1 THE GAME

Two players, $i=1, 2$
$U_i$: concave, strictly increasing
$f$: concave, continuous, $f(0) \geq 0$
$\beta \in (0,1)$: discount factor, common

With
Law of motion of the state:
$$k_{t+1} = f(k_t) - c_1^t - c_2^t$$

Payoff:
$$\sum_{t=0}^{\infty} \beta^t U_i(C_t^i), i = 1, 2$$

Histories:
$$H_t = \{(c_1^1, c_1^2, ..., c_t^1, c_t^2)\}$$

Strategies:
$$\sigma_i^t : H_t \mapsto [0, \infty)$$

A game is naturally defined.
Equilibria are Subgame Perfect Equilibria.


2 ALLOCATION RULE

For two consumption rates, $c^1, c^2$:

$$A_1(c^1, c^2, k) \equiv \begin{cases} c^1, & \text{if } c^1 + c^2 \leq f(k), \text{ or } c^1 \leq \frac{f(k)}{2} \\ f(k) - c^2, & \text{if } c^1 + c^2 \geq f(k) \text{ and } c^1 \geq \frac{f(k)}{2} \geq c^2 \\ \frac{f(k)}{2}, & \text{if } c^1, c^2 \geq \frac{f(k)}{2} \end{cases}$$

$A_2$ is defined similarly.
3 FAST CONSUMPTION STRATEGY

$\hat{c}^i(k) = f(k), i = 1, 2$

NOTE: $(\hat{c}^1, \hat{c}^2)_{t \geq 0}$ is a SPE.

Let $\bar{V}(k)$ be the total utility to the ith player of the fast consumption SPE.
4 THE SECOND BEST PROBLEM

Given $k$, initial stock, the Second Best Value is given by

\[ V_{sb}(k) \equiv \sup \sum_t \beta^t [\alpha_1 U_1(c^1_t) + \alpha_2 U_2(c^2_t)] \]

subject to

\[ f(k_t) - c^1_t - c^2_t = k_{t+1}, \text{ and } k_0 = k, \]

where the supremum is taken over the sequences $(c^1_t, c^2_t)_{t\geq0}$ of outcomes of SPE.

THE FIRST BEST (FB) value is defined as the supremum over feasible consumption paths $(c^1_t, c^2_t)_{t\geq0}$, and is denoted $\hat{V}$. 
5 TRIGGER STRATEGY PAIR

Trigger Strategy Pair is described by:

1. An agreed consumption path, \((c^1_t, c^2_t)_{t\geq 0}\);

2. The threat to switch to a fast consumption, if a defection of either player from the agreed consumption path is detected.
6 INDIVIDUAL RATIONALITY CONSTRAINT

If \((c^1_t, c^2_t)_{t \geq 0}\) is a SPE OUTCOME, then:

\[
\sum_t \beta^t U_i(c^i_t) \geq V(k)
\]

The DEFECTION VALUE

\[
V_i^D(k, c) = \max \{ \sup_{c: c \geq 0} U_i(c^1) + \beta U_i(\frac{f(f(k) - c - c^1)}{2}) + \frac{\beta^2}{1 - \beta} U_i(\frac{f(0)}{2}), V^i(k) \}
\]

The solution to this problem is denoted as \(c^D_i(k, c)\)

**Lemma** Any SPE outcome \((c^1_t, c^2_t)_{t \geq 0}\) is the outcome of a trigger strategy equilibrium. Therefore, the Second Best Problem can be equivalently reformulated as:

\[
V_{sb}(k) \equiv \sup_{(c^1_t, c^2_t)_{t \geq 0}} \sum_t \beta^t [\alpha_1 U_1(c^1_t) + \alpha_2 U_2(c^2_t)]
\]

subject to \(f(k_t) - c^1_t - c^2_t = k_{t+1}, k_0 = k\)

\[(*) \sum_{n=0}^{\infty} \beta^n U_i(c^i_{t+n}) \geq V_i^D(k_t, c^i_t) \quad t = 1, 2, \ldots i = 1, 2.\]

The constraint (*) is referred to as Incentive Compatibility Constraint.
7 THE SYMMETRIC CASE

$U_i = U$, $\alpha_i = \frac{1}{2}$, $i = 1,2$;

The Bellman operator for The Second Best Problem:

$$T_V(k) \equiv \max_{c \in A(k, \nu)} \{U(c) + \beta V(f(k) - 2c)\}$$

where

$$A(k, V) \equiv \{c : c \geq 0, V(k) \geq V^D(k, c)\}$$

$T$ is not a contraction.
THEOREM
Assume \( \lim_{k \to \infty} f'(k) \beta < 1 \), then:
1. A solution to the Second Best Problem exists;
2. \( V_{sb} \) is upper-semicontinuous.

Remark. This is not a standard Dynamic Programming Problem.

It is true that:
\[
V_{sb}(k) = \max_{c \in A(k, \nu)} \{ U(c) + \beta V_{sb}(f(k) - 2c) \}
\]
and analogous equations for the non-symmetric case; But even if \( U, f \) are concave, \( V_{sb} \) is not necessarily concave.

The symmetric second-best equilibrium is the greatest growing one.

PROPOSITION: For a given \( k \), let \( (c'_{1}, c'_{2})_{t \geq 0} \) be a second best equilibrium outcome, starting from \( k \), which is symmetric and such that \( V_{i}(k) = V^{D}(k, c_{i}) \) (that is second best is not first best), \( i = 1, 2 \). If \( k_{1} = f(k) - c_{1} - c_{2} \) and \( k' \) is the next period capital for any subgame perfect equilibrium, then \( k' \leq k_{1} \).
A FIRST EXAMPLE

\[ U_i(c) = \frac{c^{1-\epsilon}}{1-\epsilon}, \epsilon \in (0,1), \alpha_1 = \alpha_2 = 1/2 \]
\[ y = f(k) = ak, a \geq 1 \]

FIRST BEST SOLUTION

\[ \tilde{c}_1(y) = \tilde{c}_2(y) = \tilde{\lambda}y \]
\[ \tilde{\lambda} = \frac{1}{2}[1 - \beta^{\frac{1}{2}}a^{\frac{1-\epsilon}{\epsilon}}] \]
\[ \tilde{V}(k) = s(\tilde{\lambda})k^{1-\epsilon} \]

where \( s(\lambda) \equiv \frac{\lambda^{1-\epsilon}}{(1-\epsilon)(1-\beta(\alpha(1-2\lambda))^{1-\epsilon})} \)

In general, the value of a consumption policy \( c(y) = \lambda y \) for \( \lambda \in (0, \frac{1}{2}) \) is

\[ V(k) = s(\lambda)k^{1-\epsilon} \]

Defection from a policy \( c(y) = \lambda y \) is

\[ c^D(k, \lambda y) = \lambda_D y, \lambda_D = M(1-\lambda) \]
\[ V^D(k, \lambda y) = s_D(\lambda)k^{1-\epsilon}, s_D(\lambda) = \frac{(1-\lambda)^{1-\epsilon}M^\epsilon}{1-\epsilon} \]

where

\[ M^{-1} = 1 + \left(\frac{a\beta}{2}\right)^{\frac{1}{2}} \frac{2}{a} > 1 \]
**Proposition.** The Second Best Consumption policy is given by:

1. \( c_{sb}(y) = \bar{\lambda}y \) if \( s(\bar{\lambda}) \geq s_{D}(\bar{\lambda}) \)
2. \( c_{sb}(y) = \lambda_{sb}y \) if \( s(\bar{\lambda}) < s_{D}(\bar{\lambda}) \)

where
\[
\lambda_{sb} \equiv \min\{\lambda : \lambda \in [\bar{\lambda}, \frac{N}{1+M}] \text{ and } s(x) = s_{D}(\lambda)\}
\]

If \( a = 3.3, \beta = 0.325, \epsilon = 0.5, \) then
\[
V_{D}(k, \bar{\lambda}(k)) > \hat{V}(k) \text{ for } k > 0
\]

\( \hat{\lambda} = 0.326, \lambda_{sb} = 0.349 \)

**FIRST BEST:** Growth at 15 % ,  **SECOND BEST:** Contraction at 0.0015 %

**Another example**
If \( a = 1.058, \beta = 0.95, \epsilon = 0.1, a\beta > 1 \) then
**FIRST BEST:** Growth at 5.2 %  **SECOND BEST:** Contraction at -99 %
In general:
PROPOSITION Assume
1. for some $k$, $\tilde{V}(k) < V^D(k, \tilde{\lambda}k)$
2. there exists a $\tilde{c}$ such that:

$$U(\tilde{c} + \beta \tilde{\nu}(f(k) - 2\tilde{c})) = \nu^D(k, \tilde{c})$$, and

$$f(k) - 2\tilde{c} \leq k;$$

then,

$$f(k) - 2c_{ab}(k) \leq k$$
9 EXAMPLE OF WEALTH DEPENDENT GROWTH

Let:
\[ a = 1.875; b = 0.2; y = ak + be; \]
\[ \beta = 0.55; e = 0.1; \epsilon = 0.45 \]

FIRST BEST SOLUTION: Growth at 7 % for \( k > 10^{-3} \)
But,
\[ \tilde{V}(k) \geq V^D(k, \tilde{\lambda}y) \text{ for } k \geq 1 \]
\[ \tilde{V}(k) < V^D(k, \tilde{\lambda}y) \text{ for } k \leq 0.9 \]

for \( k \in [0.1, 0.4] \) the proposition applies, so
SECOND BEST SOLUTION: No growth for \( k \in [0.1, 0.4] \)

Another example
\[ a = 1.0575; b = 0.2\beta = 0.95; e = 0.2a\beta > 1 \]
No growth for \( k \in [0.3, 0.9] \)
First best for \( k \geq 1.2 \)
10 A CLASSICAL TRAP

\[ f(k) = \begin{cases} 
Ak & \text{if } k < 1 \\
A + B(k - 1) & \text{if } k \geq 1
\end{cases} \]

\[ U(c) = \begin{cases} 
c & \text{if } c \leq 1 \\
1 + b(c - 1) & \text{if } c \geq 1
\end{cases} \]

with \( A\beta > 1 > b > B\beta \)

\( A = \frac{5}{2}; \beta = \frac{1}{2}; B > 2 \)
10.1 AN OLSON ECONOMY

\[ U(c) = c \]
\[ f(k) = k^\alpha \]

FIRST BEST SOLUTION
Steady state: \( k^* = (\alpha \beta)^\frac{1}{1-\alpha} \)
Consumption:

\[ \tilde{c}(k) = \begin{cases} 
0 & \text{if } k \leq (k^*)^{\frac{1}{\alpha}} \\
\frac{k^\alpha - k^*}{2} & \text{if } k \geq (k^*)^{\frac{1}{\alpha}}
\end{cases} \]

DEFECTION from FIRST BEST: Consumption:

\[ c^D(k, c) = \begin{cases} 
0 & \text{if } f(k^\alpha - c)^{\alpha - 1} \geq \frac{2}{\alpha \beta} \\
k^\alpha - c - \gamma & \text{otherwise}
\end{cases} \]

where \( \gamma = (\frac{\alpha \beta}{2})^{\frac{1}{1-\alpha}} \)
11 INCENTIVE COMPATIBLE STEADY STATE

\( k \) such that:

\[
\frac{1}{1 - \beta} \frac{f(k) - k}{2} \geq V^D(k, \frac{f(k) - k}{2}) \tag{*}
\]

The set of \( k \)'s which satisfies (*) is an interval, \([k, \bar{k}]\).

Two cases:
FIRST BEST = SECOND BEST
Failure of Blackwell with incentive constraints that depend on the control:

1. Standard Case (Discounting condition)

\[ T(v) = \max_{c_t \geq 0} U(c_t) + \beta v(f(k_t - 2c_t)) \]

\[ T(v + \alpha) = \max_{c_t \geq 0} U(c_t) + \beta v(f(k_t - 2c_t)) + \alpha \]

So Blackwell holds because equality suffices:

\[ T(v + \alpha) \leq T(v) + \beta \alpha \]
2. Case with incentive constraints:

\[ T(v) = \max_{c_t \in R(0)} U(c_t) + \beta v(f(k_t - 2c_t)) \]

where

\[ R(\alpha) \text{ is defined by the set } c(k) \text{ satisfying} \]

\[ R(\alpha) = \{ c_t | v(k) + \alpha > v^d(k, c) \} \]

\[ \frac{\partial v^d(k, c)}{\partial k} > 0, \quad \frac{\partial v^d(k, c)}{\partial c} < 0 \]

Note here that \( \frac{\partial v^d(k, c)}{\partial c} < 0 \) because deviating from agreed upon consumption path \( \{c_t\} \) is more beneficial when \( c_t \) is lower-see Benhabib-Rustichini, JOEG, 1996.
\[ T(v + \alpha) = \max_{c_t \in R(\alpha)} U(c_t) + \beta (v (f(k_t - 2c_t)) + \alpha) \]
\[ = \max_{c_t \in R(\alpha)} U(c_t) + v (f(k_t - 2c_t)) + \beta \alpha \]

But we cannot claim, as in the standard case

\[ T(v) = \max_{c_t \in R(0)} U(c_t) + \beta v (f(k_t - 2c_t)) \]
\[ = \max_{c_t \in R(\alpha)} U(c_t) + v (f(k_t - 2c_t)) \]
\[ T(v + \alpha) \leq T(v) \]

because the set \( c_t \in R(\alpha) \) is larger than \( c_t \in R(0) > 0 \). This is because \( \{ c_t \mid v(k) + \alpha > v^d(k,c) \} \) is larger with \( \alpha > 0 \), since \( v^d(k,c) \) decreases with \( c \). Furthermore, if incentive constraint is binding, you are at the corner: you would like smaller \( c \) and accumulate more, but you do not in order to curtail other player from grabbing what you don’t consume. Thus when \( \alpha > 0 \) constraint is relaxed and you can lower \( c \). Result is Blackwell fails: continuity of \( v \) cannot be assured.
Intuition for the discontinuity: if the value function jumps at $k$, why not restrain consumption by $\varepsilon$, and take a jump in value by driving $k$ a little higher. Answer, If you were to curtail consumption even just a bit, you create incentives for the other player to grab more.