Motivation

- Technology diffusion is an important component of growth
- We provide a micro foundation for technology diffusion in a parsimonious model
- Questions:
  - How does the distribution of productivity affect growth?
  - How does the distribution of productivity evolve endogenously as the economy grows?

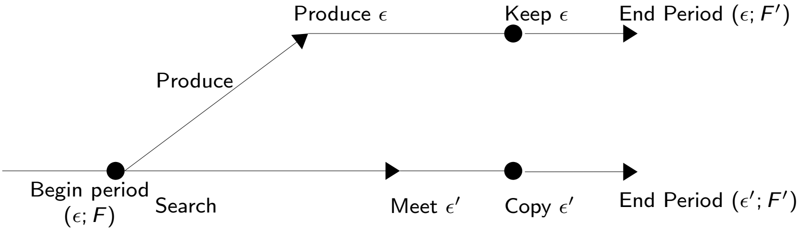
Literature

- Macro Technology Diffusion
  - Nelson & Phelps 1966, Benhabib & Spiegel 2005, many more
  - Specify parametric form for diffusion equation
- Search and the Technology Frontier
  - Kortum 1997, Bental & Peled 1996, some others
  - Growth through search; (semi) exogenous tech distribution
- Micro Technology Diffusion and Endogenous Distributions
  - Lucas & Moll 2011 is most similar to us
    - Growth through search; endogenous distributions
    - Intensive margin and computational solutions

Key Mechanism

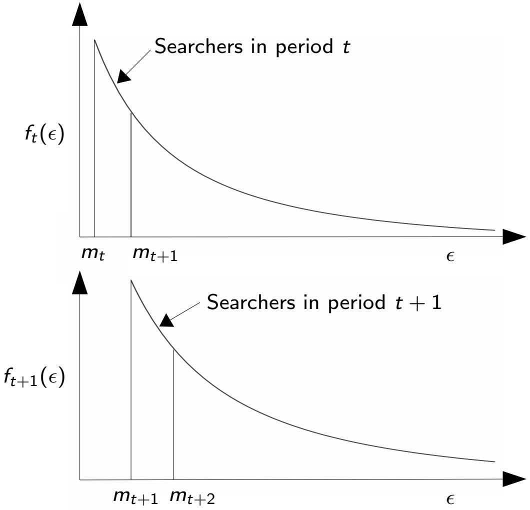
- Heterogeneous firms: produce or search for a new productivity
- Searchers randomly meet and copy a producing firm in the existing productivity distribution
- Selective search endogenously evolves distribution, shifting weight to more productive
- Aggregate state = productivity distribution, F, where $\min \mathrm{support}\{F\} = m$

Intra-Period Timing



Evolution of the Productivity Distribution

f_t = productivity pdf, $m_t := \min \operatorname{support}\{f_t\}$



Consumers

- $t = 0, 1, \ldots, \infty$
- Infinitely lived agents
- Representative consumer owns aggregate output Y_t
- Utility: $\sum_{t=0}^{\infty} \beta^t \frac{Y_t^{1-\sigma}}{1-\sigma}$, $\sigma \geq 0$
- Interest rate: $\frac{1}{1+r_t} := \beta \left(\frac{Y_{t+1}}{Y_t} \right)^{-\sigma}$

Firm Problem

Measure 1, linear production, aggregate state F, idiosyncratic ϵ

$$V(\epsilon, F) = \max_{\{produce, search\}} \Big\{ \epsilon + \tfrac{1}{1+r(F,\Gamma)} V(\epsilon, F'),$$
$$\tfrac{1}{1+r(F,\Gamma)} \int V(\epsilon', F') dF(\epsilon' | \epsilon' \geq h^e(F)) \Big\}$$
$$\text{s.t. } F' = \Gamma(F), \quad \text{law of motion for } F \text{ forecast by agent}$$

- Solution is reservation productivity function: $h(F)$
- Agent uses forecast for minimum producer: $h^e(F)$
- Discount with consumer's interest rate

BGP Proof Sketch - Existence by Construction

Guesses:

- Pareto(m_0, k) will fulfill BGP requirements for f_0
- Reservation productivity is linear in m: $h(m) = gm$
- The value of search is linear in m: $V(\epsilon, m) = mW, \forall \epsilon < h(m)$

Verify and Solve:

i) Plug the Pareto guess into the indifference equation

ii) Simplify to a system of 2 equations containing g and W

iii) Solve for g and W, confirming they are not functions of m

Firm Problem with Guesses

Use Pareto f_0 and $h(m) = gm$

$$V(\epsilon, m) = \max\left\{\epsilon + \tfrac{1}{1+r}V(\epsilon, gm), \tfrac{1}{1+r}k(gm)^k\int_{gm}^{\infty}V(\epsilon', gm)\epsilon'^{-k-1}d\epsilon'\right\}$$

Indifference at $\epsilon = gm$:

$$\begin{aligned} V(gm, m) =& gm + \tfrac{1}{1+r}V(gm, gm) \\ =& \tfrac{1}{1+r}k(gm)^k\int_{gm}^{\infty}V(\epsilon', gm)\epsilon'^{-k-1}d\epsilon' \end{aligned}$$

Linear value of search guess gives 2 equalities:

$$mW = gm + \tfrac{1}{1+r}gmW \tag{EQ1}$$

$$= \tfrac{1}{1+r}k(gm)^k\int_{gm}^{\infty}V(\epsilon', gm)\epsilon'^{-k-1}d\epsilon' \tag{EQ2}$$

BGP Solution

Proposition

Given the following initial condition and parameters:

i) F_0, *is Pareto, i.e.* $F_0(\epsilon) = 1 - \left(\frac{m_0}{\epsilon}\right)^k$

ii) *Parameter restrictions*

An equilibrium exists with the following properties

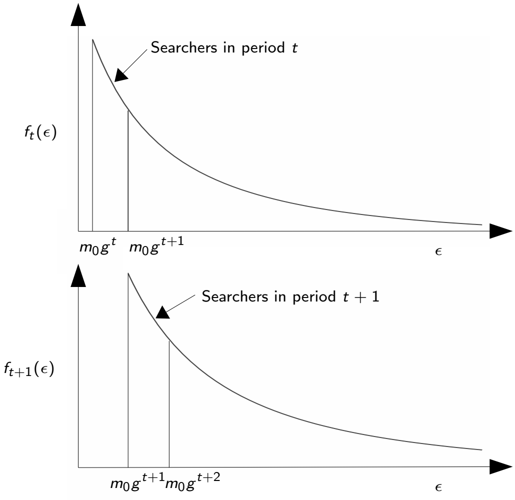
i) *The growth rate is:* $g = \left(\beta \frac{k}{k-1}\right)^{\frac{1}{\sigma - 1 + k}}$

ii) $h(m) = gm$

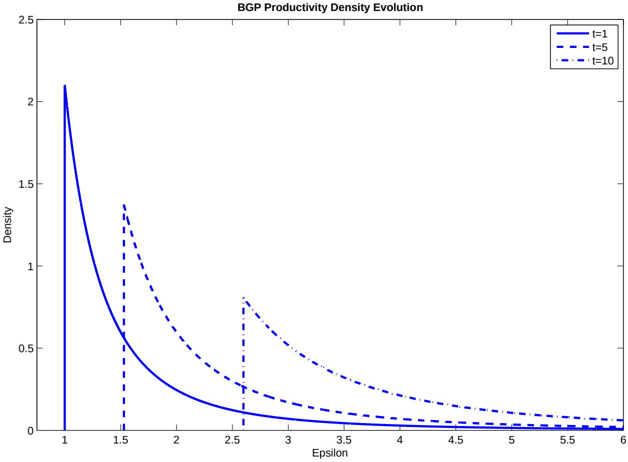
iii) *The production of the economy is linear in the aggregate state:* $Y(m) = \frac{k}{k-1} g^{1-k} m$

BGP Evolution of the Productivity Distribution

$$f_t(\epsilon) = k m_t^k \epsilon^{-k-1},\ m_t = m_0 g^t$$



BGP Evolution of the Productivity Distribution



Comparative Statics of Growth Rate

Proposition

The following properties hold for a solution to the BGP:

i) $\frac{\partial g}{\partial \beta} > 0$ *and* $\frac{\partial g}{\partial \sigma} < 0$

ii) g *is independent of* $\min \text{support}\{F_0\}$

iii) $\frac{\partial g}{\partial k} < 0$

- *$\downarrow k$ is $\uparrow$ inequality in Pareto*
- *Interpret $\downarrow k$ as broader opportunities in the economy*
- *Fatter tail generates higher growth*

BGP Proof Sketch

Guesses:

- Pareto(m_0, k) will fulfill BGP requirements for f_0
  - PDF: $f_0(\epsilon; m_0, k) = k m_0^k \epsilon^{-k-1}$ with $\text{support}\{f_0\} = [m_0, \infty)$
  - $\implies$ from LOM PDF at state m: $f_m(\epsilon) = k m^k \epsilon^{-k-1}$ with $\text{support}\{f_m\} = [m, \infty)$
- Reservation value is linear in m: $h(m) = gm$
- The value of search is linear in m: $V(\epsilon, m) = mW, \forall \epsilon < h(m)$

Verify and Solve:

i) Plug the Pareto guess into the indifference equation

ii) Simplify system of 2 equations containing g and W

iii) Solve for g and W, confirming they are not functions of m

Firm Problem with Guesses

Use Pareto f_0 and $h(m) = gm$

$$V(\epsilon, m) = \max\left\{\epsilon + \frac{1}{1+r}V(\epsilon, gm), \frac{1}{1+r}k(gm)^k \int_{gm}^{\infty} V(\epsilon', gm)\epsilon'^{-k-1}d\epsilon'\right\}$$

Indifference at $\epsilon = gm$:

$$\begin{aligned} V(gm, m) =& gm + \frac{1}{1+r}V(gm, gm) \\ =& \frac{1}{1+r}k(gm)^k \int_{gm}^{\infty} V(\epsilon', gm)\epsilon'^{-k-1}d\epsilon' \end{aligned}$$

Linear value of search guess gives 2 equalities:

$$mW = gm + \frac{1}{1+r}gmW \quad \text{(EQ1)}$$

$$= \frac{1}{1+r}k(gm)^k \int_{gm}^{\infty} V(\epsilon', gm)\epsilon'^{-k-1}d\epsilon' \quad \text{(EQ2)}$$

Solving EQ1

$$mW = gm + \frac{1}{1+r} gmW$$

Solving for W

$$W = \frac{g}{1 - g/(1+r)}$$

... independent of m, as required.

EQ2: Split Integral

Trick: Split the integral in EQ2 at *next* period's indifference point $(g^2 m)$ and use decision rule:

$$gm + \tfrac{1}{1+r} gmW = \tfrac{1}{1+r} k(gm)^k \int_{gm}^{g^2 m} V(\epsilon', gm)\epsilon'^{-k-1} d\epsilon' + \tfrac{1}{1+r} k(gm)^k \int_{g^2 m}^{\infty} V(\epsilon', gm)\epsilon'^{-k-1} d\epsilon'$$

. . . computing the integrals separately.

EQ2: First Integral

Firms will search *next* period if $\epsilon \leq g^2 m$ with value gmW.

$$\int_{gm}^{g^2 m} V(\epsilon', gm)\epsilon'^{-k-1} d\epsilon' = gmW \int_{gm}^{g^2 m} \epsilon'^{-k-1} d\epsilon'$$
$$= \frac{gmW}{k}(gm)^{-k}(1 - g^{-k})$$

EQ2: Second Integral

Firms will produce *next* period if $\epsilon > g^2 m$. Use the Bellman.

$$\int_{g^2 m}^{\infty} V(\epsilon', gm)\epsilon'^{-k-1} d\epsilon' = \int_{g^2 m}^{\infty} \left[\epsilon' + \tfrac{1}{1+r} V(\epsilon', g^2 m)\right] \epsilon'^{-k-1} d\epsilon'$$
$$= \frac{1}{k-1}(g^2 m)^{1-k} + \tfrac{1}{1+r} \int_{g^2 m}^{\infty} V(\epsilon', g^2 m)\epsilon'^{-k-1} d\epsilon'$$

...one last integral at state $g^2 m$.

EQ2: Third Integral

Trick: Use the indifference equation *next* period.
The aggregate state is gm and the indifference point is g^2m.

$$V(g^2m, gm) = g^2m + \tfrac{1}{1+r} g^2 mW \quad \text{(EQ1')}$$

$$= \tfrac{1}{1+r} k (g^2m)^k \int_{g^2m}^{\infty} V(\epsilon', g^2m) \epsilon'^{-k-1} d\epsilon' \quad \text{(EQ2')}$$

Rearrange

$$\int_{g^2m}^{\infty} V(\epsilon', g^2m) \epsilon'^{-k-1} d\epsilon' = \left(\tfrac{1}{1+r} k (gm)^k \right)^{-1} g^{-k} g^2 m (1 + \tfrac{1}{1+r} W)$$

EQ2: Collect Integrals

Combining all of the integrals into EQ2 and simplify

$$(1+r)g^k m = -Wm + \frac{k}{k-1} gm + gm(1 + \tfrac{1}{1+r} W)$$

Note that m drops out, helping confirm our guesses

Solve System for g, W

The system of equations is

$$W = \frac{g}{1 - g/(1+r)}$$

$$(1+r)g^k = -W + \frac{k}{k-1}g + g(1 + \tfrac{1}{1+r}W)$$

The solution, given parameter restrictions, is

$$g = \left[\frac{1}{1+r}\left(\frac{k}{k-1}\right)\right]^{\frac{1}{k-1}}$$

Finally: Use Consumer's IMRS

- Given a fixed r, this is a solution for a constant g
- Given a fixed g, consumer problem gives $\frac{1}{1+r} = \beta g^{-\sigma}$

Substitute and rearrange

$$g = \left[\beta\left(\frac{k}{k-1}\right)\right]^{\frac{1}{\sigma-1+k}}$$

$$V(\epsilon, m) = Wm = \left(\frac{g}{1-\beta g^{1-\sigma}}\right) m, \; \forall \epsilon \leq gm$$

Parameter constraints for positive, finite growth

$$1 < \beta\left(\frac{k}{k-1}\right), \quad \beta g^{1-\sigma} < 1$$