} &= \frac{-g'}{g''}\left(j + \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\Omega}(t)}{\Omega(t)}\right) \end{aligned}$$

But

$$\Omega(t) = \int_t^{\bar{T}} \pi(t)\,dt; \quad \dot{\Omega}(t) = -\pi(t)$$

$$\frac{\dot{\Omega}(t)}{\Omega(t)} = \frac{-\pi(t)}{\Omega(t)} - \pi_t(t)$$

So

$$\dot{c} = \frac{-g'}{g''}\left(j + \frac{\dot{\alpha}}{\alpha} - \pi_t(t)\right)$$

So the discount rate can be interpreted as $\left(\pi_t(t) - \frac{\dot{\alpha}}{\alpha}\right)$: the standard discount rate $\left(-\frac{\dot{\alpha}}{\alpha}\right)$ plus an adjusment for the possibility of death (hazard rate) at t.