Solow Model

$$\dot{K} = sF(K,L) - \delta K$$
$$= sLF(k,1) - \delta K$$
$$\frac{\dot{K}}{L} = sf(k) - \delta k$$
$$\dot{k} = \frac{d(\frac{K}{L})}{dt} = \frac{\dot{K}}{L} - nk$$
$$where \frac{\dot{L}}{L} = n$$
$$\dot{k} = sf(k) - (\delta + n)k$$

Golden Rule

At steady state $\dot{k} = 0,$ and

$$and$$

$$Max_k \; c = f(k) - (\delta + n)k$$
$$f'(k) = (\delta + n)$$

β –Convergence (not conditional):

$$\gamma = \frac{\dot{k}}{k} = s\frac{f(k)}{k} - (\delta + n)$$

$$\frac{d\gamma}{dk} = s\left(\frac{kf'(k) - f(k)}{k^2}\right) = \frac{s}{k}\left(f'(k) - \frac{f(k)}{k}\right)$$

$$= \frac{s}{k}(MPK - APK) < 0$$

Technical Progress:

$$\dot{K} = sF(K, A(t)L) - \delta K$$

$$\frac{\dot{L}}{L} = n; \quad \frac{\dot{A}}{A} = \theta; \quad x(t) = \frac{K(t}{A(t)L(t)}$$

$$\dot{x} = sf(x) - (\delta + n + \theta)x$$

$$\gamma = \frac{\dot{x}}{x} = s\frac{f(x)}{x} - (\delta + n + \theta)$$

The Solow Residual

Production Function (CRS):

$$Q = AF(K, L)$$

$$\dot{Q} = F_k \dot{K} + F_L \dot{L} + \dot{A}F$$

$$\frac{\dot{Q}}{Q} = \left(\frac{KF_k}{Q}\right)\frac{\dot{K}}{K} + \left(\frac{LF_L}{Q}\right)\frac{\dot{L}}{L} + \frac{\dot{A}}{A}$$

$$\frac{\dot{A}}{A} = \frac{\dot{Q}}{Q} - \left(\frac{KF_k}{Q}\right)\frac{\dot{K}}{K} - \left(\frac{LF_L}{Q}\right)\frac{\dot{L}}{L}$$

Let

$$k = \frac{K}{L}, \quad q = \frac{Q}{L} = Af(k)$$

$$\dot{q} = \dot{A}f + Af'\dot{k}$$

$$\frac{\dot{q}}{q} = \frac{\dot{A}}{A} + \left(\frac{Akf'(k)}{Q}\right)\frac{\dot{k}}{k} = \frac{\dot{A}}{A} + w_k \frac{\dot{k}}{k}$$

Solow adjusted capital for utilization (multiplied by $\left(\frac{employment}{Labor\ force}\right)$), used non-farm output for q, used capital share of 33% from the data, and set $A_{1909} = 1$.

Then

$$\frac{\Delta q}{q} = \frac{\Delta A}{A} + w_k \frac{\Delta k}{k}$$

88% of growth accounted by $\frac{\Delta A}{A}$, 12% by $\frac{\Delta k}{k}$.

If there is a random element to productivity, $\frac{\dot{A}}{A} = \mu + \theta_t$

$$SR_t = \frac{\dot{A}}{A} = \mu + \theta_t = \frac{\dot{Q}}{Q} - \left(\frac{KF_k}{Q}\right)\frac{\dot{K}}{K} - \left(\frac{LF_L}{Q}\right)\frac{\dot{L}}{L}$$

Issues: (Denison).

Now, if there are variables in the economy that affect output or employment or capital utilization, uncorrelated with θ_t,, they will be reflected through the production function relation, and the SR will be uncorrelated with them, provided we have CRS and perfect competition.

Market Power, **Cost based shares**

$$Q = F(K, L)$$

$$Max_{L,K} \quad p(Q)Q - wL - rK$$

$$w = [p + Qp'(Q)]\frac{dQ}{dL}; \quad r = [p + Qp'(Q)]\frac{dQ}{dK}$$

$$\frac{dQ}{dL} = \frac{w}{p\left[1 + \frac{p'}{p}Q\right]} = \frac{w}{p(1 - \varepsilon^{-1})}$$

$$\frac{dQ}{dK} = \frac{r}{p\left[1 + \frac{p'}{p}Q\right]} = \frac{r}{p(1 - \varepsilon^{-1})}$$

Markup and Elasticity:

Markup is μ and elasticity is $\varepsilon = \left(\frac{dP(Q)}{dQ}\frac{Q}{P}\right)^{-1}$. Then

Marginal cost: $\frac{w}{Q_L} = p(1 - \varepsilon^{-1}) = \frac{r}{Q_K}$

$$\mu = \frac{p}{MC} = \frac{p}{\frac{r}{Q_k}} = \frac{p}{\frac{w}{Q_L}} = (1 - \varepsilon^{-1})^{-1}$$

$\varepsilon = \infty$ is perfect competition: the lower the demand elasticity, the higher the markup

COST SHARES: From CRS

$$Q = \frac{dQ}{dL}L + \frac{dQ}{dK}K = \frac{wL + rK}{p(1 - \varepsilon^{-1})}$$

$$pQ = \frac{wL + rK}{(1 - \varepsilon^{-1})} > wL + rK$$

$$\frac{Q}{wL + rK} = \frac{1}{p(1 - \varepsilon^{-1})},$$

Cost Share : $\dfrac{wQ}{wL + rK} = \dfrac{w}{p(1 - \varepsilon^{-1})}$

$$\frac{Q}{wL + rK} = \frac{1}{p(1 - \varepsilon^{-1})},$$

Cost Share : $\dfrac{rQ}{wL + rK} = \dfrac{r}{p(1 - \varepsilon^{-1})}$

So it follows that (Cost Shares)

$$\frac{wQ}{wL + rK} = \frac{w}{p(1 - \varepsilon^{-1})} = \frac{dQ}{dL} \geq \frac{w}{p}$$

$$\frac{rQ}{wL + rK} = \frac{r}{p(1 - \varepsilon^{-1})} = \frac{dQ}{dK} \geq \frac{r}{p}$$

Correction for Market Power:

$$\frac{dQ}{Q} = \left(\frac{K}{Q} \frac{\partial F}{\partial K} \right) \frac{dK}{K} + \left(\frac{L}{Q} \frac{\partial F}{\partial L} \right) \frac{dL}{L} + \theta$$

Now define

$$\frac{\partial F}{\partial L} = \alpha \frac{Q}{L} = \frac{wQ}{wL + rK}$$

$$1 = \frac{\partial F}{\partial L} \frac{L}{Q} + \frac{\partial F}{\partial K} \frac{K}{Q} \quad \textit{from CRS}$$

$$1 - \alpha = \frac{\partial F}{\partial K} \frac{K}{Q}$$

So (COST BASED SHARES):

$$\alpha = \frac{\partial F}{\partial L} \frac{L}{Q} = \frac{wL}{wL + rK}$$

$$1 - \alpha = \frac{\partial F}{\partial K} \frac{K}{Q} = \frac{rK}{wL + rK}$$

$$\frac{dQ}{Q} = \alpha \frac{dL}{L} + (1 - \alpha) \frac{dK}{K} + \theta$$

These $\alpha' s$ are different from Solow's shares $\alpha_s = \frac{wL}{pQ}$:Since

$$pQ = \frac{wL + rK}{(1 - \varepsilon^{-1})}$$

$$\alpha = \frac{wL}{rK + wL} = \frac{wL}{pQ(1 - \varepsilon^{-1})} > \frac{wL}{pQ} = \alpha_s$$

$$\alpha_s = \alpha(1 - \varepsilon^{-1})$$

Because $\alpha > \alpha_s$, an increase in L does not generate enough of an increase in Q under Solow's accounting with market power:

$$\frac{dQ}{Q} = \alpha\frac{dL}{L} + (1-\alpha)\frac{dK}{K} + \theta$$

$$= \frac{\alpha_s}{1-\varepsilon^{-1}}\frac{dL}{L} + \left(1 - \frac{\alpha_s}{1-\varepsilon^{-1}}\right)\frac{dK}{K} + \theta$$

Since markup, $\mu = (1-\varepsilon^{-1})^{-1}$

$$\frac{dQ}{Q} = \alpha_s\mu\frac{dL}{L} + (1-\alpha_s\mu)\frac{dK}{K} + \theta$$

But

$$SR = \theta = \frac{dQ}{Q} - \alpha_s\frac{dL}{L} - (1-\alpha_s)\frac{dK}{K}$$

$$= (\mu - 1)\alpha_s\left(\frac{dL}{L} - \frac{dK}{K}\right)$$

Higher $\frac{L}{K}$ increases SR under Solow accounting, but not under cost shares.

Increasing Returns

$$Q = F(K, L)$$

$$\delta Q = \frac{\partial F}{\partial K} K + \frac{\partial F}{\partial L} L, \qquad \delta > 1$$

$$\delta Q = rK + wL,$$

Note: under IR There are losses.

$$Q = \frac{rK + wL}{\delta} \; : \quad \frac{Q}{rK + wL} = \frac{1}{\delta}$$

$$w \frac{\delta Q}{rK + wL} = w = \frac{\partial F}{\partial L}; \quad r \frac{\delta Q}{rK + wL} = r = \frac{\partial F}{\partial K};$$

COST BASED SHARES UNDER IR:

Define α :

$$\frac{\partial F}{\partial L} = \frac{\delta w Q}{rK + wL} = \alpha\delta\frac{Q}{L};$$

$$\alpha = \frac{wL}{rK + wL}; \quad (1 - \alpha) = \frac{rK}{rK + wL}$$

So

$$dQ = \frac{\partial F}{\partial K}dK + \frac{\partial F}{\partial L}dL$$

$$\frac{dQ}{Q} = \frac{K}{Q}\frac{\partial F}{\partial K}\frac{dK}{K} + \frac{L}{Q}\frac{\partial F}{\partial L}\frac{dL}{L} + \theta$$

Since

$$r = \frac{\partial F}{\partial K} = \frac{r\delta Q}{rK + wL}; \quad w = \frac{\partial F}{\partial L} = \frac{w\delta Q}{rK + wL}$$

$$\frac{dQ}{Q} = \frac{K}{Q}\frac{\delta r Q}{rK + wL}\frac{dK}{K} + \frac{L}{Q}\frac{\delta w Q}{rK + wL}\frac{dL}{L} + \theta$$

$$\frac{dQ}{Q} = \delta\left[\frac{rK}{rK + wL}\frac{dK}{K} + \frac{wL}{rK + wL}\frac{dL}{L}\right] + \theta$$

$$\frac{dQ}{Q} = \delta\left[(1 - \alpha)\frac{dK}{K} + \alpha\frac{dL}{L}\right] + \theta$$

Then, SR based on cost shares:

$$\frac{dQ}{Q} = \delta\left[(1-\alpha)\frac{dK}{K} + \alpha\frac{dL}{L}\right] + \theta$$

$$SR = \frac{dQ}{Q} - \alpha\frac{dL}{L} - (1-\alpha)\frac{dK}{K}$$

$$= \theta + (\delta - 1)\alpha\frac{dL}{L} + (\delta - 1)(1-\alpha)\frac{dK}{K}$$

So SR on cost based shares now correlated with changes in labor and capital

Modelling difficulties

Estimating problems:

$$Q_t = A_t F(K_t, L_t)$$

a)Variations in capacity utilization of capital biases coefficient of L upwards with short-run, high frequency data.

b) In general business cycles variations can interfere.

c) Suppose $A_t = A_{t-1} + \varepsilon_t$, so A_t is serially correlated. High A_t today implies high A_{t+1}. But high A_t causes high savings and high investment, and therefore high K_{t+1}, which implies that corr($K_{t+1}, \varepsilon_{t+1}$) > 0. This also biases estimates. Differencing does not help unless ε_t is a random walk. (Benhabib-Jovanovic, AER, 1995).

Modelling:

$$Y_t = v_t F(K_t, A_t L_t)$$

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t$$

$$C_t = (1-s)Y_t, \qquad L_t = 1$$

$$A_t = A_0 \gamma^t$$

$$v_t \sim D[\alpha, \beta], \quad iid, \quad E(v_t) = \bar{v}$$

$$\frac{Y_t}{A_t L_t} = y_t = v_t f(k_t)$$

$$K_{t+1} = (1-\delta)K_t + (1-s)v_t F(K_t, A_t L_t)$$

$$\frac{K_{t+1}}{A_t L_t} = (1-\delta)\frac{K_t}{A_t L_t} + (1-s)v_t f(k_t)$$

$$\frac{K_{t+1}}{A_{t+1}L_{t+1}} \frac{A_{t+1}L_{t+1}}{A_t L_t} = \gamma k_{t+1} = (1-\delta)k_t + (1-s)v_t f(k_t$$

Steady State:

$$\gamma\bar{k} = (1-\delta)\bar{k} + (1-s)\bar{v}f(\bar{k})$$

$$\bar{y} = \bar{v}f(\bar{k})$$

$$\hat{k}_t = \frac{k_t - \bar{k}}{\bar{k}}; \quad \hat{v}_t = \frac{v_t - \bar{v}}{\bar{v}}; \quad \hat{y}_t = \frac{y_t - \bar{y}}{\bar{y}}$$

Linearization

$$\hat{k}_{t+1} = a_1\hat{k}_t + c_1\hat{v}_t$$

$$\hat{y}_t = b_2\hat{k}_t + \hat{v}_t; \quad \hat{k}_t = (b_2)^{-1}(\hat{y}_t - \hat{v}_t)$$

$$(b_2)^{-1}(\hat{y}_{t+1} - \hat{v}_{t+1}) = a_1(b_2)^{-1}(\hat{y}_t - \hat{v}_t) + c_1\hat{v}_t$$

$$\hat{y}_{t+1} = a_1\hat{y}_t + (b_2c_1 - a_1)\hat{v}_t + \hat{v}_{t+1}$$

$$\hat{y}_{t+1} = a_1\hat{y}_t + \varepsilon_t$$

$$\varepsilon_t = (b_2c_1 - a_1)\hat{v}_t + \hat{v}_{t+1}$$

Now, $\ell n(\hat{y}_{t+1}) = \ell n\left(\frac{y_t}{\bar{y}}\right) \approx \hat{y}_t$. (expanding around 1)

$$y_t = \frac{Y_t}{A_0\gamma^t}$$

$$\hat{y}_{t+1} = \ell n\left(\frac{y_{t+1}}{\bar{y}}\right)$$

$$\hat{y}_{t+1} = \ell n(Y_{t+1}) - (t+1)\ell n(\gamma) - \ell n(A_0\bar{y})$$

$$= a\ell n(Y_t) - at\ell n(\gamma) - a\ell n(A_0\bar{y}) + \varepsilon_t$$

$$\tilde{y}_{t+1} = \ell n(Y_{t+1}) = a\ell n(Y_t) + bt + c + \varepsilon_t$$

But ε_t is serially correlated.

So, Using Cobb-Douglas with $\alpha, 1-\alpha$ shares

$$\ell n(Y_{t+1}) = a_0 + \alpha \ell n(K_t) + (1-\alpha)\ell n(L_t) + a_1 t + a_2 v_t$$

$$v_t = a_4 v_{t-1} + c_4 t + e_t$$

$$\tilde{y}_t = a_1 \tilde{y}_{t-1} + bt + (b_2 c_1 - a_1)\hat{v}_{t-1} + \hat{v}_t$$

$$\hat{v}_t = a_4 \hat{v}_{t-1} + c_4 t + \varepsilon_{t-1}$$