1 THE GAME

Two players, i=1, 2
U_i: concave, strictly increasing
f: concave, continuous, $f(0) \geq 0$
$\beta \in (0,1)$: discount factor, common

With
Law of motion of the state:

$$k_{t+1} = f(k_t) - c_t^1 - c_t^2$$

Payoff:

$$\sum_{t=0}^{\infty} \beta^t U_i(C_t^1), i = 1, 2$$

Histories:

$$H_t = \{(c_1^1, c_1^2, ..., c_t^1, c_t^2)\}$$

Strategies:

$$\sigma_t^i : H_t \mapsto [0, \infty)$$

A game is naturally defined.
Equilibria are Subgame Perfect Equilibria.

2 ALLOCATION RULE

For two consumption rates, c^1, c^2:

$$A_1(c^1, c^2, k) \equiv \begin{cases} c^1, & if\ c^1 + c^2 \leq f(k), or\ c^1 \leq \frac{f(k)}{2} \\ f(k) - c^2 & if c^1 + c^2 \geq f(k)\ and\ c^1 \geq \frac{f(k)}{2} \geq c^2 \\ \frac{f(k)}{2} & if\ c^1, c^2 \geq \frac{f(k)}{2} \end{cases}$$

A_2 is defined similarly.

3 FAST CONSUMPTION STRATEGY

$\vec{c}^i(k) = f(k), i = 1, 2$
NOTE: $(\vec{c}^1, \vec{c}^2)_{t\geq 0}$ is a SPE.

Let $\overline{\underline{V}}(k)$ be the total utility to the ith player of the fast consumption SPE.

4 THE SECOND BEST PROBLEM

Given k, initial stock, the Second Best Value is given by

$$V_{sb}(k) \equiv sup \sum_t \beta^t [\alpha_1 U_1(c_t^1) + \alpha_2 U_2(c_t^2)]$$

subject to $f(k_t) - c_t^1 - c_t^2 = k_{t+1}$, and $k_0 = k$,
where the supremum is taken over the sequences $(c_t^1, c_t^2)_{t \geq 0}$ of outcomes of SPE.

THE FIRST BEST (FB) value is defined as the supremum over feasible consumption paths $(c_t^1, c_t^2)_{t \geq 0}$, and is denoted $\widehat{V}$.

5 TRIGGER STRATEGY PAIR

Trigger Strategy Pair is described by:

1. An agreed consumption path, $(c_t^1, c_t^2)_{t \geq 0}$;

2. The threat to switch to a fast consumption, if a defection of either player from the agreed consumption path is detected.

6 INDIVIDUAL RATIONALITY CONSTRAINT

If $(c_t^1, c_t^2)_{t\geq 0}$ is a SPE OUTCOME, then:

$$\sum_t \beta^t U_i(c_t^i) \geq \underline{V}(k)$$

The DEFECTION VALUE

$$V_i^D(k,c) = \max\{sup_{c^1\geq 0} U_i(c^1) + \beta U_i(\frac{f(f(k)-c-c^1)}{2}) + \frac{\beta^2}{1-\beta} U_i(\frac{f(0)}{2}), \underline{V}^i(k)\}$$

The solution to this problem is denotes as $c_i^D(k,c)$

Lemma Any SPE outcome $(c_t^1, c_t^2)_{t\geq 0}$ is the outcome of a trigger strategy equilibrium. Therefore, the Second Best Problem can be equivalently reformulated as:

$$V_{sb}(k) \equiv sup_{\{(c_t^1, c_t^2)_{t\geq 0}\}} \sum_t \beta^t [\alpha_1 U_1(c_t^1) + \alpha_2 U_2(c_t^2)]$$

$$subject\ to\ f(k_t) - c_t^1 - c_t^2 = k_{t+1}, k_0 = k$$

$$(*) \sum_{n=0}^{\infty} \beta^n U_i(c_{t+n}^i) \geq V_i^D(k_t, c_t^{i'}) \quad t = 1, 2, ... i = 1, 2.$$

The constraint (*) is referred to as Incentive Compatibility Constraint.

7 THE SYMMETRIC CASE

$U_i = U$, $\alpha_i = \frac{1}{2}$, $i = 1{,}2$;

The Bellman operator for The Second Best Problem:

$$T_V(k) \equiv \max_{c \in A(k,\nu)} \{U(c) + \beta V(f(k) - 2c)\}$$

where

$$A(k, V) \equiv \{c : c \geq 0, V(k) \geq V^D(k, c)\}$$

T is not a contraction.

THEOREM

Assume $\lim_{k \to \infty} f'(k)\beta < 1$, then:

1. A solution to the Second Best Problem exists;
2. V_{sb} is upper-semicontinuous.

Remark. This is not a standard Dynamic Programming Problem.

It is true that:

$$V_{sb}(k) = \max_{c \in A(k,\nu)} \{U(c) + \beta V_{sb}(f(k) - 2c)\}$$

and analogous equations for the non-symmetric case; But even if U, f are concave, V_{sb} is not necessarily concave.

The symmetric second-best equilibrium is the greatest growing one.

PROPOSITION: For a given k, let $(c_1^t, c_2^t)_{t \geq 0}$ be a second best equilibrium outcome, starting from k, which is symmetric and such that $V_i(k) = V^D(k, c_i)$ (that is second best is not first best), $i = 1, 2$. If $k_1 = f(k) - c_1 - c_2$ and k' is the next period capital for any subgame perfect equilibrium, then $k' \leq k_1$.

8 A FIRST EXAMPLE

$$U_i(c) = \frac{c^{1-\epsilon}}{1-\epsilon}, \epsilon \in (0,1), \alpha_1 = \alpha_2 = 1/2$$
$$y = f(k) = ak, a \geq 1$$

FIRST BEST SOLUTION

$$\widehat{c_1}(y) = \widehat{c_2}(y) = \widehat{\lambda} y$$

$$\widehat{\lambda} = \frac{1}{2}[1 - \beta^{\frac{1}{\epsilon}} a^{\frac{1-\epsilon}{\epsilon}}]$$
$$\widehat{V}(k) = s(\widehat{\lambda}) k^{1-\epsilon},$$

where $s(\lambda) \equiv \frac{\lambda^{1-\epsilon}}{(1-\epsilon)[1-\beta(a(1-2\lambda))^{1-\epsilon}]}$

In general, the value of a consumption policy $c(y) = \lambda y$ for $\lambda \in (0, \frac{1}{2})$ is

$$V(k) = s(\lambda) k^{1-\epsilon}$$

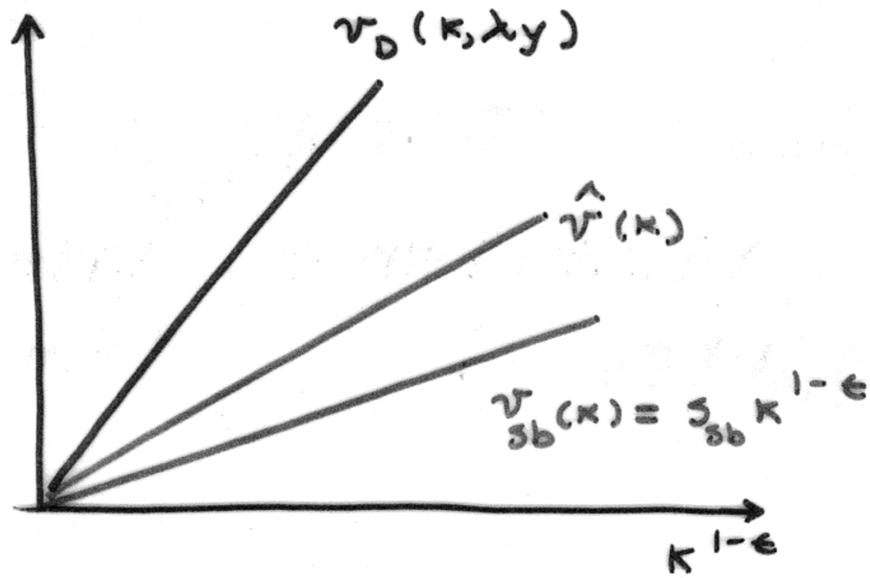
Defection from a policy $c(y) = \lambda y$ is

$$c^D(k, \lambda y) = \lambda_D y, \quad \lambda_D = M(1-\lambda)$$

$$V^D(k, \lambda_y) = s_D(\lambda) k^{1-\epsilon}; \quad s_D(\lambda) = \frac{(1-\lambda)^{1-\epsilon} M^{\epsilon}}{1-\epsilon}$$

where

$$M^{-1} = 1 + (\frac{a\beta}{2})^{\frac{1}{\epsilon}} \frac{2}{a} > 1$$



Proposition. The Second Best Consumption policy is given by:
1. $c_{sb}(y) = \widehat{\lambda} y$ if $s(\widehat{\lambda}) \geq s_D(\widehat{\lambda})$
2. $c_{sb}(y) = \lambda_{sb} y$ if $s(\widehat{\lambda}) < s_D(\widehat{\lambda})$
where

$$\lambda_{sb} \equiv \min\{\lambda : \lambda \in [\widehat{\lambda}, \frac{N}{1+M}] \ and \ s(x) = s_D(\lambda)\}$$

If $a = 3.3$, $\beta = 0.325$, $\epsilon = 0.5$, then

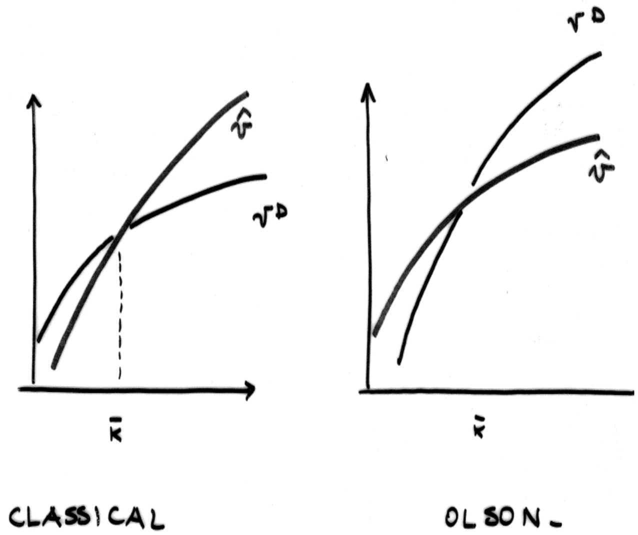
$$V^D(k, \widehat{c}(k)) > \widehat{V}(k) \ for \ k > 0$$

$$\widehat{\lambda} = 0.326, \lambda_{sb} = 0.349$$

FIRST BEST: Growth at 15 % , SECOND BEST: Contraction at 0.0015 %

Another example
If $a = 1.058$, $\beta = 0.95$, $\epsilon = 0.1$, $a\beta > 1$ then
FIRST BEST: Growth at 5.2 % SECOND BEST: Contraction at -99 %

In general:



PROPOSITION Assume

1. for some k, $\widehat{V}(k) < V^D(k, \widehat{\lambda}k)$
2. there exists a $\tilde{c}$ such that:

$$U(\tilde{c} + \beta\widehat{\nu}(f(k) - 2\tilde{c})) = \nu^D(k, \tilde{c}), \ \textit{and}$$

$$f(k) - 2\tilde{c} \leq k;$$

then,

$$f(k) - 2c_{sb}(k) \leq k$$

9 EXAMPLE OF WEALTH DEPENDENT GROWTH

Let:
$a = 1.875; b = 0.2; y = ak + be;$
$\beta = 0.55; e = 0.1; \epsilon = 0.45$

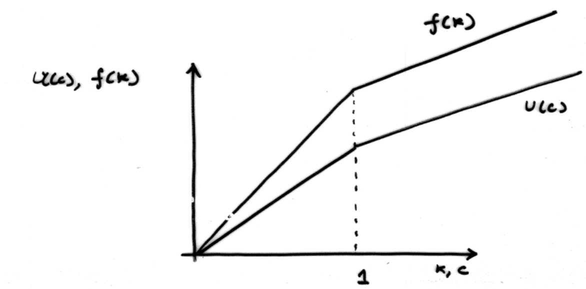
FIRST BEST SOLUTION: Growth at 7 % for $k > 10^{-3}$
But,

$$\widehat{V}(k) \geq V^D(k, \widehat{\lambda} y) \; for \; k \geq 1$$

$$\widehat{V}(k) < V^D(k, \widehat{\lambda} y) \; for \; k \leq 0.9$$

for $k \in [0.1, 0.4]$ the proposition applies, so
SECOND BEST SOLUTION: No growth for $k \in [0.1, 0, 4]$

Another example
$a = 1.0575; b = 0.2\beta = 0.95; e = 0.2a\beta > 1$
No growth for $k \in [0.3, 0.9]$
First best for $k \geq 1.2$

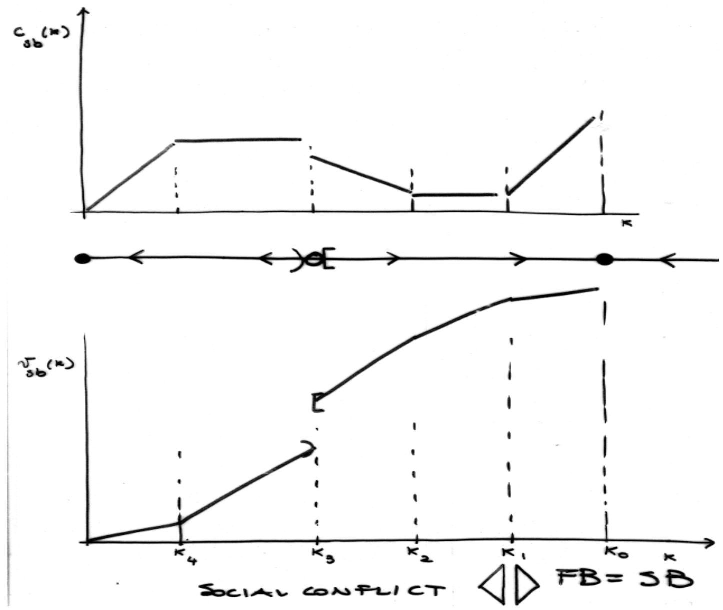


10 A CLASSICAL TRAP

$$f(k) = \begin{cases} Ak & if\ k < 1 \\ A + B(k-1) & if\ k \geq 1 \end{cases}$$

$$U(c) = \begin{cases} c & if\ c \leq 1 \\ 1 + b(c-1) & if\ c \geq 1 \end{cases}$$

with $A\beta > 1 > b > B\beta$
$A = \frac{5}{2}; \beta = \frac{1}{2}; B > 2$



10.1 AN OLSON ECONOMY

$U(c) = c$
$f(k) = k^{\alpha}$

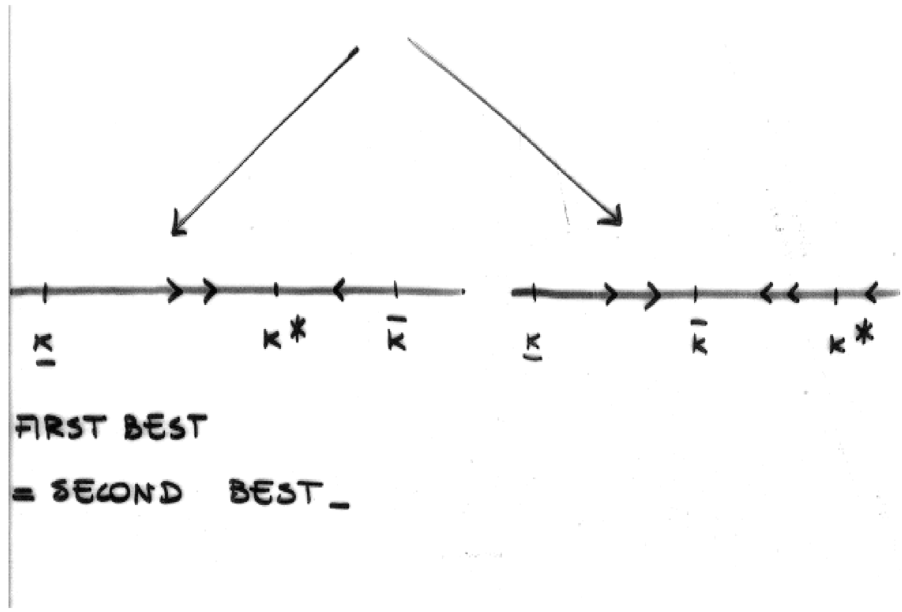
FIRST BEST SOLUTION
Steady state: $k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}$
Consumption:

$$\widehat{c}(k) = \begin{cases} 0 & if \quad k \leq (k^*)^{\frac{1}{\alpha}} \\ \frac{k^{\alpha} - k^*}{2} & if \ k \geq (k^*)^{\frac{1}{\alpha}} \end{cases}$$

DEFECTION from FIRST BEST: Consumption:

$$c^D(k, c) = \begin{cases} 0 & iff \ (k^{\alpha} - c)^{\alpha - 1} \geq \frac{2}{\alpha\beta} \\ k^{\alpha} - c - \gamma & otherwise \end{cases}$$

where $\gamma = (\frac{\alpha\beta}{2})^{\frac{1}{1-\alpha}}$



11 INCENTIVE COMPATIBLE STEADY STATE

k such that:

$$\frac{1}{1-\beta}\frac{f(k)-k}{2} \geq V^D(k, \frac{f(k)-k}{2}) \qquad (*)$$

The set of k's which satisfies (*) is an interval, $[\underline{k}, \overline{k}]$.

Two cases:
FIRST BEST = SECOND BEST

Failure of Blackwell with incentive constraints that depend on the control:

1. Standard Case (Discounting condition)

$$T(v) = Max_{c_t \geq 0}\, U(c_t) + \beta v(f(k_t - 2c_t))$$

$$\begin{aligned} T(v+\alpha) &= Max_{c_t \geq 0}\, U(c_t) + \beta(v(f(k_t - 2c_t)) + \alpha) \\ &= Max_{c_t \geq 0}\, U(c_t) + \beta v(f(k_t - 2c_t)) + \beta\alpha \end{aligned}$$

So Blackwell holds because equality suffices:

$$T(v+\alpha) \leq T(v) + \beta\alpha$$

2.Case with incentive constaints:

$$T(v) = Max_{c_t \in R(0)}\, U(c_t) + \beta v(f(k_t - 2c_t))$$

where

$$\begin{aligned} & R(\alpha)\ \ is\ defined\ by\ the\ set\ c(k)\ \ satisfying \\ R(\alpha) &= \left\{c_t \mid v(k) + \alpha > v^d(k,c)\right\} \\ \frac{\partial v^d(k,c)}{\partial k} &> 0, \ \ \frac{\partial v^d(k,c)}{\partial c} < 0 \end{aligned}$$

Note here that $\frac{\partial v^d(k,c)}{\partial c} < 0$ because deviating from agreed upon consumption path $\{c_t\}$ is more beneficial when c_t is lower-see Benhabib-Rustichini, JOEG, 1996.

$$
\begin{aligned}
T\left(v+\alpha\right) &= Max_{c_t \in R(\alpha)}\ U\left(c_t\right)+\beta\left(v\left(f\left(k_t-2c_t\right)\right)+\alpha\right) \\
&= Max_{c_t \in R(\alpha)}\ U\left(c_t\right)+v\left(f\left(k_t-2c_t\right)\right)+\beta\alpha
\end{aligned}
$$

But we cannot claim, as in the standard case

$$
\begin{aligned}
T\left(v\right) &= Max_{c_t \in R(0)}\ U\left(c_t\right)+\beta v\left(f\left(k_t-2c_t\right)\right) \\
&= Max_{c_t \in R(\alpha)}\ U\left(c_t\right)+v\left(f\left(k_t-2c_t\right)\right) \\
T\left(v+\alpha\right) &\leq T\left(v\right)
\end{aligned}
$$

because the set $c_t \in R\left(\alpha\right)$ is larger than $c_t \in R\left(0\right) > 0$ This is because $\left\{c_t \mid v(k)+\alpha > v^d(k,c)\right\}$ is larger with $\alpha > 0$, since $v^d(k,c)$ decreases with c. Furthermore, if incentive constraint is binding, you are at the corner: you would like smaller c and accumulate more, but you do not in order to curtail other player from grabbing what you don't consume. Thus when $\alpha > 0$ constraint is relaxed and you can lower c. Result is Blackwell fails: continuity of v cannot be assured.

Intuition for the discontinuity: if the value function jumps at k, why not restrain consumption by ε, and take a jump in value by driving k a little higher. Answer, If you were to curtail consumption even just a bit, you create incentives for the other player to grab more.