0.1 Vintage Model

Vintages last three periods

$$Max_{\{k,l\}} \sum_{t=2}^{\infty} \beta^t U \left(\begin{array}{c} \sum_{s=-2}^{0} f\left(k_s^t, l_s^t \gamma^{t+s} X^t\right) \\ -(1-\tau_I) k_{t+1} + \tau_t \end{array} \right)$$

where k_s^t is capital of vintage $t-s$ used at time t,that is k_{t-s}, and l_s^t is labor at t allocated to k_s^t. f is CRS. Note that $X > 1$ is disembodied and $\gamma > 1$ is embodied Harrod Neutral growth, τ_I is investment subsidy, τ_t is lump-sum tax.

FOC:

$$\begin{aligned} (1-\tau_I) U'(c_t) &= \beta U'(c_{t+1}) f_k\left(k_0^{t+1}, l_0^{t+1} \gamma^{t+1} X^{t+1}\right) \\ &+ \beta^2 U'(c_{t+2}) f_k\left(k_{-1}^{t+2}, l_{-1}^{t+2} \gamma^{t+2} X^{t+2}\right) \\ &+ \beta^3 U'(c_{t+3}) f_k\left(k_{-2}^{t+3}, l_{-2}^{t+3} \gamma^{t+3} X^{t+3}\right) \end{aligned}$$

Assuming interiority

$$
\begin{aligned}
& f_l\left(k_0^{t+1}, l_0^{t+1}\gamma^{t+1}X^{t+1}\right) \\
= \; & \gamma^{-1} f_l\left(k_{-1}^{t+1}, l_{-1}^{t+1}\gamma^{t}X^{t+1}\right) \\
= \; & \gamma^{-2} f_l\left(k_{-2}^{t+1}, l_{-2}^{t+1}\gamma^{t-1}X^{t+1}\right)
\end{aligned}
$$

or

$$
\begin{aligned}
& f_l\left(\frac{k_0^{t+1}}{l_0^{t+1}\gamma^{t+1}X^{t+1}}\right) \\
&= \gamma^{-1} f_l\left(\frac{k_{-1}^{t+1}}{l_{-1}^{t+1}\gamma^{t}X^{t+1}}\right) \\
&= \gamma^{-2} f_l\left(\frac{k_{-2}^{t+1}}{l_{-2}^{t+1}\gamma^{t-1}X^{t+1}}\right) \\
f_l\left(R_0^{t+1}\right) &= \gamma^{-1} f_l\left(R_{-1}^{t+1}\right) = \gamma^{-2} f_l\left(R_{-2}^{t+1}\right)
\end{aligned}
$$

There is a solution for CES, or in general, a solution that expresses R_{-i}^{t+1}as a function of $R_0^{t+1}$with $R_0^{t+1} < R_{-1}^{t+1} < R_{-2}^{t+1}$.

If

$$
\begin{aligned}
& f\left(k_s^t, l_s^t \gamma^{(t+s)} X^t\right) \\
= & \left(\alpha\left(k_s^t\right)^{1-\rho}+(1-\alpha)\left(\gamma^{t+s} X^t l_s^t\right)^{1-\rho}\right)^{\frac{1}{1-\rho}} \\
= & \gamma^{t+s} X^t\left(\alpha\left(\frac{k_s^t}{\gamma^{t+s} X^t}\right)^{1-\rho}+(1-\alpha)\left(l_s^t\right)^{1-\rho}\right)^{\frac{1}{1-\rho}} \\
= & \gamma^{t+s} X^t l_s^t\left(\alpha\left(\frac{k_s^t}{\gamma^{t+s} X^t l_s^t}\right)^{1-\rho}+(1-\alpha)\right)^{\frac{1}{1-\rho}}
\end{aligned}
$$

Then, if

$$R_s^t=\frac{k_s^t}{\gamma^{(t+s)} X^t l_s^t}$$

MPL is

$$
\begin{aligned}
& f_l\left(k_s^t, l_s^t \gamma^{(t+s)} X^t\right) \\
= & \gamma^{(t+s)} X^t(1-\alpha)\left(\alpha\left(R_s^t\right)^{1-\rho}+1-\alpha\right)^{\frac{\rho}{1-\rho}}
\end{aligned}
$$

From equating MPL's,

$$f_l\left(R_0^{t+1}\right)=\gamma^{-1}f_l\left(R_{-1}^{t+1}\right)=\gamma^{-2}f_l\left(R_{-2}^{t+1}\right)$$

we get:

$$\begin{aligned}&R_{-1}^{t}\\=\quad&\left(\left(\gamma^{\frac{1-\rho}{\rho}}\left(\alpha\left(R_0^t\right)^{1-\rho}+1-\alpha\right)-(1-\alpha)\right)\alpha^{-1}\right)^{\frac{1}{1-\rho}}\end{aligned}$$

$$\begin{aligned}&R_{-2}^{t}\\=\quad&\left(\left(\gamma^{\frac{2(1-\rho)}{\rho}}\left(\alpha\left(R_0^t\right)^{1-\rho}+1-\alpha\right)-(1-\alpha)\right)\alpha^{-1}\right)^{\frac{1}{1-\rho}}\end{aligned}$$

If for simplicity we assume Cobb-Douglas,

$$f=\left(k_s^{t+1}\right)^{\alpha}\left(l_s^{t+1}\gamma^{t+1+s}X^{t+1}\right)^{1-\alpha}$$

then

$$R_0^{t+1}=\gamma^{-\frac{1}{\alpha}}R_{-1}^{t+1}=\gamma^{-\frac{2}{\alpha}}R_{-2}^{t+1}$$

The Euler eq. becomes:

$$\begin{aligned}
& (1-\tau_I)U'(c_t) \\
= \; & \beta U'(c_{t+1}) f_k\left(R_0^{t+1}\right) + \beta^2 U'(c_{t+2}) f_k\left(R_{-1}^{t+2}\right) \\
& + \beta^3 U'(c_{t+3}) f_k\left(R_{-2}^{t+3}\right)
\end{aligned}$$

$$\begin{aligned}
& (1-\tau_I)U'(c_t) \\
= \; & \beta U'(c_{t+1}) f_k\left(R_0^{t+1}\right) \\
& + \beta^2 U'(c_{t+2}) f_k \\
& \left(\left(\begin{array}{c} \gamma^{\frac{1-\rho}{\rho}}\left(\alpha\left(R_0^{t+1}\right)^{1-\rho} + 1 - \alpha\right) \\ -(1-\alpha) \end{array}\right)\alpha^{-1}\right)^{\frac{1}{1-\rho}} \\
& + \beta^3 U'(c_{t+3}) f_k \\
& \left(\left(\begin{array}{c} \gamma^{2\left(\frac{1-\rho}{\rho}\right)}\left(\alpha\left(R_0^{t+1}\right)^{1-\rho} + 1 - \alpha\right) \\ -(1-\alpha) \end{array}\right)\alpha^{-1}\right)^{\frac{1}{1-\rho}}
\end{aligned}$$

At the steady state U' cancels, and we can solve the above for $R_0^{t+i} = R_0$, so that R_{-1}, R_{-2} are constant, with $R_0 < R_{-1} < R_{-2}$

$$\begin{aligned}
& (1-\tau_I) = \beta f_k(R_0) + \\
& \beta^2 f_k\left(\left(\begin{array}{c} \gamma^{\frac{1-\rho}{\rho}}\left(\alpha(R_0)^{1-\rho} + 1 - \alpha\right) \\ -(1-\alpha) \end{array}\right)\alpha^{-1}\right)^{\frac{1}{1-\rho}} \\
& + \\
& \beta^3 f_k\left(\left(\begin{array}{c} \gamma^{2\left(\frac{1-\rho}{\rho}\right)}\left(\alpha(R_0)^{1-\rho} + 1 - \alpha\right) \\ -(1-\alpha) \end{array}\right)\alpha^{-1}\right)^{\frac{1}{1-\rho}} \\
\equiv \; & G(R_0), \qquad G'(\cdot) < 0
\end{aligned}$$

$$\begin{aligned}
\frac{dR_0^{t+1}}{d\tau_I} &= -\frac{1}{G'\left(R_0^{t+1}\right)} > 0 \, , \\
\frac{d^2 R_0^{t+1}}{d\tau_I d\gamma} &\gtrless 0 \textit{ depending on } 1-\rho \gtrless 0.
\end{aligned}$$

Steady state R_0^{t+1}does not depend on X. If $\rho = 1$, steady state R_0 does not depend on γ. NOTE: Result is identical outside steady state if $U' = 1$.

Since labor must add up to a L (where total labor supply could either be fixed or determined in the labor market):

$$\begin{aligned}
L &= l_0^{t+1} + l_{-1}^{t+1} + l_{-2}^{t+1} \\
L &= k_0^{t+1}\gamma^{-(t+1)}X^{-(t+1)}\left(\frac{l_0^{t+1}}{k_0^{t+1}\gamma^{-(t+1)}X^{-(t+1)}}\right) \\
&\quad + k_{-1}^{t+1}\gamma^{-t}X^{-(t+1)}\left(\frac{l_{-1}^{t+1}}{k_{-1}^{t+1}\gamma^{-t}X^{-(t+1)}}\right) \\
&\quad + k_{-2}^{t+1}\gamma^{-(t-1)}X^{-(t+1)}\left(\frac{l_{-2}^{t+1}}{k_{-2}^{t+1}\gamma^{-(t-1)}X^{-(t+1)}}\right) \\
L &= k_0^{t+1}\gamma^{-(t+1)}X^{-(t+1)}\left(R_0^{t+1}\right)^{-1} \\
&\quad + k_{-1}^{t+1}\gamma^{-t}X^{-(t+1)}\left(R_{-1}^{t+1}\right)^{-1} \\
&\quad + k_{-2}^{t+1}\gamma^{-(t-1)}X^{-(t+1)}\left(R_{-2}^{t+1}\right)^{-1}
\end{aligned}$$

Let

$$v_s^t = k_s^t \gamma^{-(t+s)} X^{-t}$$

Note: This is capital at t, discounted by X^{-t}, the disembodied component that affects all vintages at t in the same way, and also by $\gamma^{-(t+s)}$, the embodied component that depends at t not on t, but on the birth date of this capital at $t + s \leq t$.

$$L = v_0^{t+1} \left(R_0^{t+1}\right)^{-1} + v_{-1}^{t+1} X^{-1} \left(R_{-1}^{t+1}\right)^{-1} + v_{-2}^{t+1} X^{-2} \left(R_{-2}^{t+1}\right)^{-1}$$

But since for $m = 0, 1, 2$ we have $k_s^t = k_{s-m}^{t+m}$, then $v_s^t = v_{s-m}^{t+m} X^{-m}$ so that

$$L = v_0^{t+1} \left(R_0^{t+1}\right)^{-1} + v_0^t \left(R_{-1}^{t+1}\right)^{-1} + v_0^{t-1} \left(R_{-2}^{t+1}\right)^{-1},$$

$$(R_0)\, L = v_0^{t+1} + (R_0)\,(R_{-1})^{-1}\, v_0^t + (R_0)\,(R_{-2})^{-1}\, v_0^{t-1}$$

where R_0, R_1, R_2 are constant at steady state.

Note: $1 > (R_0)\,(R_{-1})^{-1} > (R_0)\,(R_{-2})^{-1}$, so this difference equation in v_0^τ, by Kakeya's theorem has only with negative real parts inside the unit circle→oscillatory.)

So since v_0^t converges to $\bar{v}$,

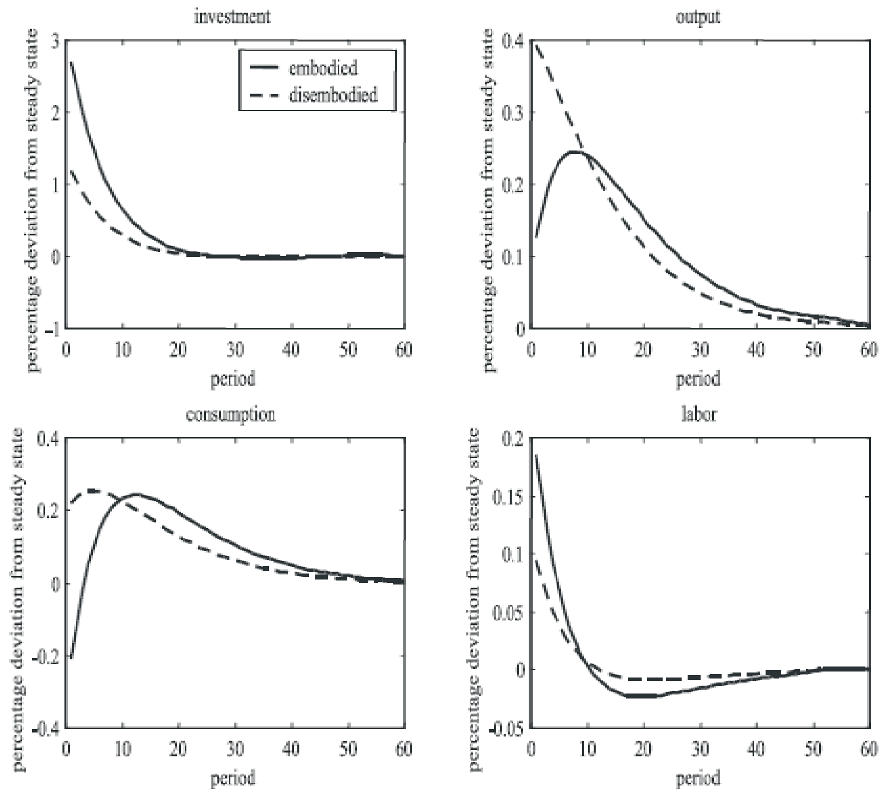
$$\begin{aligned}
\bar{v} &= (R_0)\,L\left(1+\left((R_0)\,(R_{-1})^{-1}\right)+\left((R_0)\,(R_{-2})^{-1}\right)\right)^{-1}\\
\bar{v} &= L\left((R_0)^{-1}+(R_{-1})^{-1}+(R_{-2})^{-1}\right)^{-1}
\end{aligned}$$

NOW NOTE THAT THE PROPERLY DISCOUNTED STEADY STATE CAPITAL v IS INDEPENDENT OF X. IF WE HAVE COBB-DOUGLAS $(\rho = 1)$ IT IS ALSO INDEPENDENT OF γ BECAUSE THEN R_0 IS INDEPENDENT OF γ, AS SHOWN ABOVE. SO WE HAVE PHELPS' RESULT. BUT IT IS NOT INDEPENDENT OF γ IF WE HAVE CES, BECAUSE THEN R_0 DEPENDS ON γ , SO COMPOSITION OF GROWTH MATTERS.

IN PARTICULAR $\frac{dR_0^{t+1}}{d\tau_I}$ WOULD DEPEND ON γ and $(1-\rho)$.

Does it matter if the growth of productivity is driven by disembodied technological change that raises the productivity of all factors of production, including all capital in place, or if it is driven by continuous improvements in the quality of new capital goods? This is the central question of the embodiment issue.

Short run considerations focus on business cycle implications of embodied versus disembodied technological change. Recent works by Greenwood, Hercowitz, and Krusell (2000), Gilchrist and Williams (2000), and DeJong, Ingram, and Whiteman (2000), study the transitional dynamics of various vintage models. Our main focus is also on the short run implications of embodied technological change. In particular, we are interested in whether investment driven economic expansions stimulated by significant innovations and quality improvements in capital goods tend to produce subsequent economic slowdowns.



The solid line in figure depicts the impulse response to an initial shock. It exhibits a sharp spike in investment, a real investment boom. Consumption, after dropping on impulse, smoothly climbs and settles at a higher level. The initial drop in consumption is caused by the increased labor supply that is induced by the embodied shock. Since consumption and leisure are complements a decrease in leisure lowers the marginal utility of consumption and consumption drops on impulse. Employment tracks the investment dynamics. There is a sharp increase in labor supply in order to produce a lot of output that can be converted into new and cheap capital goods. The most interesting dynamic is exhibited by output. It rises with the initial increase in investment and continues to climb because the new vintages of capital come online while older less efficient vintages are scrapped. As the investment vintage profile evens out, a humped shape response in output emerges, where on the downside of the hump output and employment contracts as if in a recession.

The persistence of the transient component of the innovation, ρ_Q, affects not only the size of the impact effect of the shock on investment and output, but also the shape of the hump in the response of output. The effects are small for $\rho_Q = 0$, because in that case the investment response is not sufficient to increase the capital stock and output after the initial period. The effects do become large as ρ_Q approaches unity. Thus a more persistent rate of innovation, typical in the early phases of the introduction of a new technology, is likely to generate a larger hump-shaped response of output. Note that the hump-shaped response of output, caused by the echo-effects, is in stark contrast to the monotonically declining impulse responses of output to embodied shocks reported in Greenwood, Hercowitz, and Krusell (1999) and DeJong, Ingram, and Whiteman (2000).

Vintage II Utility:

$$U\left(c_t, L_t\right) = \left((1-\varepsilon)\right)^{-1}\left(c_t^{((1-\varepsilon))} - 1\right)$$

Production:

$$y = z\left(a_1 k_1^{1-\varepsilon} + a_2 k_2^{1-\varepsilon} + a_3 k_3^{1-\varepsilon} + a_4\left(H-h\right)^{1-\varepsilon}\right)^{\frac{1}{(1-\varepsilon)}}$$

Value function

$$\begin{aligned} & V\left(k_1, k_2, k_3, H, z\right) \\ = \; & \underset{c,h}{Max}\left((1-\varepsilon)\eta\right)^{-1}\left(c_t^{((1-\varepsilon))} - 1\right) \\ & +\delta EV\left(\begin{array}{c} z\left(\begin{array}{c} a_1 k_1^{1-\varepsilon} + a_2 k_2^{1-\varepsilon} \\ +a_3 k_3^{1-\varepsilon} + a_4\left(H-h\right)^{1-\varepsilon} \end{array}\right)^{\frac{1}{(1-\varepsilon)}} \\ -c, \mu_1 k_1, \mu_2 k_2, gh, z' \end{array}\right) \end{aligned}$$

FOC:
wrt c :

$$c^{-\varepsilon}=\delta EV_1'\left(\begin{array}{c} z\left(\begin{array}{c} a_1k_1^{1-\varepsilon}+a_2k_2^{1-\varepsilon} \\ +a_3k_3^{1-\varepsilon}+a_4\left(H-h\right)^{1-\varepsilon}\end{array}\right)^{\frac{1}{(1-\varepsilon)}} \\ -c,\mu_1k_1,\mu_2k_2,gh,z'\end{array}\right)$$

wrt h :

$$\begin{array}{rl} & \delta EV_1'\cdot z\left(\begin{array}{c} a_1k_1^{1-\varepsilon}+a_2k_2^{1-\varepsilon} \\ +a_3k_3^{1-\varepsilon}+a_4\left(H-h\right)^{1-\varepsilon}\end{array}\right)^{\frac{\varepsilon}{(1-\varepsilon)}} \\ & \cdot a_4\left(H-h\right)^{-\varepsilon} \\ = & Eg\delta V_4'\left(\begin{array}{c} z\left(\begin{array}{c} a_1k_1^{1-\varepsilon}+a_2k_2^{1-\varepsilon}+a_3k_3^{1-\varepsilon} \\ +a_4\left(H-h\right)^{1-\varepsilon}\end{array}\right)^{\frac{1}{(1-\varepsilon)}} \\ -c,\mu_1k_1,\mu_2k_2,gh,z'\end{array}\right) \\ = & c^{-\varepsilon}z\left(\begin{array}{c} a_1k_1^{1-\varepsilon}+a_2k_2^{1-\varepsilon}+a_3k_3^{1-\varepsilon} \\ +a_4\left(H-h\right)^{1-\varepsilon}\end{array}\right)^{\frac{\varepsilon}{1-\varepsilon}} a_4\left(H-h\right)^{-\varepsilon}\end{array}$$

$$\begin{array}{rcl} V_4 & = & \delta EV_1'z\left(\begin{array}{c} a_1k_1^{1-\varepsilon}+a_2k_2^{1-\varepsilon} \\ +a_3k_3^{1-\varepsilon}+a_4\left(H-h\right)^{1-\varepsilon}\end{array}\right)^{\frac{\varepsilon}{(1-\varepsilon)}} \\ & & \cdot a_4\left(H-h\right)^{-\varepsilon} \\ & = & c^{-\varepsilon}z\left(\begin{array}{c} a_1k_1^{1-\varepsilon}+a_2k_2^{1-\varepsilon} \\ +a_3k_3^{1-\varepsilon}+a_4\left(H-h\right)^{1-\varepsilon}\end{array}\right)^{\frac{\varepsilon}{(1-\varepsilon)}} \\ & & \cdot a_4\left(H-h\right)^{-\varepsilon}\end{array}$$

So

$$\begin{array}{rl} & c^{-\varepsilon}z\left(\begin{array}{c} a_1k_1^{1-\varepsilon}+a_2k_2^{1-\varepsilon} \\ +a_3k_3^{1-\varepsilon}+a_4\left(H-h\right)^{1-\varepsilon}\end{array}\right)^{\frac{\varepsilon}{(1-\varepsilon)}} a_4\left(H-h\right)^{-\varepsilon} \\ = & Eg\delta\left(c'\right)^{-\varepsilon}z\left(\begin{array}{c} a_1\left(k'\right)_1^{1-\varepsilon}+a_2\left(k'\right)_2^{1-\varepsilon} \\ +a_3\left(k'\right)_3^{1-\varepsilon}+a_4\left(H'-h'\right)^{1-\varepsilon}\end{array}\right)^{\frac{\varepsilon}{(1-\varepsilon)}} \\ & \cdot a_4\left(H'-h'\right)^{-\varepsilon}\end{array}$$

Now differentiating $V(k_1, k_2, k_3, z)$,where $'$, $''$, $'''$indicates variables one, two, three periods ahead:

$$V_3 = \delta E V_1' \cdot MPK_3 = c^{-\varepsilon} MPK_3$$

$$\begin{aligned} V_2 &= c^{-\varepsilon} MPK_2 + \mu_2 \delta E V_3' \\ &= c^{-\varepsilon} MPK_2 + \mu_2 \delta E (c')^{-\varepsilon} MPK_3' \end{aligned}$$

$$\begin{aligned} V_1 &= c^{((1-\varepsilon)\omega)-1} MPK_1 + \mu_1 \delta E V_2' \\ &= c^{-\varepsilon} MPK_1 + \mu_1 \delta E \begin{pmatrix} (c')^{-\varepsilon} MPK_2' \\ +\mu_2 \delta E (c'')^{-\varepsilon} MPK_3'' \end{pmatrix} \end{aligned}$$

Updating V_1 :

$$\begin{aligned} V_1' &= (c')^{-\varepsilon} MPK_1' + \mu_1 \delta E V_2'' \\ &= (c')^{-\varepsilon} MPK_1' + \mu_1 \delta E \begin{pmatrix} (c'')^{-\varepsilon} MPK_2'' \\ +\mu_2 \delta E (c''')^{-\varepsilon} MPK_3''' \end{pmatrix} \end{aligned}$$

So FOC becomes:

$$c^{-\varepsilon} = \delta E \begin{bmatrix} (c')^{-\varepsilon} MPK_1' + \mu_1 \delta (c'')^{-\varepsilon} MPK_2'' \\ +\mu_1 \mu_2 \delta^2 (c''')^{-\varepsilon} MPK_3''' \end{bmatrix}$$

Now we note that:

$$y = z \begin{pmatrix} a_1 k_1^{1-\varepsilon} + a_2 k_2^{1-\varepsilon} + a_3 k_3^{1-\varepsilon} \\ +a_4 (H' - h')^{1-\varepsilon} \end{pmatrix}^{\frac{1}{(1-\varepsilon)}}$$

$$\begin{aligned} MPK_i &= z \begin{pmatrix} a_1 k_1^{1-\varepsilon} + a_2 k_2^{1-\varepsilon} + a_3 k_3^{1-\varepsilon} \\ +a_4 (H' - h')^{1-\varepsilon} \end{pmatrix}^{\frac{\varepsilon}{(1-\varepsilon)}} a_i k_i^{-\varepsilon} \\ &= a_i z^{(1-\varepsilon)\omega} \frac{y^{\varepsilon}}{k_i^{\varepsilon}} \end{aligned}$$

So FOC is

$$c^{-\varepsilon} = \delta E \begin{bmatrix} (c')^{-\varepsilon} a_1 (z')^{(1-\varepsilon)} \frac{(y')^{\varepsilon}}{(k_1')^{\varepsilon}} \\ +\mu_1 \delta (c'')^{-\varepsilon} a_2 (z'')^{(1-\varepsilon)} \frac{(y'')^{\varepsilon}}{(k_2'')^{\varepsilon}} \\ +\mu_1 \mu_2 \delta^2 (c''')^{-\varepsilon} a_3 (z''')^{(1-\varepsilon)} \frac{(y''')^{\varepsilon}}{(k_3''')^{\varepsilon}} \end{bmatrix}$$

Now let $c = \lambda y$:

$$(\lambda y)^{-\varepsilon} = \delta E \begin{bmatrix} (\lambda')^{-\varepsilon} a_1 (z')^{(1-\varepsilon)} (k_1')^{-\varepsilon} \\ +\mu_1 \delta (\lambda'')^{-\varepsilon} a_2 (z'')^{(1-\varepsilon)} (k_1'')^{-\varepsilon} \\ +\mu_1 \mu_2 \delta^2 (\lambda''')^{-\varepsilon} a_3 (z''')^{(1-\varepsilon)} (k_1''')^{-\varepsilon} \end{bmatrix}$$

If z is *iid* or constant, then set $\lambda = \lambda' = \lambda'' = \lambda'''$:

$$y^{-\varepsilon} = \delta E \left[\begin{array}{c} a_1 (z')^{(1-\varepsilon)} ((1-\lambda) y)^{-\varepsilon} \\ +\mu_1 \delta a_2 (z'')^{(1-\varepsilon)} (\mu_1 (1-\lambda) y)^{-\varepsilon} \\ +\mu_1 \mu_2 \delta^2 a_3 (z''')^{(1-\varepsilon)} (\mu_1 \mu_2 (1-\lambda) y)^{-\varepsilon} \end{array} \right]$$

$$1 = \delta E \left[\begin{array}{c} a_1 (z')^{(1-\varepsilon)} ((1-\lambda))^{-\varepsilon} \\ + (\mu_1)^{1-\varepsilon} \delta a_2 (z'')^{(1-\varepsilon)} ((1-\lambda))^{-\varepsilon} \\ + (\mu_1 \mu_2)^{1-\varepsilon} \delta^2 a_3 (z''')^{(1-\varepsilon)} ((1-\lambda) y)^{-\varepsilon} \end{array} \right]$$

$$(1-\lambda)^{\varepsilon} = \delta E \left[\begin{array}{c} a_1 (z')^{(1-\varepsilon)} + (\mu_1)^{1-\varepsilon} \delta a_2 (z'')^{(1-\varepsilon)} \\ + (\mu_1 \mu_2)^{1-\varepsilon} \delta^2 a_3 (z''')^{(1-\varepsilon)} \end{array} \right]$$

Now for H, h :

$$\begin{aligned} & c^{-\varepsilon} z \begin{pmatrix} a_1 k_1^{1-\varepsilon} + a_2 k_2^{1-\varepsilon} + a_3 k_3^{1-\varepsilon} \\ + a_4 (H-h)^{1-\varepsilon} \end{pmatrix}^{\frac{\varepsilon}{(1-\varepsilon)}} a_4 (H-h)^{-\varepsilon} \\ = \; & \delta E g (c')^{-\varepsilon} (z') \begin{pmatrix} a_1 (k')_1^{1-\varepsilon} + a_2 (k')_2^{1-\varepsilon} + a_3 (k')_3^{1-\varepsilon} \\ + a_4 (H'-h')^{1-\varepsilon} \end{pmatrix}^{\frac{\varepsilon}{(1-\varepsilon)}} \\ & \cdot a_4 (H'-h')^{-\varepsilon} \end{aligned}$$

If z is iid, λ is constant. So, if $h = \eta H$,

$$(\lambda)^{-\varepsilon} z^{1-\varepsilon} \left(\frac{1}{H-h} \right)^{\varepsilon} = E g \delta (\lambda)^{-\varepsilon} (z')^{1-\varepsilon} \left(\frac{1}{H'-h'} \right)^{\varepsilon}$$

$$z^{1-\varepsilon} \left(\frac{1}{H(1-\eta)} \right)^{\varepsilon} = E g^{1-\varepsilon} \delta (z')^{1-\varepsilon} \eta^{-\varepsilon} \left(\frac{1}{H(1-\eta)} \right)^{\varepsilon}$$

$$z^{1-\varepsilon} = \eta^{-\varepsilon} \delta E g^{1-\varepsilon} (z')^{1-\varepsilon}$$

Therefore

$$\eta = \delta^{\frac{1}{\varepsilon}} \left(\frac{E\left(g^{1-\varepsilon} (z')^{1-\varepsilon} \right)}{z^{1-\varepsilon}} \right)^{\frac{1}{\varepsilon}}$$

NOTE: Is η constant?? Only if z is constant and not stochastic! Assume that.

Let $x_t = (k_t)^{1-\varepsilon}$. Then, since

$$y = z \begin{pmatrix} a_1 k_1^{1-\varepsilon} + a_2 k_2^{1-\varepsilon} + a_3 k_3^{1-\varepsilon} \\ + a_4 (H'-h')^{1-\varepsilon} \end{pmatrix}^{\frac{1}{(1-\varepsilon)}}$$

we have a linear difference equation system:

$$\begin{aligned} x_{t+1} &= ((1-\lambda) z_t)^{1-\varepsilon} \begin{pmatrix} a_1 x_t + a_2 x_{t-1} \\ + a_3 x_{t-2} + a_4 ((1-\eta) H_t)^{1-\varepsilon} \end{pmatrix} \\ H_{t+1} &= g \eta H_t \end{aligned}$$

Let $s_t = H_t^{1-\varepsilon}$. Then:

$$x_{t+1} = ((1-\lambda)\, z_t)^{1-\varepsilon} \left(\begin{array}{c} a_1 x_t + a_2 x_{t-1} \\ +a_3 x_{t-2} + a_4 (1-\eta)^{1-\varepsilon} s_t \end{array} \right)$$

and

$$\begin{aligned} H_{t+1} &= g\eta H_t \\ (s_{t+1})^{\frac{1}{1-\varepsilon}} &= (g\eta)\, (s_t)^{\frac{1}{1-\varepsilon}} \\ (s_{t+1}) &= (g\eta)^{1-\varepsilon} (s_t) \end{aligned}$$

So

$$\begin{aligned} \frac{x_{t+1}}{s_{t+1}} &= \left(\frac{((1-\lambda)\, z_t)^{1-\varepsilon}}{(g\eta)^{1-\varepsilon}} \right) \\ &\quad \cdot \left(\begin{array}{c} a_1 \left(\frac{x_t}{s_t} \right) + \frac{a_2}{(g\eta)^{(1-\varepsilon)}} \left(\frac{x_{t-1}}{s_{t-1}} \right) \\ + \frac{a_3}{(g\eta)^{2(1-\varepsilon)}} \left(\frac{x_{t-2}}{s_{t-2}} \right) + a_4 (1-\eta)^{1-\varepsilon} \end{array} \right) \end{aligned}$$

$$\begin{aligned} \frac{x_{t+1}}{s_{t+1}} &= \left(((1-\lambda)\, z_t)^{1-\varepsilon} \right) \\ &\quad \cdot \left(\begin{array}{c} \frac{a_1}{(g\eta)^{1-\varepsilon}} \left(\frac{x_t}{s_t} \right) + \frac{a_2}{(g\eta)^{2(1-\varepsilon)}} \left(\frac{x_{t-1}}{s_{t-1}} \right) \\ + \frac{a_3}{(g\eta)^{3(1-\varepsilon)}} \left(\frac{x_{t-2}}{s_{t-2}} \right) + \frac{a_4}{(g\eta)^{(1-\varepsilon)}} (1-\eta)^{1-\varepsilon} \end{array} \right) \end{aligned}$$