Optimal Taxes Without Commitment

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1 The Consumer Problem:

$$Max \sum_{t=0}^{\infty} \beta^t \left(u(c_t) - v(L_t)\right)$$

$$r_t(k_t + b_t) + w_t L_t - k_{t+1} - b_{t+1} - c_t \geq 0$$

- k_0, b_0 given; b are bonds;

  r_t, w_t are profits and wages, **net** of taxes.

First order conditions:

$$w_t u'(c_t) - v'(L_t) = 0 \qquad t = 0, 1, ...$$

$$u'(c_t) - \beta r_{t+1} u'(c_{t+1}) = 0 \qquad t = 0, 1, ...$$

1.1 The Optimal Taxation Problem:

$$\max_{\{r_t,w_t,c_t,k_t,L_t,b_t\}} \sum_{t=0}^{\infty}(u(c_t) - v(L_t))\beta^t$$

subject to:

$$u'(c_t) - \beta r_{t+1} u'(c_{t+1}) = 0,$$

$$w_t u'(c_t) - v'(L_t) = 0,$$

$$r_t(k_t + b_t) + w_t L_t - k_{t+1} - b_{t+1} - c_t \geq 0,$$

$$f(k_t, L_t) - c_t - k_{t+1} - G \geq 0,$$

$$\sum_{t=i}^{\infty} \beta^{t-i} (u(c_t) - v(L_t)) - V^D(k_t, b_t)$$

$$r_t \geq 0,\ t = 0, 1, \ldots,\ k_0, b_0 \text{ given.}$$

1.2 Lagrangean:

$$\sum_{t=0}^{\infty} \beta^t u(c_t) - v(L_t)$$

$$+\lambda_t \left(u'(c_t) - \beta r_{t+1} u'(c_{t+1})\right)$$

$$+\mu_t \left(w_t u'(c_t) - v'(L_t)\right)$$

$$+\eta_t \left(k_{t+1} + c_t + G - f(k_t, L_t)\right)$$
$$+\xi_t \begin{pmatrix} r_t(k_t + b_t) + w_t L_t + G \\ -f(k_t, L_t) - b_{t+1} \end{pmatrix}$$

$$+\kappa_t r_t$$

$$+\sum_{i=1}^{\infty} \gamma_i \left(\sum_{t=i}^{\infty} \beta^{t-i} \left(u\left(c_t\right) - v(L_t)\right) - V^D(k_t, b_t) \right)$$

2 FOC

wrt k_t :

$$\gamma_t \beta^{-t} V_k^D(k_t, b_t) = -\eta_t f_K(k_t, L_t)$$
$$+\beta^{-1}\eta_{t-1} + \xi_t \left(r_t - f_K(k_t, L_t)\right)$$

This condition can be expressed as :

$$\left(\xi_t + \eta_t\right)\left(r_t - f_K(k_t, L_t)\right) \quad (1)$$
$$= \gamma_t \beta^{-t} V_k^D(k_t, b_t) - \beta^{-1}\left(-\eta_t + \eta_{t-1}\right)$$

The foc with respect to c_t is:

$$u''(c_t)(\lambda_t - \lambda_{t-1} r_t + \mu_t w_t) + u'(c_t) + \eta_t$$
$$= -\beta^{-t}\left(\gamma * \beta\right)_t u'(c_t) \quad (2)$$

where we define $(\gamma * \beta)_t = \sum_{i=1}^{t} \gamma_i \beta^{t-i}$.

The foc wrt b_t, r_t, w_t, L_t are:

$$-\xi_{t-1} + \xi_t \beta r_t = \gamma_t \beta^{-t} V_b^D(k_t, b_t), \tag{3}$$

$$-\lambda_{t-1} u'(c_t) + \xi_t (k_t + b_t) + \kappa_t = 0, \tag{4}$$

$$\mu_t u'(c_t) + \xi_t L_t = 0, \tag{5}$$

$$\begin{aligned} & (\xi_t + \eta_t)(w_t - f_L(k_t, L_t)) \\ & -\mu_t v''(L_t) - \big(\eta_t + u'(c_t)\big) w_t \\ = \; & \beta^{-t} (\gamma * \beta)_t v'(L_t) \end{aligned} \tag{6}$$

The above equations, together with the initial conditions and transversality conditions, define the system to be studied.

3 Value of a Steady State:

Fix k, and find the value of keeping k as a steady state, at equilibrium:

$$W(k) = \max u(c) - v(L)$$

subject to:

$$k + c + G = f(k, L) \tag{7}$$

$$\left(\beta^{-1} - 1\right) b + k\beta^{-1} + \left(\frac{v'(L)}{u'(c)}\right) L + G = f(k, L) \tag{8}$$

where we have used:

$$r\beta = 1, w = \left(\frac{v'(L)}{u'(c)}\right). \tag{9}$$

In addition, if the incentive constraint is binding:

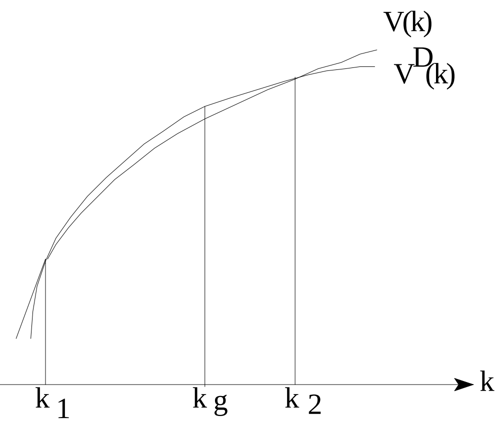
$$(1-\beta)^{-1}\left(u\left(\left(f(k,l)-G-k\right)-v(L)\right)\right.$$

$$\left.-V^{D}(k,b)=0 \qquad (10)\right.$$

If not the condition is replaced by the Chamley-Judd result:

$$r=\beta^{-1}=f_k(k,l) \qquad (11)$$

Definition 1 *: Let* $x(b)=\{r(b),w(b),c(b),k(b),L(b)\}$ *satisfy 7,8,9 and 10 (11). Then it is a candidate steady state.*



RESULT: Suppose that the constrained capital stock k^* is larger than k_g: then capital is *subsidized* at the steady state.

Note: $\beta f_k(k_g, L(k_g) = 1; \beta r = 1$. The result follows from showing that

$$f_k(k_g, L(k_g)) > f_k(k_2, L(k_2))$$

because $k_2 > k_g$ and because it can be shown that $L(k)$ is decreasing on (k_g, k_2).

4 Lagrange Multipliers:

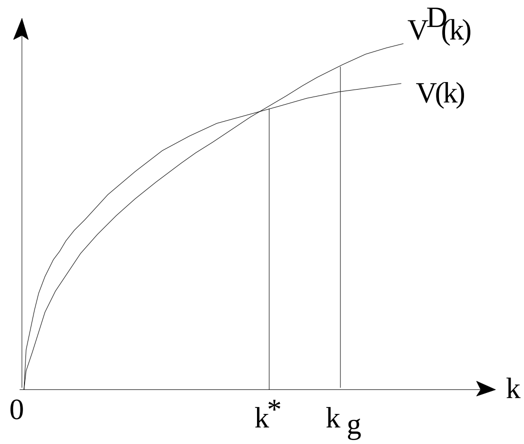
At steady state, one finds, (multiplier for budget constraint):

$$\zeta_t = (1+a)^t$$

and (multiplier for incentive constraints)

$$\gamma_t = \zeta_t \beta^t$$

- $\gamma_t \geq 0$, ζ_t depends on steady state values and is of constant sign; $\gamma_t \in \ell^1$, so $(1+a) \in (0, \frac{1}{\beta})$.



Linear utility: (identifies L as largest)

$W(k) = u(c) - v(L) = c - v(L)$

RESULT: If the defection value function V^D is strictly concave, and

$$\{k : V^D(k) \leq \frac{W(k)}{1-\beta}\} = [k_1, k_2],$$

and $k_g > k_2$,then the only constrained steady state which satisfies the necessary condition for optimality is k_2. (k^*above)

5 An Example

$$u(c) - v(L) = c - (1+e)^{-1} L^{1+e}$$

$$f(k, L) = A(\epsilon)k + BL + \epsilon k^{\alpha} L^{1-\alpha};$$

$$A(\epsilon) = \beta^{-1} - \varphi(\epsilon); \quad (A = \beta^{-1} - \epsilon^{7/8})$$

6 Deviation (Extreme):

Budget constraints and feasibility gives

$$\varepsilon f\left(k,l\right)+Ak+BL=G+wL+rk$$

or, using $e=1, w=V'(L)=L$, and the deviation $r=0$, we get:

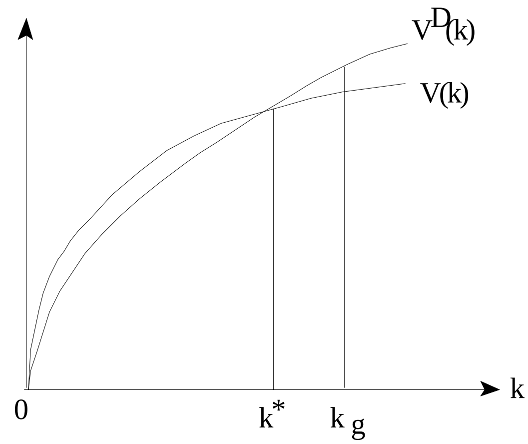
$$L=A\frac{k}{L}+B-\frac{G}{L}$$

which, given k has one positive solution if $Ak\geq G$, and two if $Ak<G$; the two solutions are:

$$L_{2(1)}(k)=\frac{B+\left(B^{2}+(-)4[Ak-G]\right)^{1/2}}{2}.$$

$L_2(k)$ is chosen (it is the better one u is linear) and we obtain the value of deviation:

$$V^D(k) = Ak + BL_2(k) - G - (1/2)L_2(k)^2 + \frac{\beta}{1-\beta}[BL_2(0) - G - (1/2)L_2(0)^2].$$



Let $e = 1, \epsilon = 0$.

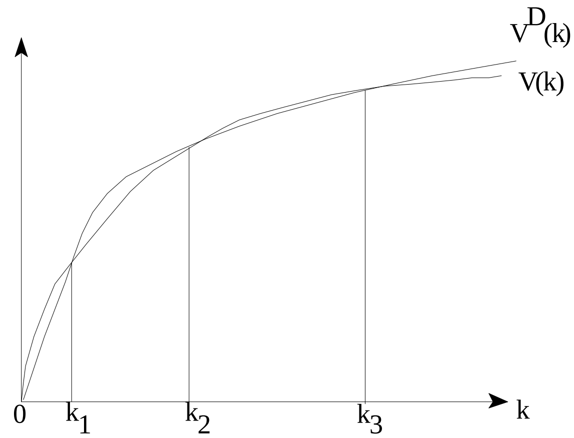
RESULT: For both steady states $k = 0$ and $k = k^*$, $1 + a \in (0, \beta^{-1})$.

RESULT: For values of G large enough, and A, k_0 small enough, the optimal path does not converge to k^*.

RESULT: For ϵ's small, $k^*(\epsilon) > k_g(\epsilon)$ and taxes on capital are negative.

7 A PARAMETRIZED FAMILY:

$$f(k, L) = A(\epsilon)k + BL + \epsilon k^{\alpha} L^{1-\alpha};$$
$$A(\epsilon) = \beta^{-1} - \epsilon^{r}, r = 7/8; \alpha = 0.33;$$
$$B = 3;$$
$$\beta = 0.95;$$
$$G = 2$$



NOTE: $k_g \in (0, k_1)$ or $k_g \in (k_1, k_2)$

If $k_g \in (k_1, k_2)$, both k_1 or k_2 are admissible as SS.

If $k_g \in (0, k_1)$, only k_2 is admissible as SS.

e	ϵ	k_g	k_1	k_2	$L(k_g)$	$L(k_1)$	$L(k_2)$	$TAX(k$
1.15	.00001	.0001	.00001	6.05	1.6678	1.6678	1.6676	.01033
1.15	.001	.0087	.0087	10.1	1.6683	1.6681	1.6428	.00876
1.15	.002	.1000	.0236	24.4	1.6688	1.6686	1.5228	.00700
1.15	.0025	.1043	.0324	—	1.6689	1.6689	—	.00620
1.00	.00001	.0046	.0002	5.7	2.0000	2.0000	1.9995	.00526
1.00	.001	.1053	.0128	8.5	2.0005	2.0003	1.9827	.00736
1.00	.002	.1199	.0340	13.2	2.0011	2.0009	1.9471	.00575
1.00	.0025	.1250	.0464	17.7	2.0013	2.0012	1.9077	.00497
.500	.00001	.1665	.00024	4.6	7.4641	7.4641	7.4640	.00333
.500	.001	.3933	.1736	5.3	7.4658	7.4657	7.4586	.00172
.500	.002	.4477	.4428	6.3	7.4677	7.4677	7.4519	.00003
.500	.0025	.5933	.5933	6.9	7.4687	7.4686	7.4474	-.00078
.450	.00001	.2129	.00039	4.3	9.8283	9.8283	9.8382	.00285
.450	.001	.5179	.3028	5.0	9.8307	9.8306	9.8245	.00100
.450	.002	.5895	.7902	5.6	9.8333	9.8332	9.8207	-.0007
.450	.0025	.6147	1.073	5.9	9.8346	9.8342	9.8181	-.0016

TABLE 1

- Increasing ϵ (curvature) causes subsidy at k_2 and the tax at $k_1$to go up.
- As labor becomes more inelastic (e increases) subsidies at k_2 increase: it is easier to finance capital because wuth inelastic labor the distortion of labor taxes is lower.
- However since k_1(and $\frac{k_1}{L(k_1)}$)declines while the marginal product at k_g is β^{-1}, capital taxes at k_1 increase.

8 Productive Public Capital and Optimal Taxes Without Commitment

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TECHNOLOGY

- k is the private good;
  g is the public good;
  y is the output, equal to:

$$A\left(a(1-\tau)^{-\rho}k^{-\rho}+(1-a)g^{-\rho}\right)^{-\frac{1}{\rho}}+Bk$$

The public good is produced in quantity:

$$g=\tau k.$$

Social Rate Of Return on Capital:

$$y = Ak\left(a(1-\tau)^{-\rho} + (1-a)\tau^{-\rho}\right)^{-\frac{1}{\rho}} + Bk$$
$$\equiv \left(\phi_1(\tau) + B\right)k \equiv \phi(\tau)k.$$

Social rate of return: $\phi(\tau)$
Private Return on Capital:

$$\frac{\partial y}{\partial k} \equiv R(\tau) = Aa\phi_1(\tau)^{1+\rho}(1-\tau)^{-\rho} + B$$

Furthermore:

$$R(\tau) \leq \phi(\tau)$$

The Consumer:

$$\sum_{t=0}^{\infty}(\frac{\sigma}{\sigma - 1})c_t^{\frac{\sigma-1}{\sigma}}\beta^t$$

Budget Constraint

$$k_{t+1} = R(\tau_t)k_t + M_t - c_t = \phi(\tau_t)k_t - c_t$$

where M_t is a government transfer.

The Equilibrium

Fix a sequence of tax rates. At equilibrium:

$$V(k_0, \tau)$$

$$= (\frac{\sigma}{\sigma - 1})\,(k_0)^{\frac{\sigma-1}{\sigma}}\,(\phi(\tau_0))^{\frac{\sigma-1}{\sigma}}\,h\,(\tau_1, \tau_2, \dots)\,.$$

$$(\phi(\tau_0))^{\frac{\sigma-1}{\sigma}}\,h\,(\tau_1, \tau_2, \dots) \equiv H(\tau_0, \tau_1, \dots).$$

$$H^*(\tau) \equiv H(\tau, \tau, \dots).$$

Derivation

The first order condition of the agent gives:

$$c_{t+1} = c_t(\beta R(\tau_{t+1}))^{\sigma} \tag{12}$$

Iteration of the feasibility condition $k_{t+1} = \phi(\tau_t)k_t - c_t$ implies

$$c_0 + \sum_{t=1}^{T} c_t \prod_{s=1}^{t} \phi^{-1}(\tau_s) + \prod_{s=1}^{T} \phi^{-1}(\tau_s) k_{T+1} = \phi(\tau_0)k_0. \tag{13}$$

As is conventional we assume that

$$\lim_{T\to\infty} \prod_{s=1}^{T} \phi^{-1}(\tau_s) k_{T+1} = 0 \tag{14}$$

so that 13 becomes

$$c_0 + \sum_{t=1}^{\infty} c_t \prod_{s=1}^{t} \phi^{-1}(\tau_s) = \phi(\tau_0)k_0 \tag{15}$$

Iterating the first order conditions for the agent we get

$$c_t = c_0 \prod_{s=1}^{t} (\beta R(\tau_s))^{\sigma} \tag{16}$$

which, substituted into 15 gives

$$c_0 \left(1 + \sum_{t=1}^{\infty} \prod_{s=1}^{t} (\beta R(\tau_s))^{\sigma} \phi^{-1}(\tau_s) \right) = \phi(\tau_0)k_0 \tag{17}$$

If we now substitute the 16 into the expression for the utility of the agent we get that the utility from an initial capital k_0 and a tax rate sequence $\tau = (\tau_0, \tau_1, \dots)$, denoted by $V(k_0, \tau)$, is

$$V(k_0, \tau) \qquad (18)$$

$$= (\frac{\sigma}{\sigma - 1}) c_0^{\frac{\sigma-1}{\sigma}} \left(1 + \sum_{t=1}^{\infty} \prod_{s=1}^{t} (\beta^{\sigma} (R(\tau_s))^{\sigma-1}) \right)$$

Now we can use the equation 17 to substitute for c_0, and obtain the value to the agent in terms of k_0 and τ only. To lighten notation, we introduce:

$$X(\tau_1, \dots) \equiv \left(1 + \sum_{t=1}^{\infty} \prod_{s=1}^{t} (\beta R(\tau_s))^{\sigma} \phi^{-1}(\tau_s) \right)^{\frac{1-}{\sigma}} \qquad (19)$$

and

$$Y(\tau_1, \tau_2, \dots) \equiv \left(1 + \sum_{t=1}^{\infty} \prod_{s=1}^{t} (\beta^{\sigma} (R(\tau_s))^{\sigma-1}) \right) \tag{20}$$

so that we define

$$h(\tau_1, \tau_2, \dots) \equiv X(\tau_1, \tau_2, \dots) Y(\tau_1, \tau_2, \dots) \tag{21}$$

Now we can write

$$V(k_0, \tau) = (\frac{\sigma}{\sigma - 1}) (k_0)^{\frac{\sigma-1}{\sigma}} (\phi(\tau_0))^{\frac{\sigma-1}{\sigma}} h(\tau_1, \dots). \tag{22}$$

as we have in the text.

Optimal Tax with Commitment

$$\lim_{m\to\infty} R'(\tau_m^*) = 0.$$

FOC wrt τ_m of X and Y are:

$$\frac{\partial X}{\partial \tau_m}$$

$$= \frac{1-\sigma}{\sigma} X^{\frac{1-2\sigma}{1-\sigma}} \sum_{t=m}^{\infty} \prod_{s=1, s\neq m}^{t} (\beta R(\tau_s))^{\sigma} \phi^{-1}(\tau_s)$$

$$\times \left(\beta \frac{\begin{array}{c} \sigma\phi(\tau_m)\left(R(\tau_m)\right)^{\sigma-1} R'(\tau_m) \\ -\phi'(\tau_m)\left(R(\tau_m)^{\sigma}\right) \end{array}}{\left(\phi(\tau_m)\right)^2} \right)$$

$$\frac{\partial X}{\partial \tau_m} = \frac{1-\sigma}{\sigma} X^{\frac{1-2\sigma}{1-\sigma}} \sum_{t=m}^{\infty} \left(\prod_{s=1}^{t} (\beta R(\tau_s))^{\sigma} \phi^{-1}(\tau_s) \right.$$

$$\left(\sigma \frac{R'(\tau_m)}{R(\tau_m)} - \frac{\phi'(\tau_m)}{\phi(\tau_m)} \right)$$

For the Y term we get

$$\frac{\partial Y}{\partial \tau_m} \tag{23}$$

$$= \sum_{t=m}^{\infty} \prod_{s=1, s\neq m}^{t} (\beta^{\sigma} R(\tau_s)^{\sigma-1})$$

$$(\sigma - 1)(R(\tau_m)^{\sigma-2} R'(\tau_m)$$

or

$$\frac{\partial Y}{\partial \tau_m} = \sum_{t=m}^{\infty} \prod_{s=1}^{t} (\beta^\sigma R(\tau_s)^{\sigma-1})(\sigma-1)\left(\frac{R'(\tau_m)}{R(\tau_m)}\right) \tag{24}$$

We can now substitute in the equation giving the value to the agent:

$$\frac{\partial V(k_0,\tau)}{\partial \tau_m}$$
$$= (\frac{\sigma}{\sigma-1})\,(\phi(\tau_0)k_0)^{\frac{\sigma-1}{\sigma}} \times$$
$$\left(Y^{\frac{1-\sigma}{\sigma}} X^{\frac{1-2\sigma}{1-\sigma}} \sum_{t=m}^{\infty} \left(\prod_{s=1}^{t} (\beta R(\tau_s))^{\sigma} \phi^{-1}(\tau \right. \right.$$
$$\left(\sigma \frac{R'(\tau_m)}{R(\tau_m)} - \frac{\phi'(\tau_m)}{\phi(\tau_m)} \right)$$
$$+(\frac{\sigma}{\sigma-1})\,(\phi(\tau_0)k_0)^{\frac{\sigma-1}{\sigma}}$$
$$\left(X \sum_{t=m}^{\infty} \prod_{s=1}^{t} (\beta^{\sigma} R(\tau_s)^{\sigma-1})(\sigma-1)\frac{R'(\tau_m)}{R(\tau_m)} \right)$$

Now we assume that $\beta^{\sigma} R(\tau_s)^{\sigma} \phi^{-1}(\tau_s) < 1$ (it can be shown this condition is necessary for the value of the program to be bounded and for an optimum to exist). Therefore,

$$\lim_{m\to\infty} \sum_{t=m}^{\infty} \left(\prod_{s=1}^{t} (\beta R(\tau_s))^{\sigma} \phi^{-1}(\tau_s) \right) = 0 \tag{25}$$

so that

$$\lim_{m\to\infty} \frac{\partial V(k_0, \tau)}{\partial \tau_m}$$

$$= \left((\frac{\sigma}{\sigma - 1}) \, (\phi(\tau_0) k_0)^{\frac{\sigma-1}{\sigma}} \right)$$

$$\left(X \sum_{t=m}^{\infty} \prod_{s=1}^{t} (\beta^{\sigma} R(\tau_s)^{\sigma-1})(\sigma - 1) \frac{R'(\tau_m)}{R(\tau_m)} \right)$$

This, together with the optimality condition

$$\frac{\partial V(k_0, \tau)}{\partial \tau_m} = 0 \tag{26}$$

implies that

$$\lim_{m\to\infty} R'(\tau_m) = 0, \qquad (27)$$

as claimed.

The Third Best

$$\max_{\tau=(\tau_0,\tau_1,\dots)} V(k_0,\tau)$$

$$V^C(k_t,\tau) \geq V^D(k_t) \text{ for all } t \geq 1$$

This equivalent to the following simple problem, which is independent of the initial capital stock:

$$\max_{\tau=(\tau_0,\tau_1,\dots)} \left(\frac{\sigma}{\sigma-1}\right) H(\tau_0,\tau_1,\dots)$$

subject to:

$$\left(\frac{\sigma}{\sigma-1}\right) H(\tau_t,\tau_{t+1},\dots) \geq \left(\frac{\sigma}{\sigma-1}\right) H^*(\tau_p)$$
$$\text{for all } t \geq 1.$$

9 Existence

Proposition *For any* $\sigma \geq 0$ *and any* $\rho \in [-1, +\infty]$*, if* $\beta^{\sigma} R(\tau)^{\sigma-1} < 1$ *for every* τ*, then:*

(1) a second best optimal tax exists;

(2) a third best (incentive compatible) optimal tax exists

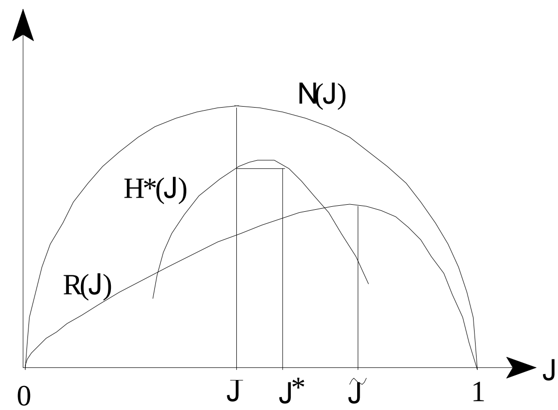
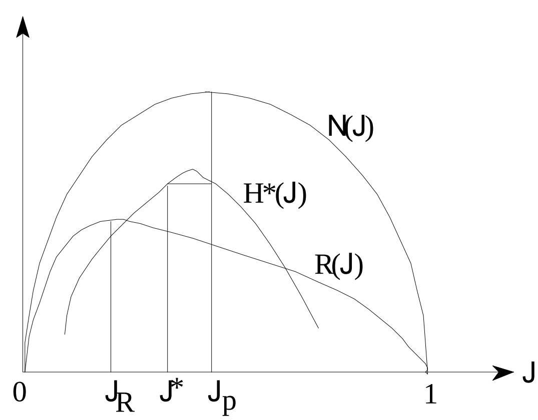


Figure 1: $\rho > 0$



$\rho < 0$

10 Calibration and Tax Rates

$$A = 1, B = 1, \sigma = 0.5, \beta = 0.95$$

$$y = Ak\left(a(1-\tau)^{-\rho} + (1-a)\tau^{-\rho}\right)^{-\frac{1}{\rho}} + Bk$$

First Number: Second Best; Second Number: Third Best; SUS: Second best IC

$\rho \backslash a$	0.3	0.5	0.7
-0.5	.303, .379	.146, .293	.040, .112
-0.1	.694, .697	.494, .497	.295, .297
0.1	.705, .702	.504, .502	.304, .302
0.5	.803, .748	.641, .578	.469, .408
2.0	.729, .697	.654, .611	.573, .521
8.0	.606, SUS	.582, SUS	.558, .556
10.0	.591, SUS	.578, SUS	.556, SUS